Symplectic diffeomorphisms and Weinstein 1-form

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Abstract. In [5] the authors showed that the Liouville 1-form lying on the cotangent bundle is derived from physical potential and is related to the symplectomorphism through the flux homomorphism. On the other hand, in [7, 8], A. Weinstein constructed a chart from the group of symplectic diffeomorphisms isotopic to the identity by using Lagrangian sub-manifolds geometry and from which he derived a closed 1-form called the Weinstein 1-form. In this paper, we establish a relation between the Liouville 1-form and the Weinstein 1-form through an explicit formula from which we derive a new characterization of symplectomorphism and a new formula of the flux homomorphism.

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1 Introduction

Symplectic geometry, as defined by *Dusa M. Duff*, is an even dimensional geometry living on an even manifold. It is a geometry of a non degenerated and closed 2-form. It arose in the 1800's with the work of *Joseph Louis Lagrange*, *Simeon Denis Poisson* and *William Rowan Hamilton.* The word symplectic have been introduced to the mathematical society by *Hermann Weyl* in 1946. In 1953, *J.M. Souriau* introduced symplectic geometry as a strong tool to study mechanic by geometrical methods. Concerning symplectic diffeomorphisms, they appeared the first time in the work of *H. Poincaré* studying celestial mechanic. After then, they become a subject of further studies in symplectic geometry. In this paper, we mainly focus on their modern development. Especially, on one hand, we establish the relationship between the *Weinstein* 1-form and symplectomorphisms. On the other hand, we establish the relationship between the *Liouville* 1-form and symplectic diffeomorphism by using the flux homomorphism.

The relation between symplectomorphism isotopic to the identity and *Weinstein* 1-form appears in *A. Banyaga*'s monography [1] from which the *Weinstein* 1-form

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has been deduced from Lagrangian submanifolds and from the *Kostant-Weinstein-Sternberg* theorem. Unfortunately, the problem of the *A. Banyaga* and *A. Bounemoura* [2] presentation of the *Weinstein* 1-form doesn't give explicitly its existence. Herein, we exhibit explicit formulas related to the *Weinstein* 1-form and study the local geometry of the *Weinstein* chart at the identity. Related to the group of symplectic diffeomorphisms is the flux homomorphism introduced by *E. Calabi* and studied by *T. Rybicki* [4] to characterize *Poisson* isotopies.

In this paper, we give a new formula of the flux homomorphism from the composition of the universal cover of the group of symplectomorphisms with the *Weinstein* 1-form and the projection of the space of closed 1-forms on the *De Rham* cohomology.

In the context of the flux homomorphism, the relationship between the *Liouville* 1-form and symplectic diffeomorphisms is a measure to the obstruction of diffeomorphisms to preserve the *Liouville* 1-form. In other words, this obstruction is expressed by the non triviality of the cohomology class of the *Weinstein* 1-form defined by the use of the flux homomorphism.

This work is organized as follows:

1. A review of symplectic geometry and the *Weinstein* chart.
2. Statements of the main results.
3. Flux homomorphism associated with the *Weinstein* chart.
4. Conclusion and perspective.
5. References.

2 A brief review of the Weinstein chart and symplectic geometry

Definition 2.1. A symplectic form on the manifold M of even dimension is a 2-form Ω on M such that:

1. for $x \in M$,

$$\begin{aligned} \widetilde{\Omega}_x : T_xM &\longrightarrow T_x^*M \\ X_x &\longmapsto \widetilde{\Omega}_x(X_x), \end{aligned}$$

   with $\widetilde{\Omega}_x(X_x) : T_xM \longrightarrow \mathbb{R},\ Y_x \longmapsto \widetilde{\Omega}_x(X_x)(Y_x) = \Omega_x(X_x, Y_x)$, is an isomorphism i.e. Ω is non degenerated.

2. $d\Omega = 0$ i.e Ω is a closed 2-form.

The pair (M, Ω) with M a C^∞ manifold of even dimension and Ω a symplectic form on M is called a symplectic manifold.

Proposition 2.1. *Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds. Then, the product $(M_1 \times M_2, \Omega_{\lambda,\mu})$ with*

$$\Omega_{\lambda,\mu} = \lambda\pi_1^*\Omega_1 + \mu\pi_2^*\Omega_2$$

is a symplectic manifold.

In particular, $(M \times M,\, \Omega_{1,-1})$ *is a symplectic manifold with* $\Omega_{1,-1} = \pi_1^*\Omega - \pi_2^*\Omega$ *its symplectic form.*

Among the examples of symplectic manifolds, the cotangent bundle plays a crucial role. In fact, let the projection $q : T^*M \longrightarrow M$ be given.
The *Liouville* 1-form denoted λ_M on T^*M is defined by $\lambda_M(a) = \langle \theta_x,\, (d_a q)(\xi)\rangle$ with $\xi \in T_a(T^*M)$ and $a = (x, \theta_x)$, with $x \in M$ and $\theta_x \in T_x^*M$.
Locally, the *Liouville* 1-form is given by the proposition below:

Proposition 2.2. *There exists local coordinates of* T^*M *such that relative to these coordinates* $(x_1, \ldots, x_n, y_1, \ldots, y_n)$*, the* Liouville 1*-form is locally written:*

$$\lambda_M = \sum_{i=1}^{n} y_i dx_i.$$

Hence, $\Omega_M = d\lambda_M$ *is a symplectic form on* T^*M*.*

Example 2.2. The pair $(T^*M,\, \Omega_M)$ is a symplectic manifold.

Definition 2.3. Let (M, Ω) be a symplectic manifold. A diffeomorphism $\phi : M \longrightarrow M$ is said to be a *symplectomorphism* if $\phi^*\Omega = \Omega$.
The set of symplectic diffeomorphisms is a group of infinite dimension denoted by $Diff_\Omega^\infty(M)$.

A. Weinstein has shown [7, 8] that this group is locally arcwise connected by using Lagrangian submanifolds geometry, as follows.

Denote by

$$\Gamma_\phi = \big\{(x, y) \in M \times M,\ y = \phi(x)\big\} \tag{2.1}$$

the graph of the diffeomorphism ϕ.
Regarding Lagrangian submanifolds, we have:

Definition 2.4. Let N be a submanifold of (M, Ω). An immersion $j : N \hookrightarrow M$ is said to be Lagrangian if $j^*\Omega = 0$ and $dimN = \frac{1}{2}\, dimM$.

The submanifold $j(N)$ is called a Lagrangian submanifold.
Sniatyki and *W. M. Tulczjew* obtained the characterization of symplectic diffeomorphisms by Lagrangian submanifolds. Precisely, they stated the following theorem:

Theorem 2.3 (Sniatyki, Tulczjew)**.** *A diffeomorphism is symplectic iff its graph* Γ_ϕ *is a Lagrangian submanifold.*

Proof. 1. The condition is necessary.
Let $j : M \hookrightarrow \Gamma_\phi \subset M \times M$ be the immersion of the graph Γ_ϕ in $M \times M$. Then, setting $\underline{\Omega} = \pi^*\Omega - \pi^*\Omega$, we have:

$$\begin{aligned} j^*\underline{\Omega} &= j^*(\pi_1^*\Omega - \pi_2^*\Omega) = j^*\pi^*\omega - j^*\pi_2^*\Omega \\ &= (\pi_1 \circ j)^*\Omega - (\pi_2 \circ j)^*\Omega = \Omega - \phi^*\Omega = 0. \end{aligned}$$

2. Conversely, suppose the graph Γ_ϕ is a Lagrangian submanifold of $M \times M$. Then, by a straightforward calculation, we have:

$$0 = j^*\underline{\Omega} = \Omega - \phi^*\Omega.$$

Hence, $\phi^*\Omega = \Omega$ i.e. ϕ is a symplectomorphism. A particular case of Lagrangian submanifold of $M \times M$ is provided by the diagonal

$$\Delta = \big\{(x, x) \in M \times M,\ \phi = id\big\}. \tag{2.2}$$

We herein call the first characterization of symplectomorphism the *Sniatyki - Tulczjew* theorem.

When ϕ is a symplectic diffeomorphism C^0-close to the identity, the Lagrangian immersion will be denoted by the pair (id, ϕ).

□

2.1 One-forms as sections of the cotangent bundles

Lagrangian submanifolds of the cotangent bundle are obtained this way:

Theorem 2.4. *The image $\alpha(M) \subset T^*M$ of a 1-form α, seen as section, is a Lagrangian submanifold of T^*M if and only if α is a closed 1-form.*

Proof. Let $v \in T_xM$. The pull-back of α gives the following:

$$\begin{aligned}\alpha^*(\lambda_M)(x)(v) &= \lambda_M(\alpha(x))\big((d\alpha)_x(v)\big)\\ &= \alpha(x)\big(d(q \circ \alpha)_x(v)\big)\\ &= \alpha(x)(v).\end{aligned}$$

Therefore, $\alpha^*\lambda_M = \alpha$, $x \in M$ and $v \in T_xM$. □

Corollary 2.5. *Let $(T^*M, \Omega_M = d\lambda_M)$ be the symplectic structure on T^*M. Then $\alpha^*\Omega_M = d\alpha$.*

Proof. As $\alpha^*\lambda_M = \alpha$; we have:

$$d\alpha^*\lambda_M = d\alpha \implies \alpha^* d\lambda_M = d\alpha \implies \alpha^*\Omega_M = d\alpha,$$

which proves that $\alpha(M)$ is a Lagrangian submanifold of T^*M if and only if α is a closed 1-form. □

Example 2.5. The zero section $\mathcal{O}_M$ of the cotangent bundle is a Lagrangian submanifold.

2.2 Weinstein's chart

A. Weinstein, *Kostant* and *Sternberg* have related the above theorems relying on the first characterization of symplectic diffeomorphism and that of the characterization of Lagrangian submanifold by closed 1-forms. Mainly, they stated the following theorem:

Theorem 2.6. *(Kostant - Weinstein - Sternberg) Let S be a Lagrangian submanifold of a symplectic manifold (M, Ω). Let S be regarded as the zero section in (T^*S, Ω_S). There exists a diffeomorphism k of a neighborhood $U(S)$ of S in $M \times M$ into a neighborhood $\mathcal{W}(\mathcal{O}(S)) \subset T^*S$ such that $k/S = id$ and $k^*\Omega_S = \Omega$.*

In fact, S can be regarded both as a graph of $M \times M$ and a Lagrangian submanifold of T^*M by means of the *Kostant* map k.

Inspired by Theorem 2.3 ([5]), Theorem 2.4 ([2]) and the Kostant - Weinstein - Sternberg Theorem 2.6 ([1]), we have the following construction due to *A. Banyaga* [1] and related to the existence of the *Weinstein* chart and hence the *Weinstein* 1-form.

Theorem 2.7 (A. Banyaga)**.** *Let ϕ be a symplectic diffeomorphism isotopic to the identity in the C^0-topology. Then, there exists a chart i.e*

$$\begin{aligned} \mathcal{W} : \mathcal{V} \subset Diff^{\infty}_{\Omega}(M)_0 &\longrightarrow \mathcal{Z}^1_c(M) \\ \phi &\longrightarrow \mathcal{W}(\phi). \end{aligned}$$

Proof. Let ϕ be a symplectic diffeomorphism C^1-close to the identity and Γ_ϕ its Lagrangian submanifold C^1-close to the diagonal in $M \times M$. By the *Kostant-Weinstein-Sternberg* theorem 2.6 and the preservation of Lagrangian submanifolds by symplectomorphism, $k(\Gamma(\phi))$ is a Lagrangian submanifold in T^*M. Hence, by the theorem 2.4 on the characterization of Lagrangian submanifolds in T^*M by closed 1-form, there exists a closed 1-form whose Lagrangian submanifold is $k\big(\Gamma(\phi)\big)$ and denoted by $\mathcal{W}(\phi)$. In [3] and [1], the authors, from the above proof, deduced the *Weinstein* chart denoted too by the correspondence:

$$\begin{aligned} \mathcal{W} : \mathcal{V} \subset Diff^{\infty}_{\Omega}(M)_0 &\longrightarrow \mathcal{Z}^1_c(M) \\ \phi &\longrightarrow \mathcal{W}(\phi). \end{aligned}$$

□

Let U_0 be a neighborhood of the zero section in $\mathcal{Z}^1_c(M)$ and $\mathcal{V} = \mathcal{W}^{-1}(U_0)$ the *Weinstein* domain at id_M. In the sequel, we will be studying the local geometry of the group $Diff^{\infty}_{\Omega}(M)_0$ using the identity of the *Weinstein* domain.

3 Main results

In this section, we explicitly show the existence of the *Weinstein* 1-form. After that, we study the local geometry of the group of symplectic diffeomorphism isotopic to the identity which lies in the *Weinstein* domain.

The local geometry of the group of symplectic diffeomorphisms isotopic to the identity defined by the *Weinstein* chart is well understood by the following proposition.

Proposition 3.1. *Let ϕ be a symplectomorphism C^0-close to the identity and $(\mathcal{W}, \mathcal{V})$ be the* Weinstein *chart with $\mathcal{V}$ the* Weinstein *domain. Then,*

$$\mathcal{W}(id) = 0_M$$

with 0_M the zero section of the cotangent bundle.

Proof. A straightforward calculation gives:

$$\begin{aligned}\mathcal{W}(id) &= \big(\gamma^{-1}\circ(id,id)\big)\circ\big(\pi\circ\gamma^{-1}\circ(id,id)\big)\\ &= (\gamma^{-1}\circ 0_M)\circ(\pi\circ\gamma^{-1}\circ 0_M) = 0_M.\end{aligned}$$

□

Moreover, we have the following result, which relates the *Weinstein* 1-form to the *Liouville* 1-form by an explicit formula:

Proposition 3.2.

1. *The* Weinstein 1*-form* $\mathcal{W}(\phi)$ *is d-exact.*
2. *The pull-back of the* Liouville 1*-form* θ_M *on the Lagrangian submanifold* $L = \mathcal{W}(\phi)(M)$ *is d-exact on the Lagrangian submanifold* $\Gamma(\phi)$ *of* ϕ *i.e.* $(id,\phi)^*\theta_1$ *is d-exact on* M*.*

Proof. The proof relies on *Sniatyki-Tulczjew* theorem, the theorem on the characterization of Lagrangian submanifolds in T^*M by means of closed 1-form and the *Kostant-Weinstein-Sternberg* theorem. Set

$$\theta_1 = (\gamma)^*\theta_M.$$

A straightforward computation gives the following:

$$\begin{aligned}\mathcal{W}(\phi)^*\theta_1 &= \mathcal{W}(\phi)^*\gamma^*\theta_M = [\gamma\circ\mathcal{W}(\phi)]^*\theta_M = (id,\phi)^*\theta_M\\ &= \phi^*(\pi_1\theta_M - \pi_2^*\theta_M) = \phi^*\pi_1\theta_M - \phi^*\pi_2^*\theta_M)\\ &= (\pi_1\circ\phi)^*\theta_M - (\pi_2\circ\phi)^*\theta_M = \theta_M - \phi^*\theta_M = \mathcal{W}(\phi).\end{aligned}$$

Therefore, the relation between the *Liouville* 1-form and the *Weinstein* 1-form is given by the formula:

$$\mathcal{W}(\phi) = \theta_M - \phi^*\theta_M. \tag{3.1}$$

Hence, the *Weinstein* 1-form is d-exact, i.e., there exists $S \in C^\infty(M)$, $\mathcal{W}(\phi) = dS$, if and only if $\theta_M - \phi^*\theta_M$ is d-exact if and only if θ_M is d-exact on the Lagrangian submanifold $L = \mathcal{W}(\phi)(M)$. □

Theorem 3.3. *Let* ϕ *be a symplectic diffeomorphism isotopic to the identity in the* C^1*-topology. Then, there exists a closed* 1*-form denoted by* $\mathcal{W}(\phi)$ *and whose graph is a Lagrangian submanifold* $\Gamma_{\mathcal{W}(\phi)}$*.*

Proof. The existence of the *Weinstein* 1-form is guaranteed by the commutative diagram on Figure 1:

Set

$$\mathcal{W}(\phi) = \Big(\gamma^{-1}\circ(id,\phi)\circ\big(\pi\circ\gamma^{-1}\circ(id,\phi)\big)\Big).$$

The 1-form $\mathcal{W}(\phi)$ is closed since $\phi \in Diff^\infty_\Omega(M)_0$ and satisfies Proposition 3.1. □

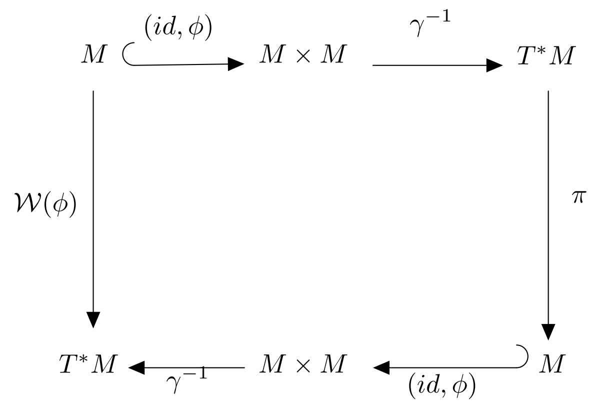


Figure 1: Existence of the Weinstein chart.

More, relying on the *Sniatyki* and *W. M. Tulczjew* theorem characterization of the symplectic diffeomorphisms, the theorem on the characterization of the Lagrangian submanifolds by closed 1-forms and *Kostant-Weinstein-Sternberg* theorem, we have established the formula (3.1).

From this formula, we have restated and proved the new characterization of symplectic diffeomorphisms by means of the *Weinstein* 1-form. We have obtained the following result, thanks to *A. Weinstein*;

Theorem 3.4. *Let ϕ be a diffeomorphism C^0-close to the identity so that its graph is close enough with the diagonal.*

Then ϕ is a symplectomorphism if and only if the Weinstein 1-*form $\mathcal{W}(\phi)$ is a closed* 1-*form.*

Proof. The *Weinstein* 1-form $\mathcal{W}(\phi)$ is closed if and only if $d\mathcal{W}(\phi) = 0$; and

$$\begin{aligned} d\mathcal{W}(\phi) = 0 &\Longleftrightarrow d\theta_M - d\phi^*\theta_M = 0 \\ &\Longleftrightarrow \Omega_M - \phi^*\Omega = 0 \\ &\Longleftrightarrow \phi^*\Omega = \Omega, \end{aligned}$$

and hence ϕ is a symplectomorphism. □

Thus, the *De Rham* cohomology class of the *Weinstein* 1-form $\mathcal{W}(\phi)$ is non trivial. Hence, this non-trivial class of the *De Rham* cohomology is an obstruction to the diffeomorphism ϕ to be a symplectomorphism. To decide whether the above formula agrees with the local geometry of the *Weinstein* chart, calculating at the identity, we have

Corollary 3.5.

$$\mathcal{W}(id) = 0_M.$$

Proof. As $\mathcal{W}(\phi) = \theta_M - \phi^*\theta_M$, at the identity, we still have:

$$\mathcal{W}(id) = \theta_M - id^*\theta_M = 0_M.$$

□

Hence we have proved that the symplectomorphism in the formula (3.1) lies in the *Weinstein* domain. In other words, we have shown that it agrees with the local geometry induced by the *Weinstein* chart.

4 The flux homomorphism associated with the Weinstein chart

We introduce herein the relation between the *Weinstein* 1-form and the flux homomorphism studied in great details by *A. Banyaga* in [1], *C. Viterbo* in [6], *A. Bounemoura* in [2] and *T. Rybicki* in [4]. A new formulation of the flux homomorphism is given.

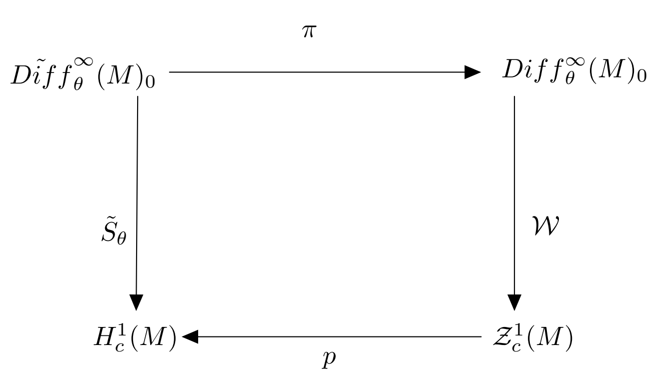
The link between the flux homomorphism and the *Weinstein* 1-form is summarized in the following statement:

Proposition 4.1. *Let θ be a closed 1-form on M and $\tilde{S}_\theta$ the flux homomorphism. Denote by $D\tilde{iff}_\theta^{\infty}(M)_0$ the universal cover of $Diff_\theta^\infty(M)_0$ and $\pi : D\tilde{iff}_\theta^{\infty}(M)_0 \longrightarrow Diff_\theta^\infty(M)_0$ the projection of $D\tilde{iff}_\theta^{\infty}(M)_0$ into $Diff_\theta^\infty(M)_0$.*

Let $\mathcal{Z}_c^1(M)$ be the space of closed 1-forms and $p : \mathcal{Z}_c^1(M) \longrightarrow H_c^1(M)$ the projection of $\mathcal{Z}_c^1(M)$ into the De Rham *cohomology $H_c^1(M)$ with compact support. We denote by $\mathcal{W}$ the* Weinstein *parametrization. The following formula holds:*

$$\tilde{S}_\theta = p \circ \mathcal{W} \circ \pi.$$

Proof. We have to prove that the diagram below is commutative:



In other words;

$$\tilde{S}_\theta = p \circ \mathcal{W} \circ \pi.$$

So, let $\{\phi_t\}$ be the homotopy class of the symplectic isotopy (ϕ_t). We have by direct computation:

$$\begin{aligned}(p \circ \mathcal{W} \circ \pi)\big(\{\phi_t\}\big) &= (p \circ \mathcal{W})\big(\pi(\phi_t)\big)\\ &= p \circ \mathcal{W}(\phi_t)\\ &= [\mathcal{W}(\phi_t)]\\ &= \tilde{S}_\theta(\{\phi_t\}),\end{aligned}$$

i.e.,

$$\tilde{S}_\theta = p \circ \mathcal{W} \circ \pi.$$

Since the Calabi invariant $\tilde{S}_\theta$ descends to the homomorphism S_θ and the relation $\pi' \circ \tilde{S}_\theta = S_\theta \circ \pi$ holds, by a straightforward calculation, we have

$$\begin{aligned}\pi' \circ \tilde{S}_\theta &= \pi' \circ \big(p \circ \mathcal{W} \circ \pi\big)\\ &= \big(\pi' \circ p \circ \mathcal{W}\big) \circ \pi\\ &= S_\theta \circ \pi.\end{aligned}$$

Therefore, $S_\theta = \pi' \circ p \circ \mathcal{W}$. □

A. Banyaga in his marvelous monograph [1] obtained the same result using the *Moser* 1-form and Lagrangian immersions. *A. Bounemoura* obtained the same formula in [2] and *C. Viterbo* in [6].

5 Conclusion and perspective

In this paper, we have mainly stated and proved the characterization of symplectic diffeomorphism by means of the Weinstein 1-form and Lagrangian submanifolds. In the counterpart of this work, the characterization of symplectic diffeomorphism have been obtained before by Śniatycki and Tulczyjew in their joint work.

We also found new formulas which link the flux homomorphism to the Weinstein chart. However, we suspect that these formulas can be used to show that the flux homomorphism kernel is arc wise connected, and hope that our results greatly contribute to the development of symplectic topology.

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