

Harmonic vector fields on vertical rescaled generalized Cheeger-Gromoll metrics

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Abstract. In this paper, we introduce the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM over an m -dimensional Riemannian manifold (M, g) , as a natural metric on TM . We establish a necessary and sufficient conditions under which a vector field is harmonic with respect to the vertical rescaled generalized Cheeger-Gromoll metric. We also construct some examples of harmonic vector fields.

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1 Introduction

The geometry of the tangent bundle TM equipped with the Sasaki metric has been studied by many authors such as Sasaki [22], Yano and Ishihara [24], Dombrowski [8], Salimov and Gezer [19], [20], etc. The rigidity of the Sasaki metric has incited some geometers to construct and study other metrics on TM . Musso and Tricerri have introduced the notion of Cheeger-Gromoll metric [17], which has been studied also by many authors (e.g., see [12], [21], [23]).

Consider a smooth map $\phi : (M^m, g) \rightarrow (N^n, h)$ between two Riemannian manifolds. The energy functional of ϕ is defined by

$$(1.1) \quad E(\phi) = \int_K e(\phi) dv_g,$$

where K is compact subset in M , where

$$(1.2) \quad e(\phi) = \frac{1}{2} \text{trace}_g h(d\phi, d\phi),$$

is the energy density of ϕ .

A map is called *harmonic* if it is a critical point of the energy functional E . For any smooth variation $\{\phi_t\}_{t \in I}$ of ϕ with $\phi_0 = \phi$ and $V = \left. \frac{d}{dt} \phi_t \right|_{t=0}$, we have

$$(1.3) \quad \left. \frac{d}{dt} E(\phi_t) \right|_{t=0} = - \int_K h(\tau(\phi), V) dv_g.$$

Here

$$(1.4) \quad \tau(\phi) = \text{trace}_g \bar{\nabla} d\phi$$

is the tension field of ϕ and $\bar{\nabla} d\phi$ is the second fundamental form of ϕ . Then ϕ is harmonic if and only if $\tau(\phi) = 0$. One can refer to [10], [9] for background on harmonic maps.

The main idea of this note consists of the study of harmonicity with respect to the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM [2]. We establish necessary and sufficient conditions under which a vector field is harmonic (Theorem 4.3 and Theorem 4.4). We also construct examples of harmonic vector fields and we give a formula for the construction of non trivial examples of vector fields (Theorem 4.7 and Corollary 4.9). We further study the harmonicity of the map $\sigma : (M, g) \rightarrow (TN, h^f)$ (Theorem 4.11 and Theorem 4.12) and the map $\phi : (TM, g^f) \rightarrow (N, h)$ (Theorem 4.14 and Theorem 4.15).

2 Basic notions and definitions on TM

Let (M^m, g) be an m -dimensional Riemannian manifold and let (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1, \dots, m}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1, \dots, m}$ on TM . Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g . Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and let $\Gamma(TM)$ be the module over $C^\infty(M)$ of C^∞ -vector fields on M .

We have two complementary distributions on TM , the vertical distribution \mathcal{V} and the horizontal distribution \mathcal{H} , defined by:

$$\begin{aligned} \mathcal{V}_{(x,u)} &= \text{Ker}(d\pi_{(x,u)}) = \left\{ a^i \frac{\partial}{\partial y^i} \Big|_{(x,u)}; \quad a^i \in \mathbb{R} \right\}, \\ \mathcal{H}_{(x,u)} &= \left\{ a^i \frac{\partial}{\partial x^i} \Big|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big|_{(x,u)}; \quad a^i \in \mathbb{R} \right\}, \end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M . The vertical and the horizontal lifts of X are defined by

$$(2.1) \quad X^V = X^i \frac{\partial}{\partial y^i},$$

$$(2.2) \quad X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\}.$$

Consequently, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$, $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, and $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1, \dots, m}$ is a local adapted frame on TTM .

If $w = w^i \frac{\partial}{\partial x^i} + \bar{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$(2.3) \quad w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)},$$

$$(2.4) \quad w^v = (\bar{w}^k + w^i u^j \Gamma_{ij}^k) \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}.$$

Lemma 2.1. [24] *Let (M, g) be a Riemannian manifold and R its curvature tensor. Then for all vector fields $X, Y \in \Gamma(TM)$, we have:*

1. $[X^H, Y^H]_p = [X, Y]_p^H - (R_x(X, Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

3 Vertical rescaled generalized Cheeger-Gromoll metric

Definition 3.1. [2] Let (M, g) be a Riemannian manifold and $f : M \rightarrow]0, +\infty[$ be a strictly positive smooth function. We define a vertical rescaled generalized Cheeger-Gromoll metric g^f on the tangent bundle TM by

1. $g^f(X^H, Y^H)_{(x,u)} = g_x(X, Y)$,
2. $g^f(X^H, Y^V)_{(x,u)} = 0$,
3. $g^f(X^V, Y^V)_{(x,u)} = f(x)\omega^p [g_x(X, Y) + qg_x(X, u)g_x(Y, u)]$,

for all vector fields $X, Y \in \Gamma(TM)$, $(x, u) \in TM$ and $r = \|u\| = \sqrt{g(u, u)}$, where $\omega = \frac{1}{1 + \|u\|^2}$, $p, q \in \mathbb{R}$, and q positive ensures non-degeneracy.

Theorem 3.1. [2] *Let (M, g) be a Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If ∇ (resp. $\bar{\nabla}$) denote the Levi-Civita connections of (M, g) (resp. (TM, g^f)), then we*

have:

$$\begin{aligned}
1. (\bar{\nabla}_{X^H} Y^H) &= (\nabla_X Y)^H - \frac{1}{2}(R(X, Y)u)^V, \\
2. (\bar{\nabla}_{X^H} Y^V) &= \frac{f}{2}\omega^p(R(u, Y)X)^H + \frac{X(f)}{2f}Y^V + (\nabla_X Y)^V, \\
3. (\bar{\nabla}_{X^V} Y^H) &= \frac{f}{2}\omega^p(R(u, X)Y)^H + \frac{Y(f)}{2f}X^V, \\
4. (\bar{\nabla}_{X^V} Y^V) &= -\frac{p\omega^{(-p+1)}}{f(1+qr^2)} \left[g^f(X^V, U^V)Y^V + g^f(Y^V, U^V)X^V \right] \\
&\quad + \frac{(p\omega + q)\omega^{-p}}{f(1+qr^2)} g^f(X^V, Y^V)U^V \\
&\quad - \frac{q^2\omega^{-2p}}{f^2(1+qr^2)^3} g^f(X^V, U^V)g^f(Y^V, U^V)U^V \\
&\quad - \frac{1}{2f} g^f(X^V, Y^V)(\text{grad } f)^H,
\end{aligned}$$

for all vector fields $X, Y, U \in \Gamma(TM)$, $U_x = u \in T_x M$ and $(x, u) \in TM$, where R denotes the curvature tensor of (M, g) .

4 Vertical rescaled generalized Cheeger-Gromoll metric and harmonicity.

4.1 Harmonicity of a vector field $X : (M, g) \longrightarrow (TM, g^f)$

Lemma 4.1. *Let (M, g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields on M and $(x, u) \in TM$ such that $Y_x = u$, then we have:*

$$d_x Y(X_x) = X_{(x,u)}^H + (\nabla_X Y)_{(x,u)}^V.$$

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $\pi^{-1}(U), x^i, y^j$ be the induced chart on TM . If $X_x = X^i(x) \frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x) \frac{\partial}{\partial x^i}|_x = u$, then

$$d_x Y(X_x) = X^i(x) \frac{\partial}{\partial x^i}|_{(x,u)} + X^i(x) \frac{\partial Y^k}{\partial x^i}(x) \frac{\partial}{\partial y^k}|_{(x,u)},$$

and thus the horizontal part is given by:

$$\begin{aligned}
(d_x Y(X_x))^h &= X^i(x) \frac{\partial}{\partial x^i}|_{(x,u)} - X^i(x) Y^j(x) \Gamma_{ij}^k(x) \frac{\partial}{\partial y^k}|_{(x,u)} \\
&= X_{(x,u)}^h,
\end{aligned}$$

and the vertical part, by:

$$\begin{aligned}
(d_x Y(X_x))^v &= \left\{ X^i(x) \frac{\partial Y^k}{\partial x^i}(x) + X^i(x) Y^j(x) \Gamma_{ij}^k(x) \right\} \frac{\partial}{\partial y^k}|_{(x,u)} \\
&= (\nabla_X Y)_{(x,u)}^v.
\end{aligned}$$

□

Lemma 4.2. *Let (M^m, g) be an m -dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the energy density associated to X is given by:*

$$(4.1) \quad e(X) = \frac{m}{2} + \frac{f\omega^p}{2} \text{trace}_g [g(\nabla X, \nabla X) + qg(\nabla X, X)^2],$$

where $\omega = \frac{1}{1 + \|X\|^2}$ and $\|X\|^2 = g(X, X)$.

Proof. Let $(x, u) \in TM$, $X \in \Gamma(TM)$, $X_x = u$ and let (E_1, \dots, E_m) be a local orthonormal frame on M . Then:

$$\begin{aligned} e(X)_x &= \frac{1}{2} \text{trace}_g g^f(dX, dX)_{(x,u)} \\ &= \frac{1}{2} \sum_{i=1}^m g^f(dX(E_i), dX(E_i))_{(x,u)}. \end{aligned}$$

Using Lemma 4.1, we obtain:

$$\begin{aligned} e(X) &= \frac{1}{2} \sum_{i=1}^m g^f(E_i^H + (\nabla_{E_i} X)^V, E_i^H + (\nabla_{E_i} X)^V) \\ &= \frac{1}{2} \sum_{i=1}^m [g^f(E_i^H, E_i^H) + g^f((\nabla_{E_i} X)^V, (\nabla_{E_i} X)^V)] \\ &= \frac{1}{2} \sum_{i=1}^m [g(E_i, E_i) + f\omega^p [g(\nabla_{E_i} X, \nabla_{E_i} X) + qg(\nabla_{E_i} X, X)^2]] \\ &= \frac{m}{2} + \frac{f\omega^p}{2} \text{trace}_g [g(\nabla X, \nabla X) + g(\nabla X, X)^2]. \end{aligned}$$

□

Theorem 4.3. *Let (M^m, g) be an m -dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the tension field associated to X is given by:*

$$(4.2) \quad \tau(X) = [\text{trace}_g A(X)]^H + [\text{trace}_g B(X)]^V,$$

where $A(X)$ and $B(X)$ are the bilinear maps defined by:

$$\begin{aligned} A(X) &= f\omega^p R(X, \nabla X) * -\frac{\omega^p}{2} [g(\nabla X, \nabla X) + qg(\nabla X, X)^2] \text{grad } f, \\ B(X) &= \nabla^2 X + \left[\frac{1}{f} df(*) - 2p\omega g(\nabla X, X) \right] \nabla X \\ &\quad + \left[\frac{p\omega + q}{1 + q\|X\|^2} g(\nabla X, \nabla X) + \frac{p\omega}{1 + q\|X\|^2} g(\nabla X, X)^2 \right] X, \end{aligned}$$

where $\omega = \frac{1}{1 + \|X\|^2}$ and $\|X\|^2 = g(X, X)$.

Proof. Let $(x, u) \in TM$, $X \in \Gamma(TM)$, $X_x = u$ and let $\{E_i\}_{i=1, \dots, m}$ be a local orthonormal frame on M such that $(\nabla_{E_i}^M E_i)_x = 0$. Then

$$\begin{aligned}
\tau(X)_x &= \sum_{i=1}^m \{(\nabla_{E_i}^X dX(E_i))_x - dX(\nabla_{E_i}^M E_i)_x\} \\
&= \sum_{i=1}^m \{\bar{\nabla}_{dX(E_i)} dX(E_i)\}_{(x,u)} \\
&= \sum_{i=1}^m \{\bar{\nabla}_{(E_i^H + (\nabla_{E_i} X)^V)} (E_i^H + (\nabla_{E_i} X)^V)\}_{(x,u)} \\
&= \sum_{i=1}^m \{\bar{\nabla}_{E_i^H} E_i^H + \bar{\nabla}_{E_i^H} (\nabla_{E_i} X)^V + \bar{\nabla}_{(\nabla_{E_i} X)^V} (E_i^H) \\
&\quad + \bar{\nabla}_{(\nabla_{E_i} X)^V} (\nabla_{E_i} X)^V\}_{(x,u)}.
\end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned}
\tau(X) &= \sum_{i=1}^m \left[(\nabla_{E_i} E_i)^H - \frac{1}{2} (R(E_i, E_i)X)^V + \frac{f}{2} \omega^p (R(X, \nabla_{E_i} X)E_i)^H \right. \\
&\quad + \frac{1}{2f} E_i(f) (\nabla_{E_i} X)^V + (\nabla_{E_i} \nabla_{E_i} X)^V + \frac{f}{2} \omega^p (R(X, \nabla_{E_i} X)E_i)^H \\
&\quad + \frac{1}{2f} E_i(f) (\nabla_{E_i} X)^V - \frac{2p\omega^{(-p+1)}}{f(1+q\|X\|^2)} g^f((\nabla_{E_i} X)^V, X^V) (\nabla_{E_i} X)^V \\
&\quad + \frac{(p\omega+q)\omega^{-p}}{f(1+q\|X\|^2)} g^f((\nabla_{E_i} X)^V, (\nabla_{E_i} X)^V) X^V \\
&\quad - \frac{q^2\omega^{-2p}}{f^2(1+q\|X\|^2)^3} g^f((\nabla_{E_i} X)^V, X^V)^2 X^V \\
&\quad \left. - \frac{1}{2f} g^f((\nabla_{E_i} X)^V, (\nabla_{E_i} X)^V) (\text{grad } f)^H \right] \\
&= \sum_{i=1}^m \left[f\omega^p (R(X, \nabla_{E_i} X)E_i)^H + \frac{1}{f} E_i(f) (\nabla_{E_i} X)^V + (\nabla_{E_i} \nabla_{E_i} X)^V \right. \\
&\quad - 2p\omega g(\nabla_{E_i} X, X) (\nabla_{E_i} X)^V + \frac{p\omega+q}{1+q\|X\|^2} g(\nabla_{E_i} X, \nabla_{E_i} X) X^V \\
&\quad + \frac{(p\omega+q)q}{1+q\|X\|^2} g(\nabla_{E_i} X, X)^2 X^V - \frac{q^2}{1+q\|X\|^2} g(\nabla_{E_i} X, X)^2 X^V \\
&\quad \left. - \frac{\omega^p}{2} [g(\nabla_{E_i} X, \nabla_{E_i} X) + qg(\nabla_{E_i} X, X)^2] (\text{grad } f)^H \right] \\
&= \left[\text{trace}_g \left[f\omega^p R(X, \nabla X) * - \frac{\omega^p}{2} [g(\nabla X, \nabla X) + qg(\nabla X, X)^2] \text{grad } f \right] \right]^H \\
&\quad + \left[\text{trace}_g \left[\nabla^2 X + \left[\frac{1}{f} df(*) - 2p\omega g(\nabla X, X) \right] \nabla X \right. \right. \\
&\quad \left. \left. + \left[\frac{p\omega+q}{1+q\|X\|^2} g(\nabla X, \nabla X) + \frac{p\omega}{1+q\|X\|^2} g(\nabla X, X)^2 \right] X \right] \right]^V.
\end{aligned}$$

□

Theorem 4.4. *Let (M^m, g) be an m -dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then X is a harmonic vector field if and only the following conditions are verified:*

$$(4.3) \text{trace}_g \left(f\omega^p R(X, \nabla X) * -\frac{\omega^p}{2} [g(\nabla X, \nabla X) + qg(\nabla X, X)^2] \text{grad } f \right) = 0,$$

and

$$(4.4) \text{trace}_g \left(\nabla^2 X + \left[\frac{1}{f} df(*) - 2p\omega g(\nabla X, X) \right] \nabla X + \left[\frac{p\omega + q}{1 + q\|X\|^2} g(\nabla X, \nabla X) + \frac{p\omega}{1 + q\|X\|^2} g(\nabla X, X)^2 \right] X \right) = 0,$$

where $\omega = \frac{1}{1 + \|X\|^2}$ and $\|X\|^2 = g(X, X)$.

Proof. The statement is a direct consequence of Theorem 4.3. \square

Corollary 4.5. *Let (M^m, g) be an m -dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$ such that X is a parallel vector field (i.e, $\nabla X = 0$), then X is harmonic.*

Example 4.1. Let \mathbb{R}^3 be endowed with the cylindrical Riemannian metric given by:

$$g = dr^2 + r^2 d\theta^2 + dt^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r.$$

Then, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} &= 0, \quad \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial t} = 0, \quad \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial r} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} = -r \frac{\partial}{\partial r}, \\ \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial t} &= 0, \quad \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial r} = 0, \quad \nabla_{\partial t} \partial \theta = 0, \quad \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0, \end{aligned}$$

the vector field $X = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}$ is harmonic because X is parallel, indeed,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial r}} X &= \sin \theta \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} - \frac{1}{r^2} \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{r} \cos \theta \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta} + \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial t} = 0, \\ \nabla_{\frac{\partial}{\partial \theta}} X &= \cos \theta \frac{\partial}{\partial r} + \sin \theta \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r} \sin \theta \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} + \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial t} = 0, \\ \nabla_{\frac{\partial}{\partial t}} X &= \sin \theta \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial \theta} + \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0, \end{aligned}$$

i.e, $\nabla X = 0$, which yields that X is harmonic.

Theorem 4.6. *Let (M^m, g) be a compact m -dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. A vector field $X \in \Gamma(TM)$ is harmonic if and only if X is parallel (i.e., $\nabla X = 0$).*

Proof. If X is parallel, from Corollary 4.5, we infer that X is a harmonic vector field. Conversely, let φ_t be a compactly supported variation of X , defined by:

$$\begin{aligned} \varphi : \mathbb{R} \times M &\longrightarrow T_x M \\ (t, x) &\longmapsto \varphi(t, x) = \varphi_t(x) = (t+1)X_x \end{aligned}$$

From Lemma 4.2, we have:

$$\begin{aligned} e(\varphi_t) &= \frac{m}{2} + \frac{(1+t)^2}{2} f\omega^p \text{trace}_g(\nabla X, \nabla X) + \frac{(1+t)^4}{2} f\omega^p q \text{trace}_g(\nabla X, X)^2 \\ E(\varphi_t) &= \frac{m}{2} \text{Vol}(M) + \frac{(1+t)^2}{2} \int_M f\omega^p \text{trace}_g(\nabla X, \nabla X) dv_g \\ &\quad + \frac{(1+t)^4}{2} \int_M f\omega^p q \text{trace}_g(\nabla X, X)^2 dv_g \end{aligned}$$

If X is a critical point of the energy functional, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t} \left[\frac{m}{2} \text{Vol}(M) + \frac{(1+t)^2}{2} \int_M f\omega^p \text{trace}_g(\nabla X, \nabla X) dv_g \right]_{t=0} \\ &\quad + \frac{\partial}{\partial t} \left[+ \frac{(1+t)^4}{2} \int_M f\omega^p q \text{trace}_g(\nabla X, X)^2 dv_g \right]_{t=0} \\ &= \int_M f\omega^p \text{trace}_g(\nabla X, \nabla X) dv_g + 2 \int_M f\omega^p q \text{trace}_g(\nabla X, X)^2 dv_g \\ &= \int_M f\omega^p \text{trace}_g(g(\nabla X, \nabla X) + 2qg(\nabla X, X)^2) dv_g \end{aligned}$$

which gives

$$g(\nabla X, \nabla X) + 2qg(\nabla X, X)^2 = 0.$$

Hence $\nabla X = 0$. □

Example 4.2. The Riemannian compact manifold \mathbb{S}^1 can be equipped with the metric:

$$g = e^x dx^2.$$

The only Christoffel symbol of the Levi-Civita connection is given by:

$$\Gamma_{11}^1 = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1} \right) = \frac{1}{2}.$$

The vector field $X = h(x)\frac{d}{dx}$, with $h \in C^\infty(\mathbb{S}^1)$, is harmonic if and only if X is parallel,

$$\begin{aligned}\nabla X = 0 &\Leftrightarrow h'(x) + \frac{1}{2}h(x) = 0 \\ &\Leftrightarrow h(x) = k \exp\left(-\frac{x}{2}\right), k \in \mathbb{R} \\ &\Leftrightarrow X = k \exp\left(-\frac{x}{2}\right)\frac{d}{dx}, k \in \mathbb{R}.\end{aligned}$$

Remark 4.3. In general, using Corollary 4.5 and Theorem 4.6, we can construct numerous examples of harmonic vector fields.

Theorem 4.7. Let (\mathbb{R}^m, g_0) be the real Euclidean space and let $(T\mathbb{R}^m, g^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X = (X^1, \dots, X^m) \in \Gamma(T\mathbb{R}^m)$, then X is harmonic if and only if the following conditions are verified

$$(4.5) \quad X = \text{constant} \text{ or } f = \text{constant},$$

and

$$(4.6) \quad \begin{aligned} &\sum_{i=1}^m \left[\frac{\partial^2 X^k}{\partial (x^i)^2} + \frac{1}{f} \frac{\partial f}{\partial x^i} \frac{\partial X^k}{\partial x^i} \right] + \frac{p\omega X^k}{1+q\|X\|^2} \sum_{i=1}^m \left(\sum_{j=1}^m X^j \frac{\partial X^j}{\partial x^i} \right)^2 \\ &+ \sum_{i,j=1}^m \left[-2p\omega X^j \frac{\partial X^j}{\partial x^i} \frac{\partial X^k}{\partial x^i} + \frac{(p\omega+q)X^k}{1+q\|X\|^2} \left(\frac{\partial X^j}{\partial x^i} \right)^2 \right] = 0. \end{aligned}$$

for all $k = \overline{1, m}$, where $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1, m}}$ is the canonical frame on \mathbb{R}^m , $\omega = \frac{1}{1+\|X\|^2}$ and $\|X\|^2 = g(X, X)$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1, m}}$ be the canonical frame on \mathbb{R}^m . Using Theorem 4.4, we have: $\tau(X) = 0$ holds true, iff the following conditions are satisfied:

$$\begin{aligned}(4.3) \quad &\Leftrightarrow \text{trace}_g \left(-\frac{\omega^p}{2} [g(\nabla X, \nabla X) + qg(\nabla X, X)^2] \text{grad } f \right) = 0 \\ &\Leftrightarrow \sum_{i=1}^m [g(\nabla_{\frac{\partial}{\partial x^i}} X, \nabla_{\frac{\partial}{\partial x^i}} X) + qg(\nabla_{\frac{\partial}{\partial x^i}} X, X)^2] = 0 \text{ or } \text{grad } f = 0 \\ &\Leftrightarrow \sum_{i=1}^m \left[\sum_{j=1}^m \left(\frac{\partial X^j}{\partial x^i} \right)^2 + q \left(\sum_{j=1}^m \frac{\partial X^j}{\partial x^i} X^j \right)^2 \right] = 0 \text{ or } f = \text{constant} \\ &\Leftrightarrow \frac{\partial X^j}{\partial x^i} = 0, \forall i, j = \overline{1, m} \text{ or } f = \text{constant} \\ &\Leftrightarrow X = \text{constant} \text{ or } f = \text{constant}.\end{aligned}$$

$$\begin{aligned}
(4.4) \Leftrightarrow & \operatorname{trace}_g \left[\nabla^2 X + \left[\frac{1}{f} df(*) - 2p\omega g(\nabla X, X) \right] \nabla X \right. \\
& \left. + \left[\frac{p\omega + q}{1 + q\|X\|^2} g(\nabla X, \nabla X) + \frac{p\omega}{1 + q\|X\|^2} g(\nabla X, X)^2 \right] X \right] = 0 \\
\Leftrightarrow & \sum_{i=1}^m \left[\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^i}} X + \left[\frac{1}{f} df\left(\frac{\partial}{\partial x^i}\right) - 2p\omega g\left(\nabla_{\frac{\partial}{\partial x^i}} X, X\right) \right] \left(\nabla_{\frac{\partial}{\partial x^i}} X\right) \right. \\
& \left. + \left[\frac{p\omega + q}{1 + q\|X\|^2} g\left(\nabla_{\frac{\partial}{\partial x^i}} X, \nabla_{\frac{\partial}{\partial x^i}} X\right) + \frac{p\omega}{1 + q\|X\|^2} g\left(\nabla_{\frac{\partial}{\partial x^i}} X, X\right)^2 \right] X \right] = 0 \\
\Leftrightarrow & \sum_{i=1}^m \left\{ \sum_{k=1}^m \left(\frac{\partial^2 X^k}{\partial(x^i)^2} \frac{\partial}{\partial x^k} \right) + \frac{1}{f} \frac{\partial f}{\partial x^i} \sum_{k=1}^m \left(\frac{\partial X^k}{\partial x^i} \frac{\partial}{\partial x^k} \right) \right. \\
& - 2p\omega \sum_{j=1}^m \left(\frac{\partial X^j}{\partial x^i} X^j \right) \sum_{k=1}^m \left(\frac{\partial X^k}{\partial x^i} \frac{\partial}{\partial x^k} \right) \\
& + \frac{p\omega + q}{1 + q\|X\|^2} \sum_{j=1}^m \left(\frac{\partial X^j}{\partial x^i} \right)^2 \sum_{k=1}^m \left(X^k \frac{\partial}{\partial x^k} \right) \\
& \left. + \frac{p\omega}{1 + q\|X\|^2} \left(\sum_{j=1}^m X^j \frac{\partial X^j}{\partial x^i} \right)^2 \sum_{k=1}^m \left(X^k \frac{\partial}{\partial x^k} \right) \right\} = 0 \\
\Leftrightarrow & \sum_{i=1}^m \left[\frac{\partial^2 X^k}{\partial(x^i)^2} + \frac{1}{f} \frac{\partial f}{\partial x^i} \frac{\partial X^k}{\partial x^i} \right] + \frac{p\omega X^k}{1 + q\|X\|^2} \sum_{i=1}^m \left(\sum_{j=1}^m X^j \frac{\partial X^j}{\partial x^i} \right)^2 \\
& + \sum_{i,j=1}^m \left[-2p\omega X^j \frac{\partial X^j}{\partial x^i} \frac{\partial X^k}{\partial x^i} + \frac{(p\omega + q)X^k}{1 + q\|X\|^2} \left(\frac{\partial X^j}{\partial x^i} \right)^2 \right] = 0.
\end{aligned}$$

for all $k = \overline{1, m}$. □

Corollary 4.8. *Let (\mathbb{R}^m, g_0) be the real Euclidean space, let $(T\mathbb{R}^m, g_0^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric and let f be a constant function. If $X = (X^1, \dots, X^m)$ is a vector field on \mathbb{R}^m , then X is harmonic on \mathbb{R}^m if and only if X satisfies the following system of equations:*

$$\begin{aligned}
(4.7) \quad & \sum_{i=1}^m \frac{\partial^2 X^k}{\partial(x^i)^2} + \frac{p\omega X^k}{1 + q\|X\|^2} \sum_{i=1}^m \left(\sum_{j=1}^m X^j \frac{\partial X^j}{\partial x^i} \right)^2 \\
& + \sum_{i,j=1}^m \left[-2p\omega X^j \frac{\partial X^j}{\partial x^i} \frac{\partial X^k}{\partial x^i} + \frac{(p\omega + q)X^k}{1 + q\|X\|^2} \left(\frac{\partial X^j}{\partial x^i} \right)^2 \right] = 0.
\end{aligned}$$

for all $k = \overline{1, m}$, where $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1, m}}$ is the canonical frame on \mathbb{R}^m , $\omega = \frac{1}{1 + \|X\|^2}$ and $\|X\|^2 = g(X, X)$.

Corollary 4.9. *Let (\mathbb{R}^m, g_0) be the real Euclidean space, let $(T\mathbb{R}^m, g_0^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric and let $X = (X^1, \dots, X^m) \in \Gamma(T\mathbb{R}^m)$. If $f \neq \text{constant}$, then X is a harmonic if and only if X is constant.*

Remark 4.4. Using Corollary 4.8, we can construct many examples of nontrivial harmonic vector fields.

Example 4.5. If \mathbb{R}^m is endowed with the canonical metric and $T\mathbb{R}^m$ is its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric, such that f is a constant function, from Corollary 4.8, we infer that $X = (y(x_1), 0, \dots, 0)$ is a harmonic vector field if and only the function y is solution of differential equation:

$$(4.8) \quad y'' + \frac{q-p+(p+q-2pq)y^2}{(1+y^2)(1+qy^2)}(y')^2y = 0.$$

In the case $p = q = 1$, (the case of Cheeger-Gromoll metric), the solution of the differential equation (4.8) is given by $y(x) = ax_1 + b$, $a, b \in \mathbb{R}$.

4.2 Harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, h^f)$

Lemma 4.10. Let (M^m, g) , (N^n, h) be two Riemannian manifolds and let $\varphi : M \rightarrow N$ be a smooth map. If

$$\begin{aligned} \sigma : M &\longrightarrow TN \\ x &\longmapsto (\varphi(x), v) \end{aligned}$$

is a smooth map such that $\varphi = \pi_N \circ \sigma$, where $v \in T_{\varphi(x)}N$ and $\pi_N : TN \rightarrow N$ is the canonical projection, then

$$(4.9) \quad d\sigma(X) = (d\varphi(X))^H + (\nabla_X^\varphi \sigma)^V,$$

for all $X \in \Gamma(TM)$.

Proof. Let $x \in M$, $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$, such that $Y_{\varphi(x)} = v \in T_{\varphi(x)}N$. Using Lemma 4.1, we obtain:

$$\begin{aligned} d_x \sigma(X_x) &= d_x(Y \circ \varphi)(X_x) \\ &= d_{\varphi(x)} Y(d_x \varphi(X_x)) \\ &= (d\varphi(X))_{(\varphi(x), v)}^H + (\nabla_{d\varphi(X)} Y)_{(\varphi(x), v)}^V \\ &= (d\varphi(X))_{(\varphi(x), v)}^H + (\nabla_X^\varphi \sigma)_{(\varphi(x), v)}^V. \end{aligned}$$

□

Theorem 4.11. Let (M^m, g) , (N^n, h) be two Riemannian manifolds and let f be a strictly positive smooth function on N . Let (TN, h^f) be the tangent bundle of N , equipped with vertical rescaled generalized Cheeger-Gromoll metric.

Let $\varphi : M \rightarrow N$ be a smooth map and let

$$\begin{aligned} \sigma : M &\longrightarrow TN \\ x &\longmapsto (\varphi(x), v) \end{aligned}$$

be a smooth map such that $\varphi = \pi_N \circ \sigma$ and $v \in T_{\varphi(x)}N$. The tension field of σ is given by

$$(4.10) \quad \tau(\sigma) = [\tau(\varphi) + \text{trace}_g A(\sigma)]^H + [\text{trace}_g(B(\sigma))]^V,$$

where $A(\sigma)$ and $B(\sigma)$ are the bilinear maps defined by

$$\begin{aligned} A(\sigma) &= f\omega^p R^N(\sigma, \nabla^\varphi \sigma) d\varphi(*) - \frac{\omega^p}{2} [h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + qh(\nabla^\varphi \sigma, \sigma)^2] \text{grad } f \\ B(\sigma) &= (\nabla^\varphi)^2 \sigma + \left[\frac{1}{f} d\varphi(*) (f) - 2p\omega h(\nabla^\varphi \sigma, \sigma) \right] \nabla^\varphi \sigma \\ &\quad + \left[\frac{p\omega + q}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + \frac{p\omega}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \sigma)^2 \right] \sigma, \end{aligned}$$

where $\omega = \frac{1}{1 + \|\sigma\|^2}$ and $\|\sigma\|^2 = h(\sigma, \sigma)$.

Proof. Let $x \in M$ and let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla_{E_i}^M E_i)_x = 0$ and $\sigma(x) = (\varphi(x), v), v \in T_{\varphi(x)}N$. Using Lemma 4.10, we have

$$\begin{aligned} \tau(\sigma)_x &= \text{trace}_g(\nabla d\sigma)_x \\ &= \sum_{i=1}^m \{(\nabla_{E_i}^\sigma d\sigma(E_i))_x - d\sigma(\nabla_{E_i}^M E_i)_x\} \\ &= \sum_{i=1}^m \{\nabla_{d\sigma(E_i)}^{TN} d\sigma(E_i)\}_{(\varphi(x), v)} \\ &= \sum_{i=1}^m \{\nabla_{(d\varphi(E_i))^H}^{TN} (d\varphi(E_i))^H + \nabla_{(d\varphi(E_i))^H}^{TN} (\nabla_{E_i}^\varphi \sigma)^V \\ &\quad + \nabla_{(\nabla_{E_i}^\varphi \sigma)^V}^{TN} (d\varphi(E_i))^H + \nabla_{(\nabla_{E_i}^\varphi \sigma)^V}^{TN} (\nabla_{E_i}^\varphi \sigma)^V\}_{(\varphi(x), v)} \end{aligned}$$

From Theorem 3.1, we obtain:

$$\begin{aligned} \tau(\sigma) &= \sum_{i=1}^m \left[(\nabla_{d\varphi(E_i)}^N d\varphi(E_i))^H - \frac{1}{2} (R^N(d\varphi(E_i), d\varphi(E_i))\sigma)^V \right. \\ &\quad \left. + \frac{f}{2} \omega^p (R^N(\sigma, \nabla_{E_i}^\varphi \sigma) d\varphi(E_i))^H + \frac{1}{2f} d\varphi(E_i)(f) (\nabla_{E_i}^\varphi \sigma)^V \right. \\ &\quad \left. + (\nabla_{d\varphi(E_i)}^N \nabla_{E_i}^\varphi \sigma)^V + \frac{f}{2} \omega^p (R^N(\sigma, \nabla_{E_i}^\varphi \sigma) d\varphi(E_i))^H \right. \\ &\quad \left. + \frac{1}{2f} d\varphi(E_i)(f) (\nabla_{E_i}^\varphi \sigma)^V - \frac{2p\omega^{(-p+1)}}{f(1 + q\|\sigma\|^2)} h^f((\nabla_{E_i}^\varphi \sigma)^V, \sigma^V) (\nabla_{E_i}^\varphi \sigma)^V \right. \\ &\quad \left. + \frac{(p\omega + q)\omega^{-p}}{f(1 + q\|\sigma\|^2)} h^f((\nabla_{E_i}^\varphi \sigma)^V, (\nabla_{E_i}^\varphi \sigma)^V) \sigma^V \right. \\ &\quad \left. - \frac{q^2\omega^{-2p}}{f(1 + q\|\sigma\|^2)^3} h^f((\nabla_{E_i}^\varphi \sigma)^V, \sigma^V)^2 \sigma^V \right. \\ &\quad \left. - \frac{1}{2f} h^f((\nabla_{E_i}^\varphi \sigma)^V, (\nabla_{E_i}^\varphi \sigma)^V) (\text{grad } f)^H \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^m \left[(\nabla_{E_i}^\varphi d\varphi(E_i))^H + f\omega^p (R^N(\sigma, \nabla_{E_i}^\varphi \sigma) d\varphi(E_i))^H \right. \\
&\quad + \frac{1}{f} d\varphi(E_i)(f)(\nabla_{E_i}^\varphi \sigma)^V + (\nabla_{E_i}^\varphi \nabla_{E_i}^\varphi \sigma)^V \\
&\quad - 2p\omega h(\nabla_{E_i}^\varphi \sigma, \sigma)(\nabla_{E_i}^\varphi \sigma)^V + \frac{p\omega + q}{1 + q\|\sigma\|^2} h(\nabla_{E_i}^\varphi \sigma, \nabla_{E_i}^\varphi \sigma) \sigma^V \\
&\quad + \frac{p\omega}{1 + q\|\sigma\|^2} h(\nabla_{E_i}^\varphi \sigma, \sigma)^2 \sigma^V \\
&\quad \left. - \frac{\omega^p}{2} [h(\nabla_{E_i}^\varphi \sigma, \nabla_{E_i}^\varphi \sigma) + qh(\nabla_{E_i}^\varphi \sigma, \sigma)^2] (\text{grad } f)^H \right]
\end{aligned}$$

This implies:

$$\begin{aligned}
\tau(\sigma) &= \left(\tau(\varphi) + \text{trace}_g [f\omega^p R^N(\sigma, \nabla^\varphi \sigma) d\varphi(*) \right. \\
&\quad \left. - \frac{\omega^p}{2} [h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + qh(\nabla^\varphi \sigma, \sigma)^2] \text{grad } f] \right)^H \\
&\quad + \left(\text{trace}_g [(\nabla^\varphi)^2 \sigma + [\frac{1}{f} d\varphi(*) (f) - 2p\omega h(\nabla^\varphi \sigma, \sigma)] \nabla^\varphi \sigma \right. \\
&\quad \left. + [\frac{p\omega + q}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + \frac{p\omega}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \sigma)^2] \sigma \right)^V.
\end{aligned}$$

□

From Theorem 4.11 we consequently get the following

Theorem 4.12. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and let f be a strictly positive smooth function on N . Let (TN, h^f) be the tangent bundle of N , equipped with vertical rescaled generalized Cheeger-Gromoll metric.*

Let $\varphi : M \rightarrow N$ be a smooth map and let

$$\begin{aligned}
\sigma : (M, g) &\longrightarrow (TN, h^f) \\
x &\longmapsto (\varphi(x), v)
\end{aligned}$$

be a smooth map such that $\varphi = \pi_N \circ \sigma$ and $v \in T_{\varphi(x)} N$. Then σ is a harmonic if and only if the following conditions are satisfied

$$\begin{aligned}
(4.11) \quad \tau(\varphi) &= \text{trace}_g [-f\omega^p R^N(\sigma, \nabla^\varphi \sigma) d\varphi(*) \\
&\quad + \frac{\omega^p}{2} [h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + qh(\nabla^\varphi \sigma, \sigma)^2] \text{grad } f],
\end{aligned}$$

and

$$\begin{aligned}
(4.12) \quad 0 &= \text{trace}_g [(\nabla^\varphi)^2 \sigma + [\frac{1}{f} d\varphi(*) (f) - 2p\omega h(\nabla^\varphi \sigma, \sigma)] \nabla^\varphi \sigma \\
&\quad + [\frac{p\omega + q}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \nabla^\varphi \sigma) + \frac{p\omega}{1 + q\|\sigma\|^2} h(\nabla^\varphi \sigma, \sigma)^2] \sigma].
\end{aligned}$$

4.3 Harmonicity of the map $\phi : (TM, g^f) \longrightarrow (N, h)$

Lemma 4.13. *Let (M^m, g) be an m -dimensional Riemannian manifold, let f be a strictly positive smooth function on M and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. Then the tension field of the canonical projection*

$$\begin{aligned} \pi : (TM, g^f) &\longrightarrow (M, g) \\ (x, u) &\longmapsto x \end{aligned}$$

is given by:

$$(4.13) \quad \tau(\pi) = \frac{m}{2f}(\text{grad } f) \circ \pi.$$

Proof. Let $(x, u) \in TM$ and let $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = \frac{u}{\|u\|}$ is an orthonormal basis of TM at x . Then $\{E_i^H, \frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V, \frac{1}{\sqrt{f\omega^p}}E_j^V\}_{i=\overline{1,m}, j=\overline{2,m}}$ is an orthonormal basis of TM at (x, u) .

$$\begin{aligned} \tau(\pi) &= \text{trace}_{g^f} \nabla d\pi \\ &= \sum_{i=1}^m \left\{ \nabla_{E_i^H}^\pi d\pi(E_i^H) - d\pi(\nabla_{E_i^H}^{TM} E_i^H) \right\} \\ &\quad + \nabla_{\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right)}^\pi d\pi\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right) \\ &\quad - d\pi\left(\nabla_{\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right)}^{TM}\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right)\right) \\ &\quad + \sum_{j=2}^m \left\{ \nabla_{\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right)}^\pi d\pi\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right) - d\pi\left(\nabla_{\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right)}^{TM}\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right)\right) \right\} \\ &= \sum_{i=1}^m \left\{ \nabla_{d\pi(E_i^H)}^M d\pi(E_i^H) - d\pi(\nabla_{E_i^H}^{TM} E_i^H) \right\} \\ &\quad + \nabla_{d\pi\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right)}^M d\pi\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right) \\ &\quad - \frac{1}{\sqrt{f\omega^p(1+qr^2)}} d\pi\left(\nabla_{E_1^V}^{TM}\left(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V\right)\right) \\ &\quad + \sum_{j=2}^m \left\{ \nabla_{d\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right)}^M d\pi\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right) - \frac{1}{\sqrt{f\omega^p}} d\pi\left(\nabla_{E_j^V}^{TM}\left(\frac{1}{\sqrt{f\omega^p}}E_j^V\right)\right) \right\} \end{aligned}$$

as $d\pi(E_i^V) = 0$ and $d\pi(E_i^H) = E_i \circ \pi$ then:

$$\begin{aligned} \tau(\pi) &= \sum_{i=1}^m \left\{ (\nabla_{E_i}^M E_i) \circ \pi - d\pi(\nabla_{E_i}^M E_i) \right\} \\ &\quad - \frac{1}{f\omega^p(1+qr^2)} d\pi(\nabla_{E_1^V}^{TM} E_1^V) - \sum_{j=2}^m \left\{ \frac{1}{f\omega^p} d\pi(\nabla_{E_j^V}^{TM} E_j^V) \right\} \\ &= \frac{1}{2f}(\text{grad } f) \circ \pi + \frac{m-1}{2f}(\text{grad } f) \circ \pi \\ &= \frac{m}{2f}(\text{grad } f) \circ \pi. \end{aligned}$$

□

Theorem 4.14. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds, let f be a strictly positive smooth function on M and let (TM, g^f) be the tangent bundle of M equipped with the vertical rescaled generalized Cheeger-Gromoll metric.*

If $\varphi : (M, g) \longrightarrow (N, h)$ a smooth map, then the tension field of the map

$$\begin{aligned} \phi : (TM, g^f) &\longrightarrow (N, h) \\ (x, u) &\longmapsto \varphi(x) \end{aligned}$$

is given by:

$$(4.14) \quad \tau(\phi) = \left[\tau(\varphi) + \frac{m}{2f} d\varphi(\text{grad } f) \right] \circ \pi.$$

Proof. Let $(x, u) \in TM$ and let $\{E_i\}_{i=1, \dots, m}$ with $E_1 = \frac{u}{\|u\|}$ be an orthonormal basis on M at x . Then $\{E_i^H, \frac{1}{\sqrt{f\omega^p(1+qr^2)}} E_1^V, \frac{1}{\sqrt{f\omega^p}} E_j^V\}_{i=1, \dots, m, j=2, \dots, m}$ is an orthonormal basis of TM at (x, u) . For ϕ defined by:

$$\begin{aligned} \phi : (TM, g^f) &\xrightarrow{\pi} (M, g) \xrightarrow{\varphi} (N, h) \\ (x, y) &\longmapsto x \longmapsto \varphi(x) \end{aligned}$$

i.e., $\phi = \varphi \circ \pi$, we have:

$$\begin{aligned} \tau(\phi) &= \tau(\varphi \circ \pi) \\ &= d\varphi(\tau(\pi)) + \text{trace}_{g^f} \nabla d\varphi(d\pi, d\pi) \end{aligned}$$

$$\begin{aligned} \text{trace}_{g^f} \nabla d\varphi(d\pi, d\pi) &= \sum_{i=1}^m \left\{ \nabla_{d\pi(E_i^H)}^\varphi d\varphi(d\pi(E_i^H)) - d\varphi(\nabla_{d\pi(E_i^H)}^M d\pi(E_i^H)) \right\} \\ &\quad + \nabla_{d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}} E_1^V)}^\varphi d\varphi(d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}} E_1^V)) \\ &\quad - d\varphi(\nabla_{d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}} E_1^V)}^M d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}} E_1^V)) \\ &\quad + \sum_{j=2}^m \left\{ \nabla_{d\pi(\frac{1}{\sqrt{f\omega^p}} E_j^V)}^\varphi d\varphi(d\pi(\frac{1}{\sqrt{f\omega^p}} E_j^V)) \right. \\ &\quad \left. - d\varphi(\nabla_{d\pi(\frac{1}{\sqrt{f\omega^p}} E_j^V)}^M d\pi(\frac{1}{\sqrt{f\omega^p}} E_j^V)) \right\} \\ &= \sum_{i=1}^m \left\{ (\nabla_{E_i \circ \pi}^\varphi d\varphi(E_i \circ \pi)) - d\varphi(\nabla_{E_i \circ \pi}^M E_i \circ \pi) \right\} \\ &= \sum_{i=1}^m \left\{ \nabla_{E_i}^\varphi d\varphi(E_i) - d\varphi(\nabla_{E_i}^M E_i) \right\} \circ \pi \\ &= \tau(\varphi) \circ \pi \end{aligned}$$

Using Lemma 4.13, we obtain: $\tau(\phi) = \left[\tau(\varphi) + \frac{m}{2f} d\varphi(\text{grad } f) \right] \circ \pi.$

□

Theorem 4.15. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds, let f be a strictly positive smooth function on M and let (TM, g^f) the tangent bundle of M equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $\varphi : (M, g) \rightarrow (N, h)$ is a smooth map, then the map*

$$\begin{aligned} \phi : (TM, g^f) &\longrightarrow (N, h) \\ (x, u) &\longmapsto \varphi(x) \end{aligned}$$

is a harmonic if and only if

$$\tau(\varphi) = -\frac{m}{2f}d\varphi(\text{grad } f).$$

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