Anti-invariant and Lagrangian submersions from trans-Sasakian manifolds

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Abstract. We study anti-invariant and Lagrangian submersions from trans-Sasakian manifolds onto Riemannian manifolds. We prove that the horizontal distributions of such submersions are not integrable and their fibers are not totally geodesic. Consequently, they cannot be totally geodesic maps. We also check that the harmonicity of such submersions. In particular, we show that they cannot be harmonic in the case when the Reeb vector field is horizontal.

M.S.C. 2010: 53C15, 53B20.
Key words: Trans-Sasakian manifold; Riemannian submersion; anti-invariant submersion; Lagrangian submersion; horizontal distribution.

1 Introduction

One of the popular research areas in differential geometry is the theory of Riemannian submersions which was initiated by O'Neill [15] and Gray [9]. Watson [25] considered Riemannian submersions between almost Hermitian manifolds under the name of almost Hermitian submersions. Afterwards, almost Hermitian submersions have been actively studied between different subclasses of almost Hermitian manifolds. Also, Riemannian submersions were extended to several subclasses of almost contact manifolds under the name of contact Riemannian submersions. Most of the studies related to Riemannian, almost Hermitian or contact Riemannian submersions can be found in the book [8].

The theory of anti-invariant Riemannian and Lagrangian submersions has been becoming a very active research area since Şahin [18] first defined such submersions from almost Hermitian manifolds onto Riemannian manifolds. In fact, anti-invariant Riemannian and Lagrangian submersions have been studying in different kinds of structures such as Kähler [18, 20], nearly Kähler [19], almost product [11], locally product Riemannian [22], Sasakian [13, 21, 23], Kenmotsu [5, 23] and cosymplectic [14]. Note that the notion of anti-invariant Riemannian submersion was generalized

Balkan Journal of Geometry and Its Applications, Vol.25, No.2, 2020, pp. 106-123.

to the notion of conformal anti-invariant submersion [1]. Most of the studies related to the theory of anti-invariant Riemannian and Lagrangian submersions can be found in Şahin's monograph [17].

This paper is organized as follows. In section 2, we present basic notion and definition of trans-Sasakian manifolds. In section 3, we give some background for Riemannian submersions. In section 4, we recall the definition of anti-invariant and Lagrangian submersions. In section 5, we study anti-invariant submersions from trans-Sasakian manifolds onto Riemannian manifolds admitting vertical Reeb vector field, provide an example and give their some characteristic properties. The case of the Reeb vector field is horizontal is discussed in section 6. In section 7, we consider Lagrangian submersions admitting vertical Reeb vector field and investigate the geometry of the vertical and horizontal distributions. We also give a necessary and sufficient condition for such submersions to be harmonic. Similar studies for Lagrangian submersions admitting horizontal Reeb vector field are placed in the last section.

2 Trans-Sasakian manifolds

Let (M, g) be a $(2m+1)$-dimensional Riemannian manifold. Then M is called an *almost contact metric manifold* [3] if there exists a tensor φ of type $(1,1)$ and global vector field ξ which is called the *Reeb vector field* or the *characteristic vector field* such that, if η is the dual 1-form of ξ, then we have

$$\varphi\xi = 0,\ \eta(\xi) = 1,\ \varphi^2 = -I + \eta \otimes \xi,\ \ g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F)\ , \tag{2.1}$$

where E and F are any vector fields on M. Also, it can be deduced from the above axioms that $\eta \circ \varphi = 0$ and $\eta(E) = g(E, \xi)$. In this case, (φ, ξ, η, g) is called the *almost contact metric structure* of M.

An almost contact metric structure (φ, ξ, η, g) on a connected manifold M is called *trans-Sasakian manifold* [16] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [10], where J is the almost complex structure on $M \times \mathbb{R}$ given by

$$J(E, \lambda\frac{d}{dt}) = (\varphi E - \lambda\xi, -\eta(E)\frac{d}{dt})$$

for all vector fields E on M and λ is a smooth function on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This definition is equivalent to the condition [4]

$$(\nabla_E\varphi)F = \alpha[g(E, F)\xi - \eta(F)E] + \beta[g(\varphi E, F)\xi - \eta(F)\varphi E] \tag{2.2}$$

for functions α and β and the Levi-Civita connection ∇ on M. Sometimes, $(M, \varphi, \xi, \eta, g)$ is called a trans-Sasakian manifold of type (α, β). It can be deduced from (2.2) that

$$\nabla_E\xi = -\alpha\varphi E + \beta(E - \eta(E)\xi)\ . \tag{2.3}$$

3 Riemannian submersions

In this section, we give necessary background for Riemannian submersions.

Let (M,g) and (N,g_N) be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective mapping $\pi : (M,g) \to (N,g_N)$ is called a *Riemannian submersion* [15] if:
(S1) The rank of π equals $dim(N)$.

In this case, for each $q \in N$, $\pi^{-1}(q) = \pi_q^{-1}$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e. , $\pi_*(X_p) = X_{*\pi(p)}$ for all $p \in M$, where π_* is derivative or differential map of π. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $ker\pi_*^{\perp}$, respectively. As usual, the manifold (M,g) is called *total manifold* and the manifold (N,g_N) is called *base manifold* of the submersion $\pi : (M,g) \to (N,g_N)$.

(S2) π_* preserves the lengths of the horizontal vectors.

This condition is equivalent to say that the derivative map π_* of π, restricted to $ker\pi_*^{\perp}$, is a linear isometry. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{3.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{3.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V,W be vertical and X,Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{3.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \tag{3.4}$$

On the other hand, from (1) and (2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{3.5}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{3.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{3.7}$$

(3.8) $$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y,$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if X is basic, then we have $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [15] and to the book [8].

Any fiber of a Riemannian submersion $\pi : (M, g) \to (N, g_N)$ is called *totally umbilical* provided

(3.9) $$T_U V = g(U, V)H, \quad \forall \quad U, V \in \Gamma(ker\pi_*),$$

where H is the mean curvature vector field of the fiber in M, see [8].

4 Anti-invariant Riemannian and Lagrangian submersions from trans-Sasakian manifolds

We first recall the definition of an anti-invariant Riemannian submersion whose total manifold is almost contact metric manifold.

Definition 4.1. ([13]) Let M be a $(2m + 1)$-dimensional almost contact metric manifold with almost contact metric structure (φ, ξ, η, g) and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $\pi : M \to N$ such that the vertical distribution $ker\pi_*$ is anti-invariant with respect to φ, i.e., $\varphi ker\pi_* \subseteq ker\pi_*^\perp$. Then the Riemannian submersion π is called an *anti-invariant Riemannian submersion.* We shall briefly call such submersions as *anti-invariant submersions.*

In this case, the horizontal distribution $ker\pi_*^\perp$ is decomposed as

(4.1) $$ker\pi_*^\perp = \varphi ker\pi_* \oplus \mu\,,$$

where μ is the orthogonal complementary distribution of $\varphi ker\pi_*$ in $ker\pi_*^\perp$ and it is invariant with respect to φ.

We say that an anti-invariant $\pi : M \to N$ *admits vertical Reeb vector field* if the Reeb vector field ξ is tangent to $ker\pi_*$ and it *admits horizontal Reeb vector field* if the Reeb vector field ξ is normal to $ker\pi_*$. It is easy to see that μ contains the Reeb vector field ξ in the case of $\pi : M \to N$ admits horizontal Reeb vector field ξ.

For some details of the anti-invariant submersions from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N); see [5, 13, 14, 21].

Remark 4.2. Throughout this paper, as a total manifold of an anti-invariant submersion, we consider a trans-sasakian manifold $(M, \varphi, \xi, \eta, g)$ of type (α, β) such that both $\alpha \neq 0$ and $\beta \neq 0$.

The notion of Lagrangian submersion is a special case of the notion of anti-invariant submersion. We next recall the definition of a Lagrangian submersion from almost contact metric manifold onto a Riemannian manifold.

Definition 4.3. ([21]) Let π be an anti-invariant Riemannian submersion from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). If $\mu = \{0\}$ or $\mu = span\{\xi\}$, i.e., $ker\pi_*^\perp = \varphi(ker\pi_*)$ or $ker\pi_*^\perp = \varphi(ker\pi_*) \oplus < \xi >$, respectively, then we call π a *Lagrangian submersion.*

Remark 4.4. This case has been studied partially as a special case of an anti-invariant Riemannian submersion; see [5, 13, 14, 21] for some details.

5 Anti-invariant submersions admitting vertical Reeb vector field

In this section, we begin to study anti-invariant submersions admitting vertical Reeb vector field from trans-sasakian manifolds $(M, \varphi, \xi, \eta, g)$ of type (α, β) by giving a (non-trivial) example.

Example 5.1. Let M be a 3-dimensional Euclidean space given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid \; yz \neq 0\}.$$

We consider the trans-Sasakian structure (φ, ξ, η, g) on M with $\alpha = -\frac{1}{2}z^2$ and $\beta = -\frac{1}{z}$ [7] given by the following:

$$\xi = \frac{\partial}{\partial z}, \; \eta = dz, \; g = \begin{pmatrix} \frac{1}{(1+y^2)z^2} & 0 & \frac{1}{yz} \\ 0 & \frac{1}{z^2} & 0 \\ \frac{1}{yz} & 0 & 1 \end{pmatrix} \text{ and } \varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

An orthonormal φ-basis for this structure can be given by

$$\left\{ E_1 = z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z} \right\}.$$

Now, we define the map $\pi : (M, \varphi, \xi, \eta, g) \to (\mathbb{R}, g_1)$ by the following:

$$\pi(x, y, z) = \frac{x + y}{\sqrt{2}},$$

where g_1 is the usual metric on $\mathbb{R}$. Then, the Jacobian matrix of π is as follows:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Since the rank of this matrix equals 1, the map π is a submersion. After some calculation, we see that

$$ker\pi_* = span\left\{ V = \frac{E_1 - E_2}{\sqrt{2}}, \quad W = E_3 \right\},$$

and

$$ker\pi_*^\perp = span\left\{X = \frac{E_1 + E_2}{\sqrt{2}}\right\} .$$

By direct calculation, we see that π satisfies the condition **S2)**. Hence, π is a Riemannian submersion. Moreover, we have $\varphi(V) = X$. Therefore, π is an anti-invariant submersion admitting vertical Reeb vector field.

Let π be an anti-invariant submersion from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). For any $X \in ker\pi_*^\perp$, we write

$$\varphi X = \mathcal{B}X + \mathcal{C}X \,, \tag{5.1}$$

where $\mathcal{B}X \in \Gamma(ker\pi_*)$ and $\mathcal{C}X \in \Gamma(ker\pi_*^\perp)$. At first, we examine how the trans-Sasakian structure on M has effects on the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of the submersion π.

Lemma 5.1. *Let π be an anti-invariant submersion from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical Reeb vector field. Then, we have*

$$\mathcal{T}_U\varphi V - \alpha g(U, V)\xi = \mathcal{B}\mathcal{T}_U V - \eta(V)U \,, \tag{5.2}$$

$$\mathcal{H}\nabla_U\varphi V = \mathcal{C}\mathcal{T}_U V + \varphi\hat{\nabla}_U V - \beta\eta(V)\varphi U \tag{5.3}$$

$$\hat{\nabla}_V\mathcal{B}X + \mathcal{T}_V\mathcal{C}X = \mathcal{B}\mathcal{H}\nabla_V X + \beta g(\varphi V, X)\xi \tag{5.4}$$

$$\mathcal{T}_V\mathcal{B}X + \mathcal{H}\nabla_V\mathcal{C}X = \mathcal{C}\mathcal{H}\nabla_V X + \varphi\mathcal{T}_V X \tag{5.5}$$

$$\mathcal{A}_X\varphi V = \mathcal{B}\mathcal{A}_X V + \beta g(\varphi X, V)\xi - \beta\eta(V)\mathcal{B}X \tag{5.6}$$

$$\mathcal{H}\nabla_X\varphi V + \alpha\eta(V)X = \varphi(\mathcal{V}\nabla_X V) + \mathcal{C}\mathcal{A}_X V - \beta\eta(V)\mathcal{C}X \tag{5.7}$$

$$\begin{aligned}\mathcal{V}\nabla_X\mathcal{B}Y + \mathcal{A}_X\mathcal{C}Y = \quad & \mathcal{B}\mathcal{H}\nabla_X Y + \alpha g(X, Y)\xi \\ & +\beta g(\varphi X, Y)\xi - \beta\eta(Y)\mathcal{B}X\end{aligned} \tag{5.8}$$

$$\mathcal{A}_X\mathcal{B}Y + \mathcal{H}\nabla_X\mathcal{C}Y = \quad \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y \tag{5.9}$$

where $U, V \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^\perp)$.

Proof. For any $U, V \in \Gamma(ker\pi_*)$, from (2.2), we have

$$\nabla_U\varphi V = \varphi\nabla_U V + \alpha[g(U, V)\xi - \eta(V)U] + \beta[g(\varphi U, V)\xi - \eta(V)\varphi U].$$

Hence, using (3.5), (3.6) and (5.1), we obtain

$$\begin{aligned}\mathcal{H}\nabla_U\varphi V + \mathcal{T}_U\varphi V = \quad & \mathcal{B}\mathcal{T}_U V + \mathcal{C}\mathcal{T}_U V + \varphi\hat{\nabla}_V W \\ & +\alpha[g(V, W)\xi - \eta(W)V] - \beta\eta(V)\varphi U.\end{aligned} \tag{5.10}$$

In view of the fact that ξ is vertical, taking the vertical and horizontal parts of (5.10), we get (5.2) and (5.3), respectively.

Now, let X and Y be any horizontal vector fields. Again, from (2.2), we have

$$\nabla_X \varphi Y = \varphi \nabla_X Y + \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X].$$

Hence, using (3.7), (3.8) and (5.1), we obtain

$$\begin{aligned} &\mathcal{A}_X \mathcal{B}Y + \mathcal{V}\nabla_X \mathcal{B}Y + \mathcal{H}\nabla_X \mathcal{C}Y + \mathcal{A}_X \mathcal{C}Y \\ &= \mathcal{B}\mathcal{H}\nabla_X Y + \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y + \alpha[g(X,Y)\xi - \eta(Y)X] \\ &\quad + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]. \end{aligned} \tag{5.11}$$

If we take the vertical and horizontal parts of (5.11) and using the fact that ξ is vertical, we easily get (5.8) and (5.9), respectively. The other assertions can be obtained in a similar way. □

Let π be an anti-invariant submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, using (5.1) and the condition **(S2)**, we derive $g(\pi_* \varphi V, \pi_* \mathcal{C}X) = 0$, for every $X \in \Gamma(ker\pi_*{}^\perp)$ and $V \in \Gamma(ker\pi_*)$, which implies that

$$\mathcal{T}N = \pi_*(\varphi(ker\pi_*)) \oplus \pi_*(\mu) \ . \tag{5.12}$$

From (2.1) and (5.1), we have following Lemma.

Lemma 5.2. *Let π be an anti-invariant submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{B}\mathcal{C}X = 0, \qquad \varphi\mathcal{B}X + \mathcal{C}^2 X = -X$$

for any $X \in \Gamma(ker\pi_*{}^\perp)$.

Lemma 5.3. *Let π be an anti-invariant Riemannian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{C}X = -\frac{1}{\alpha}\mathcal{A}_X \xi, \tag{5.13}$$

$$g(\mathcal{A}_X \xi, \varphi U) = 0, \tag{5.14}$$

$$\begin{aligned} g(\nabla_Y \mathcal{A}_X \xi, \varphi U) = &-g(\mathcal{A}_X \xi, \phi \mathcal{A}_Y U) + \alpha\eta(U) g(\mathcal{A}_X \xi, Y) \\ &+ \beta\eta(U) g(\mathcal{A}_X \xi, \varphi X) \end{aligned} \tag{5.15}$$

$$g(X, \mathcal{A}_Y \xi) = -g(Y, \mathcal{A}_X \xi) \tag{5.16}$$

for $X, Y \in \Gamma(ker\pi_*{}^\perp)$ and $U \in \Gamma(ker\pi_*)$.

Proof. By virtue of (3.7) and (2.3) we have (5.13).
For $X \in \Gamma(ker\pi_*{}^\perp)$ and $U \in \Gamma(ker\pi_*)$, by virtue of (3.2), (5.1)and (5.13) we get

$$g(\mathcal{A}_X\xi, \phi U) = -g(\alpha\phi X - \alpha\mathcal{B}X, \phi U) \tag{5.17}$$

$$= -\alpha g(X, U) + \alpha\eta(X)\eta(U) - \alpha g(\phi\mathcal{B}X, U).$$

Since $\phi BX \in \Gamma(ker\pi_*{}^\perp)$ and $\xi \in \Gamma(ker\pi_*)$, (5.17) implies (5.14).

Now from (5.14) we get

$$g(\nabla_Y \mathcal{A}_X\xi, \varphi U) = -g(\mathcal{A}_X\xi, \nabla_Y \varphi U)$$

for $X, Y \in \Gamma(ker\pi_*{}^\perp)$ and $U \in \Gamma(ker\pi_*)$. Then using geodesic condition and (2.2) we have

$$g(\nabla_Y \mathcal{A}_X\xi, \varphi U) = -g(\mathcal{A}_X\xi, \varphi\mathcal{A}_Y U) - g(\mathcal{A}_X\xi, \varphi(\mathcal{V}\nabla_Y U))$$

$$+\alpha\eta(U)g(\mathcal{A}_X\xi, Y) + \beta\eta(U)g(\mathcal{A}_X\xi, \varphi X).$$

Since $\varphi(\mathcal{V}\nabla_Y U) \in \Gamma(\varphi ker\pi_*) = \Gamma(ker\pi_*{}^\perp)$, we obtain (5.15). Using the skew-symmetricness of $\mathcal{A}$ and (3.4), we obtain directly (5.16). □

6 Anti-invariant submersions admitting horizontal Reeb vector field

In this section, we begin to study anti-invariant submersions admitting horizonal Reeb vector field from trans-sasakian manifolds $(M, \varphi, \xi, \eta, g)$ of type (α, β) by giving a (non-trivial) example.

Example 6.1. Let $\overline{\mathbb{R}}^5$ be five-dimensional Euclidean space given by

$$\overline{\mathbb{R}}^5 = \{(x, y, z, u, v) \in \mathbb{R}^5 \mid (x, y) \neq (0, 0),\ (u, v) \neq (0, 0) \text{ and } z \neq 0\}.$$

The vector fields

$$E_1 = 2(-\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z}),\ E_2 = 2\tfrac{\partial}{\partial y},\ E_3 = 2\tfrac{\partial}{\partial z},\ E_4 = 2(-\tfrac{\partial}{\partial u} + v\tfrac{\partial}{\partial z}),\ E_5 = 2\tfrac{\partial}{\partial v}\ .$$

are linearly independent at each point of $\overline{\mathbb{R}}^5$. Then, we can choose a trans-Sasakian structure (φ, ξ, η, g) on $\overline{\mathbb{R}}^5$ such as $\xi = E_3,\ \eta = \frac{1}{2}dz,\ g$ is defined by $g(E_i, E_j) = \delta_i^j$ and φ is defined by as follows:

$$\varphi E_1 = E_2,\ \varphi E_2 = -E_1,\ \varphi E_3 = 0,\ \varphi E_4 = E_5,\ \varphi E_5 = -E_4\ .$$

Indeed, (φ, ξ, η, g) is a trans-Sasakian structure on $\overline{\mathbb{R}}^5$ with $\alpha = -1$ and $\beta = 1$, see [6].

Now, we consider the map $\pi : (\overline{\mathbb{R}}^5, \varphi, \xi, \eta, g) \to (\mathbb{R}^3, g_3)$ defined by the following:

$$\pi(x, y, z, u, v) = \left(\frac{x - y}{\sqrt{2}}, \frac{u - v}{\sqrt{2}}, z\right)\ ,$$

where g_3 is the Euclidean metric on $\mathbb{R}^3$. Then, the Jacobian matrix of π is as follows:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the rank of this matrix is equal to 3, the map π is a submersion. One can see that π satisfies the condition **S2)**. Therefore, π is a Riemannian submersion. After some computations, we have

$$ker\pi_* = span\left\{V = \frac{E_1 + E_2}{\sqrt{2}}, \quad W = \frac{E_4 + E_5}{\sqrt{2}}\right\},$$

and

$$ker\pi_*^\perp = span\left\{X = \frac{E_1 - E_2}{\sqrt{2}}, \quad Y = \frac{E_4 - E_5}{\sqrt{2}}, \quad \xi\right\}.$$

In addition, we have $\varphi(V) = -X$ and $\varphi(W) = -Y$. Hence, we see that π is an anti-invariant submersion admitting horizontal Reeb vector field.

Lemma 6.1. *Let π be an anti-invariant submersion from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting horizontal Reeb vector field. Then, we have*

$$\mathcal{T}_U \varphi V = \mathcal{B}\mathcal{T}_U V, \tag{6.1}$$

$$\mathcal{H}\nabla_U \varphi V - \alpha g(U, V)\xi = \mathcal{C}\mathcal{T}_U V + \varphi\hat{\nabla}_U V, \tag{6.2}$$

$$\hat{\nabla}_V \mathcal{B}X + \mathcal{T}_V \mathcal{C}X = \mathcal{B}\mathcal{H}\nabla_V X - \alpha\eta(X)V, \tag{6.3}$$

$$\mathcal{T}_V \mathcal{B}X + \mathcal{H}\nabla_V \mathcal{C}X = \mathcal{C}\mathcal{H}\nabla_V X + \varphi\mathcal{T}_V X + \beta g(\varphi V, X)\xi - \beta\eta(X)\varphi V, \tag{6.4}$$

$$\mathcal{A}_X \varphi V = \mathcal{B}\mathcal{A}_X V, \tag{6.5}$$

$$\mathcal{H}\nabla_X \varphi V = \varphi(\mathcal{V}\nabla_X V) + \mathcal{C}\mathcal{A}_X V + \beta g(\varphi X, V)\xi, \tag{6.6}$$

$$\mathcal{V}\nabla_X \mathcal{B}Y + \mathcal{A}_X \mathcal{C}Y = \mathcal{B}\mathcal{H}\nabla_X Y - \beta\eta(Y)\mathcal{B}X, \tag{6.7}$$

$$\begin{aligned} \mathcal{A}_X \mathcal{B}Y + \mathcal{H}\nabla_X \mathcal{C}Y = \; & \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y + \alpha g(X, Y)\xi \\ & -\alpha\eta(Y)X - \beta\eta(Y)\mathcal{C}X \end{aligned} \tag{6.8}$$

where $U, V \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^\perp)$.

Proof. The proof is very similar to the proof of Lemma 5.1. So, we omit it. □

Using (5.1), we have $\mu = \varphi\mu \oplus \{\xi\}$.

Now, we suppose that V is vertical and X is horizontal vector field. Using above relation and (2.2), we obtain

$$g(\varphi V, \mathcal{C}X) = 0.$$

From this last relation we have $g(\pi_*\varphi V, \pi_*\mathcal{C}X) = 0$ which implies that

$$TN = \pi_*(\varphi ker\pi_*) \oplus \pi_*(\mu). \tag{6.9}$$

From (2.2) and (5.1) we obtain following Lemma.

Lemma 6.2. *Let π be an anti-invariant submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $M(\varphi, \xi, \eta, g)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{B}\mathcal{C}X = 0, \qquad \varphi^2 X = \varphi\mathcal{B}X + \mathcal{C}^2 X$$

for any $X \in \Gamma(ker\pi_*{}^{\perp})$.

Lemma 6.3. *Let π be an anti-invariant Riemannian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $M(\varphi, \xi, \eta, g)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{B}X = -\frac{1}{\alpha}\mathcal{A}_X\xi, \tag{6.10}$$

$$\mathcal{T}_U\xi = \beta U, \tag{6.11}$$

$$g(\mathcal{A}_X\xi, \varphi U) = 0, \tag{6.12}$$

$$g(\nabla_Y\mathcal{A}_X\xi, \varphi U) = -g(\mathcal{A}_X\xi, \varphi\mathcal{A}_Y U), \tag{6.13}$$

$$g(\nabla_X\mathcal{C}Y, \varphi U) = -g(\mathcal{C}Y, \varphi\mathcal{A}_X U) \tag{6.14}$$

for $X, Y \in \Gamma(ker\pi_*{}^{\perp})$ and $U \in \Gamma(ker\pi_*)$.

Proof. By the virtue of (3.8), (2.3) and (5.1) we have (6.10). Using (3.6) and (2.3), we obtain (6.11). Since $\mathcal{A}_X\xi$ is vertical and φU is horizontal for $X \in \Gamma(ker\pi_*{}^{\perp})$ and $U \in \Gamma(ker\pi_*)$, we have (6.12). Now using (6.12) we get

$$g(\nabla_Y\mathcal{A}_X\xi, \varphi U) = -g(\mathcal{A}_X\xi, \nabla_Y\varphi U)$$

for $X, Y \in \Gamma(ker\pi_*{}^{\perp})$ and $U \in \Gamma(ker\pi_*)$. Then using (3.7) and (2.2) we have

$$g(\nabla_Y\mathcal{A}_X\xi, \varphi U) = -g(\mathcal{A}_X\xi, \varphi\mathcal{A}_Y U) - g(\mathcal{A}_X\xi, \varphi(V\nabla_Y U))$$

Since $\varphi(V\nabla_Y U) \in \Gamma(ker\pi_*{}^{\perp})$, we obtain (6.13).

From (4.1) we get

$$g(\mathcal{C}Y, \varphi U) = 0$$

$$0 = g(\nabla_X\mathcal{C}Y, \varphi U) + g(\mathcal{C}Y, \nabla_X\varphi U)$$

$$= g(\nabla_X\mathcal{C}Y, \varphi U) + g(\mathcal{C}Y, \varphi\nabla_X U)$$

$$g(\nabla_X\mathcal{C}Y, \varphi U) = g(\mathcal{C}Y, \varphi(\mathcal{A}_X U)).$$

Hence we obtain (6.14). □

7 Lagrangian submersions admitting vertical Reeb vector field from trans-Sasakian manifolds

In this section, we shall study the integrability and totally geodesicness of the horizontal distribution of Lagrangian submersions admitting vertical Reeb vector field from trans-Sasakian manifolds. We first investigate the behavior of the O'Neill's tensor $\mathcal{T}$ of such a submersion. From Lemma 6.1, we obtain the following results.

Corollary 7.1. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{T}_U \varphi V - \alpha g(U, V)\xi = \varphi \mathcal{T}_U V - \eta(V)U \,, \tag{7.1}$$

$$\mathcal{T}_V \varphi X = \varphi \mathcal{T}_V X \,, \tag{7.2}$$

$$\mathcal{T}_V \xi = -\alpha \varphi V \,, \tag{7.3}$$

$$\mathcal{T}_\xi X = -\alpha \varphi X \,, \tag{7.4}$$

for $U, V \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^\perp)$.

Proof. For a Lagrangian submersion, we have $\mathcal{C}X = 0$ for any $X \in \Gamma(ker\pi_*^\perp)$. Thus, the assertions (7.1) and (7.2) follows from (5.2) and (5.5), respectively. (7.3) follows from (2.3) and (3.5). The last assertion comes from (7.3). □

Remark 7.1. It is known from [24] that the fibers of a Riemannian submersion are totally geodesic if and only if the O'Neill's tensor $\mathcal{T}$ vanishes.

From Corollary 7.1, we see that the O'Neill's tensor $\mathcal{T}$ cannot vanish. Thus, in view of Remark 7.1, we immediately get the following result.

Theorem 7.2. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the fibers of π cannot be totally geodesic.*

Next, we give some results about the behaviour of the O'Neill's tensor $\mathcal{A}$ of such a submersion.

Corollary 7.3. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{A}_X \varphi V = \varphi \mathcal{A}_X V - \beta \eta(V) \varphi X \,, \tag{7.5}$$

$$\mathcal{A}_X \varphi Y = \varphi \mathcal{A}_X Y \,, \tag{7.6}$$

$$\mathcal{A}_X \xi = \beta X \,, \tag{7.7}$$

for $V \in \Gamma(ker\pi_*)$ and $X \in \Gamma(ker\pi_*^\perp)$.

Proof. The assertions (7.5) and (7.6) follows from (5.6) and (5.10), respectively. The last assertion follows from (2.3) and (3.7). □

Remark 7.2. For a Riemannian submersion, the integrability and totally geodesicness of the horizontal distribution are equivalent to each other. This fact can be seen from (3.4) and (3.8). In this case, the O'Neill's tensor $\mathcal{A}$ vanishes.

One can see that the O'Neill's tensor $\mathcal{A}$ cannot vanish for a such submersion from (7.7). Thus, we get the following result.

Theorem 7.4. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the horizontal distribution of π cannot be integrable.*

Remark 7.3. A smooth map $\pi : (M, g) \to (N, g_N)$ between Riemannian manifolds is called a *totally geodesic map* if π_* preserves parallel translation. Vilms [24] classified totally geodesic Riemannian submersions and proved that a Riemannian submersion $\pi : (M, g) \to (N, g_N)$ is totally geodesic if and only if both O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ vanish.

Thus, in view of Remark 7.3 from Theorem 7.2 or Theorem 7.4, it follows that the following result.

Theorem 7.5. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the submersion π cannot be a totally geodesic map.*

Lastly, we give a necessary and sufficient condition for such submersions to be harmonic.

Theorem 7.6. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then π is harmonic if and only if $trace\varphi\mathcal{T}_V|_{ker\pi_*} = 0$ for $V \in \Gamma(ker\pi_*)$, where $\varphi\mathcal{T}_V|_{ker\pi_*}$ is the restriction of $\varphi\mathcal{T}_V$ to $ker\pi_*$.*

Proof. From [12], we know that π is harmonic if and only if π has minimal fibers. Let $\{e_1, ..., e_k, \xi\}$ be an orthonormal frame of $ker\pi_*$. Thus π is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i + \mathcal{T}_\xi \xi = 0$. Since $\mathcal{T}_\xi \xi = 0$, it follows that π is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$. Now, we calculate $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i$. By orthonormal expansion, we can write

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = \sum_{i=1}^{k} \sum_{j=1}^{k} g(\mathcal{T}_{e_i} e_i, \varphi e_j) \varphi e_j \ ,$$

where $\{\varphi e_1, ..., \varphi e_k\}$ is an orthonormal frame of $\varphi ker\pi_*$. Since $\mathcal{T}_{e_i}$ is skew-symmetric, we obtain

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = -\sum_{i,j=1}^{k} g(\mathcal{T}_{e_i} \varphi e_j, e_i) \varphi e_j \ .$$

Here, from (7.1), we know

$$\mathcal{T}_{e_i}\varphi e_j = \varphi\mathcal{T}_{e_i}e_j + \alpha g(e_i, e_j)\xi - \eta(e_j)e_i \ .$$

Thus, we get

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = -\sum_{i,j=1}^{k} g(\varphi\mathcal{T}_{e_i}e_j, e_i)\varphi e_j \ ,$$

since both $\eta(e_j) = 0$ and $\eta(e_i) = 0$. Using (3.3), we arrive

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = -\sum_{i,j=1}^{k} g(\varphi\mathcal{T}_{e_j}e_i, e_i)\varphi e_j \ . \tag{7.8}$$

Since, $\varphi e_1, ..., \varphi e_k$ are linearly independent, from (7.8), we see that

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = 0 \Leftrightarrow \sum_{i,j=1}^{k} g(\varphi\mathcal{T}_{e_i}e_j, e_i) = 0 \ . \tag{7.9}$$

It easy to see that,

$$\sum_{i,j=1}^{k} g(\varphi\mathcal{T}_{e_j}e_i, e_i) = 0 \Leftrightarrow \sum_{i=1}^{k} g(\varphi\mathcal{T}_V e_i, e_i) = 0 \tag{7.10}$$

for any $V \in \Gamma(ker\pi_*)$. On the other hand,

$$Trace\varphi\mathcal{T}_V|_{ker\pi_*} = \sum_{i=1}^{k} g(\varphi\mathcal{T}_V e_i, e_i) + g(\mathcal{T}_V\xi, \xi)$$

and by (2.1) and (7.3),

$$Trace\varphi\mathcal{T}_V|_{ker\pi_*} = \sum_{i=1}^{k} g(\varphi\mathcal{T}_V e_i, e_i) \ . \tag{7.11}$$

Thus, by (7.9)∼(7.11), the assertion follows. □

Corollary 7.7. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, for any $U \in \Gamma(ker\pi_*)$, we have*

$$g((\nabla_\xi T)_U U, \xi) = 2(\alpha^2 - \beta^2). \tag{7.12}$$

Proof. Let $U \in \Gamma(ker\pi_*)$ such that $\|U\| = 1$. Then, we have

$$K(\xi, U) = g((\nabla_\xi\mathcal{T})_U U, \xi) + \|\mathcal{A}_\xi U\|^2 - \|\mathcal{T}_U\xi\|^2 \tag{7.13}$$

from the equation {3} of Corollay 1 of [15], where $K(\xi, V)$ is the sectional curvature of the plane section spanned by ξ and U. Here, by using (7.3) and (7.7), we get $\|\mathcal{T}_U\xi\|^2 = \alpha^2$ and $\|\mathcal{A}_\xi U\|^2 = \beta^2$, respectively. Thus, the right hand side of (7.13) is equal to $g((\nabla_\xi\mathcal{T})_U U, \xi) + \beta^2 - \alpha^2$. On the other hand, by using the eq. (2.15) of [2], we calculate $K(\xi, U) = \alpha^2 - \beta^2$. Thus, the assertion follows from (7.13). □

8 Lagrangian submersions admitting horizontal Reeb vector field from trans-Sasakian manifolds

In this section, we study Lagrangian submersions admitting horizontal Reeb vector field from trans-Sasakian manifolds onto Riemannian manifolds.

From Lemma 5.1, we obtain the following result.

Corollary 8.1. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{T}_U \varphi V = \varphi \mathcal{T}_U V \,, \tag{8.1}$$

$$\mathcal{T}_V \varphi X - \varphi \mathcal{T}_V X = \beta g(\varphi V, X)\xi - \beta\eta(X)\varphi V \,, \tag{8.2}$$

$$\mathcal{T}_V \xi = \beta V \,. \tag{8.3}$$

for $U, V \in \Gamma(ker\pi_*)$ and $X \in \Gamma(ker\pi_*^{\perp})$.

Proof. Assertions (8.1) and (8.2) follows from (6.1) and (6.4), respectively. The last assertion (8.3) follows from (2.3) and (3.6) or directly (6.11). □

From (8.3), we see that the O'Neill's tensor $\mathcal{T}$ cannot vanish, so we have the following result.

Theorem 8.2. *Let π be a Lagrangian submersion admitting horizonal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the fibers of π cannot be totally geodesic.*

Corollary 8.3. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{A}_X \varphi V = \varphi \mathcal{A}_X V \,, \tag{8.4}$$

$$\mathcal{A}_X \mathcal{B} Y = \varphi \mathcal{A}_X Y + \alpha g(X, Y)\mathcal{H}\xi - \alpha\eta(Y)X \,, \tag{8.5}$$

$$\mathcal{A}_\xi V = -\alpha\varphi V \,. \tag{8.6}$$

$$\mathcal{A}_\xi X = -\alpha\varphi X \,. \tag{8.7}$$

for $V \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^{\perp})$.

Proof. Assertions (8.4) and (8.5) follows from (6.5) and (6.8), respectively. Third assertion (8.6) follows from (2.3) and (3.7). The last one comes from (8.7). □

From (8.4) and (8.5), it is easily seen that the O'Neill's tensor $\mathcal{A}$ cannot vanish. Thus, by Remark 7.2, we have the following result.

Theorem 8.4. *Let π be a Lagrangian submersion admitting horizonal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the horizontal distribution of π cannot be integrable.*

In view of Remark 7.3 from Theorem 8.2 or Theorem 8.4, we get the following result.

Theorem 8.5. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, the submersion π cannot be a totally geodesic map.*

Finally, we give a result concerning the harmonicity of such submersions.

Theorem 8.6. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then π cannot be harmonic.*

Proof. Let $\{e_1, ..., e_k\}$ be an orthonormal frame of $ker\pi_*$. Then $\{\varphi e_1, ..., \varphi e_k, \xi\}$ be an orthonormal frame of $ker\pi_*^{\perp}$. Hence, we have

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = \sum_{i,j=1}^{k} \left\{ g(\mathcal{T}_{e_i} e_i, \varphi e_j)\varphi e_j + g(\mathcal{T}_{e_i} e_i, \xi)\xi \right\} .$$

Using the skew-symmetricness of $\mathcal{T}_{e_i}$ and (8.1), we obtain

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = -\sum_{i,j=1}^{k} \left\{ g(\varphi \mathcal{T}_{e_i} e_j, e_i)\varphi e_j - g(\mathcal{T}_{e_i} \xi, e_i)\xi \right\} .$$

By (3.3) and (8.3), we get

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = -\sum_{i,j=1}^{k} g(\varphi \mathcal{T}_{e_j} e_i, e_i)\varphi e_j - \sum_{i=1}^{k} \beta g(e_i, e_i)\xi .$$

Upon straightforward calculation, we find

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = -\sum_{i,j=1}^{k} g(\varphi \mathcal{T}_{e_j} e_i, e_i)\varphi e_j - k\beta\xi . \tag{8.8}$$

Now, we assume that π is harmonic. Then $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$. From (8.8), it follows that $\xi = -\frac{1}{k\beta} \sum_{i,j=1}^{k} g(\varphi \mathcal{T}_{e_j} e_i, e_i)\varphi e_j$. Which is a contradiction, since $\{\varphi e_1, ..., \varphi e_k, \xi\}$ are linearly independent. □

Corollary 8.7. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then, for any $U \in \Gamma(ker\pi_*)$, we have*

$$g((\nabla_\xi T)_U U, \xi) = 0. \tag{8.9}$$

Proof. Let $U \in \Gamma(ker\pi_*)$ such that $\|U\| = 1$. Then, we have

$$K(\xi, U) = g((\nabla_\xi \mathcal{T})_U U, \xi) + \|\mathcal{A}_\xi U\|^2 - \|\mathcal{T}_U \xi\|^2 \tag{8.10}$$

from the equation $\{3\}$ of Corollary 1 of [15], where $K(\xi, V)$ is the sectional curvature of the plane section spanned by ξ and U. Here, by using (8.3) and (8.6), we get $\|\mathcal{T}_U \xi\|^2 = \beta^2$ and $\|\mathcal{A}_\xi U\|^2 = \alpha^2$, respectively. Thus, the right hand side of (8.10) is equal to $g((\nabla_\xi \mathcal{T})_U U, \xi) + \alpha^2 - \beta^2$. On the other hand, by using the eq. (2.15) of [2], we calculate $K(\xi, U) = \alpha^2 - \beta^2$. Thus, the assertion follows from (8.10). □

Remark 8.1. Corollary 8.9 is a generalization of the Corollary 8.8 of [21].

Theorem 8.8. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). If $dim(ker\pi_*) \geq 2$ and the fibers are totally umbilical, then we have*

$$H = -\beta\xi \ , \tag{8.11}$$

where H is the mean curvature tensor field of the fibers.

Proof. By the hypothesis, we may take any two vector fields U and V in $ker\pi_*$ such that $g(U, V) = 0$ and $\|U\| = 1$. Since the fibers are totally umbilical, with (3.9), it follows that

$$T_U V = 0 \ . \tag{8.12}$$

Using (8.1), the skew symmetry of T and φ, and (8.12), we have

$$g(H, \varphi V) = g(T_U U, \varphi V) = -g(\varphi T_U V, U) = 0.$$

Since π is a Lagrangian submersion admitting horizontal Reeb vector field, it follows that $H = g(H, \xi)\xi$. But, using (8.3), we obtain $g(H, \xi) = g(T_U U, \xi) = -g(T_U \xi, U) = -\beta$, which complete the proof. □

Acknowledgement. The authors are grateful to the referee's for their valuable criticisms, comments and suggestions towards the improvement of this paper.

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