Anti-invariant submanifolds of locally decomposable golden Riemannian manifolds

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Abstract. In this paper, we give some properties of anti-invariant submanifolds of a golden Riemannian manifold. We obtain some necessary conditions for any submanifold in a locally decomposable golden Riemannian manifold to be anti-invariant. In these conditions, we also show that the submanifold is totally geodesic. We find a local orthonormal frame for the normal bundle of any anti-invariant submanifold of a locally decomposable golden Riemannian manifold. Finally, we demonstrate the existence of unit and mutually orthogonal normal vector fields such that their corresponding second fundamental tensors vanish identically under the assumption that the codimension of the anti-invariant submanifold is greater than its dimension.

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1 Introduction

Submanifold theory, the origins of which are in curve and surface theories, is an important research field in differential geometry. There exist two well known classes of submanifolds among all submanifolds of an ambient manifold, namely, invariant submanifolds and anti-invariant submanifolds. The differential geometry of invariant submanifolds is very different from that of anti-invariant submanifolds. Because, in general, an invariant submanifold inherits almost all properties of the ambient manifold. That is, the invariant submanifold doesn't present a completely different geometric characteristic of the ambient manifold than expected. When considered from this point view, the investigation of invariant submanifolds isn't interesting in the differential geometry of submanifolds. Therefore, this situation makes anti-invariant submanifolds become a challenging topic in differential geometry. The differential geometry of anti-invariant submanifolds has been studied by many geometers in various ambient manifolds as follows: The research on the differential geometry of

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anti-invariant submanifolds has been firstly initiated by B. Y. Chen and K. Ogiue in [5] including some fundamental properties, characterizations and classifications of those in complex space forms. Moreover, in complex space forms, two reduction theorems have been obtained for anti-invariant submanifolds by B. Y. Chen, C. S. Houh and H. S. Lue [4], then a corollary as an application of these theorems has been given. Next, it has been shown that the normal bundle of the submanifold admits no parallel isoperimetric sections if the ambient manifold is not flat. Lastly, two necessary conditions have been found for compact anti-invariant submanifolds of the complex number space to be a product submanifold. In Kaehlerian manifolds with the vanishing Bochner curvature tensor, K. Yano [21] has discussed some conditions for anti-invariant submanifolds to be conformally flat by generalizing D. E. Blair's theorem in [3]. In Sasakian manifolds, by introducing the concept of the vanishing contact Bochner curvature tensor as an analogue of the Bochner curvature tensor in Kaehlerian manifolds, anti-invariant submanifolds have been examined in [22] containing some conditions for the conformal flatness and the local productness of those. Besides, in the event that the ambient manifold is a Sasakian space form, I. Ishihara [16] has analyzed anti-invariant submanifolds with the pseudo-parallel mean curvature vector and the pseudo-flat normal connection in the same way as taken in complex space forms [24]. In locally product Riemannian manifolds, T Adati [1] has given a necessary condition for an arbitrary submanifold to be both anti-invariant and totally geodesic and shown that an anti-invariant submanifold is totally geodesic under the assumption that the dimension of the submanifold is equal to half of that of the ambient manifold. Furthermore, G. Pitis [19] has investigated algebraic conditions for any compact anti-invariant submanifold to be stable or unstable. Many geometers have also contributed to the differential geometry of anti-invariant submanifolds in other well known ambient manifolds, such as almost contact metric manifolds [17], quaternionic Kaehlerian manifolds [9], almost para contact manifolds [18], 6-spheres [8], Kenmotsu manifolds [20].

The golden ratio, also known as the golden proportion, the divine ratio, the golden section or the golden mean, is an irrational number, which appears in geometry, physics, chemistry, astrophysics, biology, anatomy, art, architecture, sculpture etc. It arises from the division problem of a line segment into two pieces of different lengths so that the ratio of the whole segment to the larger piece is equal to that of the larger piece to the smaller piece. That is, a line segment AB with a non-midpoint C is divided in the ratio $\frac{AB}{AC} = \frac{AC}{CB}$, where AC and CB are large and small pieces of the line segment AB, respectively. If putting $x = \frac{AB}{AC}$, then the problem is expressed by the quadratic equation $x^2 = x + 1$, whose roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. The former is the golden ratio. It is frequently denoted by ϕ, the first Greek letter in the name of Phidias [7, 10].

Recently, the golden ratio has been used to research on its effect on differential geometry with the help of a special geometric structure on C^∞-differentiable manifolds, called a golden structure, in [6]. Herein a golden structure has been investigated by means of the corresponding almost product structure, then by endowing the golden structure with a main geometric object, namely, a Riemannian metric, the concepts of a golden Riemannian structure and a golden Riemannian manifold have been defined and their basic properties have been obtained. Thus, the application of the golden ratio to differential geometry has been immensely successful. Since then, C^∞-

differentiable manifolds admitting golden Riemannian structures, i.e., golden Riemannian manifolds are of great interest to geometers. Particularly, their different kind of submanifolds, such as invariant submanifolds, slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds have been examined in [2, 12, 13, 14, 15]. The increasing interest of golden Riemannian manifolds related to the golden ratio, especially their submanifolds, gives an opportunity to make new examinations in the differential geometry of Riemannian manifolds endowed with special geometric structures and their submanifolds.

Motivated by the above mentioned studies and guidances, the main purpose of this paper is to investigate anti-invariant submanifolds in the case that the ambient manifold is a locally decomposable golden Riemannian manifold.

The organisation of this paper is as follows: The paper consists of three sections. Section 2 contains some fundamental facts on golden Riemannian manifolds and their submanifolds. Section 3 is concerned with a research on anti-invariant submanifolds in locally decomposable golden Riemannian manifolds. We obtain a few basic properties of an anti-invariant submanifold in golden Riemannian manifolds. We get necessary conditions for any submanifold of a locally decomposable golden Riemannian manifold to be both anti-invariant and totally geodesic. We also prove that an anti-invariant submanifold is totally geodesic in locally decomposable golden Riemannian manifolds if the dimension of the ambient manifold is equal to twice that of the submanifold. We establish a local orthonormal frame for the normal bundle of any anti-invariant submanifold in a locally decomposable golden Riemannian manifold providing that the dimension of the submanifold is less than its codimension, moreover, we show that there exist normal vector fields determined by a chosen local orthonormal frame for the tangent bundle of the submanifold as the number of its dimension.

2 Preliminaries

In this section, we give a short review of main definitions, concepts, formulas, notations and results on golden Riemannian manifolds and their submanifolds.

A non-trivial C^∞-tensor field f of type $(1,1)$ on a C^∞-differentiable manifold $\overline{M}$ is called a polynomial structure of degree n if it satisfies the algebraic equation

$$Q(x) = x^n + a_n x^{n-1} + \cdots + a_2 x + a_1 I = 0, \tag{2.1}$$

where I is the identity $(1,1)$-tensor field on $\overline{M}$ and $f^{n-1}(p), f^{n-2}(p), \ldots, f(p), I$ are linearly independent for every point $p \in \overline{M}$. Also, the monic polynomial $Q(x)$ is said to be the structure polynomial [11].

A polynomial structure $\overline{\Phi}$ of degree 2 with the structure polynomial $Q(x) = x^2 - x - 1$ on a C^∞-differentiable real manifold $\overline{M}$ is called a golden structure. That is, the golden structure $\overline{\Phi}$ is a tensor field of type $(1,1)$ satisfying the equation

$$\overline{\Phi}^2 = \overline{\Phi} + I. \tag{2.2}$$

In this case, $\overline{M}$ is called a golden manifold. We denote by $\Gamma\left(T\overline{M}\right)$ the Lie algebra of differentiable vector fields on $\overline{M}$. If there is a Riemannian metric $\overline{g}$ on $\overline{M}$ endowed with a golden structure $\overline{\Phi}$ such that $\overline{g}$ and $\overline{\Phi}$ yield the relation

$$\overline{g}\left(\overline{\Phi}X, Y\right) = \overline{g}\left(X, \overline{\Phi}Y\right) \tag{2.3}$$

for any vector fields $X, Y \in \Gamma\left(T\overline{M}\right)$, then the pair $\left(\overline{g}, \overline{\Phi}\right)$ is named a golden Riemannian structure and the triple $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is called a golden Riemannian manifold. The eigenvalues of the golden structure $\overline{\Phi}$ are $\phi = \frac{1+\sqrt{5}}{2}$ and $1 - \phi = \frac{1-\sqrt{5}}{2}$ being the roots of the algebraic equation $x^2 - x - 1 = 0$. The inverse $\overline{\Phi}^{-1}$ of the golden structure $\overline{\Phi}$ is given by $\overline{\Phi}^{-1} = \overline{\Phi} - I$ and verifies the equation $\left(\overline{\Phi}^{-1}\right)^2 = -\overline{\Phi}^{-1} + I$, so it isn't a golden structure [6, 14, 15].

Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. We denote by T_pM and $T_pM^{\perp}$ its tangent and normal spaces at a point $p \in M$, respectively. Then the tangent space $T_p\overline{M}$ has the decomposition

$$T_p\overline{M} = T_pM \oplus T_pM^{\perp} \tag{2.4}$$

for each point $p \in M$. The induced Riemannian metric g on M is given by

$$g\left(X, Y\right) = \overline{g}\left(i_*X, i_*Y\right) \tag{2.5}$$

for any vector fields $X, Y \in \Gamma(TM)$, where i_* denotes the differential of the immersion $i : M \longrightarrow \overline{M}$. We consider a local orthonormal frame $\{N_1, \ldots, N_r\}$ of the normal bundle $TM^{\perp}$. For every tangent vector field $X \in \Gamma(TM)$, the vector fields $\overline{\Phi}\left(i_*X\right)$ and $\overline{\Phi}\left(N_\alpha\right)$ on the ambient manifold $\overline{M}$ can be expressed in the following forms:

$$\overline{\Phi}\left(i_*X\right) = i_*\left(\Phi\left(X\right)\right) + \sum_{\alpha=1}^{r} u_\alpha\left(X\right) N_\alpha \tag{2.6}$$

and

$$\overline{\Phi}\left(N_\alpha\right) = \varepsilon i_*\left(\xi_\alpha\right) + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \; \varepsilon = \pm 1, \tag{2.7}$$

respectively, where Φ is a tensor field of type $(1,1)$ on M, ξ_α's are tangent vector fields on M, u_α's are differential 1-forms on M and $(a_{\alpha\beta})$ is a matrix of type $r \times r$ of real functions on M. Thus, we obtain a structure $\left(\Phi, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_{r\times r}\right)$ induced on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\overline{M}$ and M, respectively. Then the Gauss and Weingarten formulas of M in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + \sum_{\alpha=1}^{r} h_\alpha\left(X, Y\right) N_\alpha \tag{2.8}$$

and

$$\overline{\nabla}_{i_*X} N_\alpha = -i_*A_\alpha X + \sum_{\beta=1}^{r} l_{\alpha\beta}\left(X\right) N_\beta \tag{2.9}$$

for any vector fields $X, Y \in \Gamma(TM)$, where h_α's are the second fundamental tensors corresponding to N_α's, A_α's are the shape operators in the direction of N_α's and

$l_{\alpha\beta}$'s are the 1-forms on M corresponding to the normal connection $\nabla^{\perp}$ for any $\alpha, \beta \in \{1, \ldots, r\}$. Besides, the following relations are verified:

$$h(X,Y) = \sum_{\alpha=1}^{r} h_\alpha(X,Y) N_\alpha, \tag{2.10}$$

$$h_\alpha(X,Y) = h_\alpha(Y,X), \tag{2.11}$$

$$h_\alpha(X,Y) = g(A_\alpha X, Y), \tag{2.12}$$

$$\nabla^{\perp}_X N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}(X) N_\beta \tag{2.13}$$

and

$$l_{\alpha\beta} = -l_{\beta\alpha} \tag{2.14}$$

for any vector fields $X, Y \in \Gamma(TM)$ [14].

As it is well known, the submanifold M is called totally geodesic if the second fundamental form h vanishes identically. Also, the mean curvature vector H of M is defined by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{2.15}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM at a point $p \in M$. If $H = 0$, then M is named a minimal submanifold. If $h(X,Y) = g(X,Y)H$ for any vector fields $X, Y \in \Gamma(TM)$, then M is said to be a totally umbilical submanifold [23].

The triple $(\overline{M}, \overline{g}, \overline{\Phi})$ is called a locally decomposable golden Riemannian manifold if the golden structure $\overline{\Phi}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$, i.e., the covariant derivative $\overline{\nabla}\,\overline{\Phi}$ is identically zero.

The induced structure $\left(\Phi, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_{r\times r}\right)$ on the submanifold M by the golden Riemannian structure $(\overline{g}, \overline{\Phi})$ satisfies the following relations:

$$\Phi^2(X) = \Phi(X) + X - \varepsilon \sum_{\alpha=1}^{r} u_\alpha(X) \xi_\alpha, \tag{2.16}$$

$$u_\alpha(\Phi(X)) = (1 - a_{\alpha\alpha}) u_\alpha(X), \tag{2.17}$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \tag{2.18}$$

$$u_\beta(\xi_\alpha) = \varepsilon \left(\delta_{\alpha\beta} + a_{\alpha\beta} - \sum_{\gamma=1}^{r} a_{\alpha\gamma} a_{\beta\gamma} \right), \tag{2.19}$$

$$\Phi\left(\xi_\alpha\right) = \xi_\alpha - \sum_{\beta=1}^{r} a_{\alpha\beta}\xi_\beta, \tag{2.20}$$

$$u_\alpha\left(X\right) = \varepsilon g\left(X, \xi_\alpha\right), \tag{2.21}$$

$$g\left(\Phi\left(X\right), Y\right) = g\left(X, \Phi\left(Y\right)\right) \tag{2.22}$$

and

$$g\left(\Phi\left(X\right), \Phi\left(Y\right)\right) = g\left(\Phi\left(X\right), Y\right) + g\left(X, Y\right) - \sum_{\alpha=1}^{r} u_\alpha\left(X\right) u_\alpha\left(Y\right) \tag{2.23}$$

for any vector fields $X, Y \in \Gamma(TM)$, where $\delta_{\alpha\beta}$ is the Kronecker delta [14, 15]. Moreover, if $\overline{M}$ is a locally decomposable golden Riemannian manifold, then we have the following relations:

$$\left(\nabla_X \Phi\right) Y = \varepsilon \sum_{\alpha=1}^{r} h_\alpha\left(X, Y\right) \xi_\alpha + \sum_{\alpha=1}^{r} u_\alpha\left(Y\right) A_\alpha X, \tag{2.24}$$

$$\left(\nabla_X u_\alpha\right) Y = -h_\alpha\left(X, \Phi Y\right) + \sum_{\beta=1}^{r} u_\beta\left(Y\right) l_{\alpha\beta}\left(X\right) + \sum_{\beta=1}^{r} h_\beta\left(X, Y\right) a_{\alpha\beta}, \tag{2.25}$$

$$\nabla_X \xi_\alpha = -\varepsilon \Phi\left(A_\alpha X\right) + \varepsilon \sum_{\beta=1}^{r} a_{\alpha\beta} A_\beta X + \sum_{\beta=1}^{r} l_{\alpha\beta}\left(X\right) \xi_\beta \tag{2.26}$$

and

$$X\left(a_{\alpha\beta}\right) = -\varepsilon h_\beta\left(X, \xi_\alpha\right) - \varepsilon h_\alpha\left(X, \xi_\beta\right) - \sum_{\gamma=1}^{r} a_{\alpha\gamma} l_{\gamma\beta}\left(X\right) - \sum_{\gamma=1}^{r} a_{\beta\gamma} l_{\gamma\alpha}\left(X\right) \tag{2.27}$$

for any vector fields $X, Y \in \Gamma(TM)$ [14].

Let $\{N_1, \ldots, N_r\}$ and $\{N_1', \ldots, N_r'\}$ be two local orthonormal frames of the normal bundle $TM^\perp$. Then the decomposition of the normal vector field N_α' in the local orthonormal frame $\{N_1, \ldots, N_r\}$ is given by

$$N_\alpha' = \sum_{\gamma=1}^{r} k_\alpha^\gamma N_\gamma \tag{2.28}$$

for any $\alpha \in \{1, \ldots, r\}$, where (k_α^γ) is an orthogonal matrix of type $r \times r$. We write

$$u_\alpha' = \sum_{\gamma=1}^{r} k_\alpha^\gamma u_\gamma, \tag{2.29}$$

$$\xi'_{\alpha} = \sum_{\gamma=1}^{r} k_{\alpha}^{\gamma} \xi_{\gamma} \tag{2.30}$$

and

$$a'_{\alpha\beta} = \sum_{\gamma=1}^{r} \sum_{\delta=1}^{r} k_{\alpha}^{\gamma} a_{\gamma\delta} k_{\beta}^{\delta}. \tag{2.31}$$

Then using (2.28), (2.6) and (2.7) take the following forms:

$$\overline{\Phi}\left(i_{*}X\right) = i_{*}\Phi\left(X\right) + \sum_{\alpha=1}^{r} u'_{\alpha}\left(X\right) N'_{\alpha} \tag{2.32}$$

and

$$\overline{\Phi}\left(N'_{\alpha}\right) = \varepsilon i_{*}\left(\xi'_{\alpha}\right) + \sum_{\beta=1}^{r} a'_{\alpha\beta} N'_{\beta},\ \varepsilon = \pm 1, \tag{2.33}$$

respectively.

On the other hand, (2.30) shows that if the tangent vector fields $\xi_1, \ldots, \xi_r$ are linearly independent (respectively, linearly dependent), then the tangent vector fields $\xi'_1, \ldots, \xi'_r$ are also linearly independent (respectively, linearly dependent). As the matrix element $a_{\alpha\beta}$ is symmetric in the indices α and β, it can be reduced to the form $a'_{\alpha\beta} = \lambda_{\alpha}\delta_{\alpha\beta}$, where λ_{α}'s are the eigenvalues of the matrix $\left(a_{\alpha\beta}\right)_{r\times r}$ for any $\alpha, \beta \in \{1, \ldots, r\}$ [6].

Lemma 2.1. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional locally decomposable golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. If $a_{\alpha\beta} = \lambda_a \delta_{\alpha\beta}$, $\lambda_a \in (1-\phi, \phi)$ for any $\alpha, \beta \in \{1, \ldots, r\}$, then the tangent vector fields $\xi_1, \ldots, \xi_r$ are linearly independent.*

Proof. We assume that $a_{\alpha\beta} = \lambda_a \delta_{\alpha\beta}$, $\lambda_a \in (1-\phi, \phi)$ for any $\alpha, \beta \in \{1, \ldots, r\}$. By a straightforward calculation, we obtain from (2.18) and (2.19) that

$$u_{\beta}\left(\xi_{\alpha}\right) = \varepsilon \delta_{\alpha\beta}\left(1 + \lambda_a - \lambda_a^2\right). \tag{2.34}$$

On the other hand, it can be easily seen from (2.21) that $g\left(\xi_{\alpha}, \xi_{\beta}\right) = \varepsilon u_{\beta}\left(\xi_{\alpha}\right)$ for any $\alpha, \beta \in \{1, \ldots, r\}$. Thus, we get

$$g\left(\xi_{\alpha}, \xi_{\beta}\right) = \delta_{\alpha\beta}\left(1 + \lambda_a - \lambda_a^2\right). \tag{2.35}$$

If we write $\sum_{\beta=1}^{r} \rho_{\beta}\xi_{\beta} = 0$, then it follows from (2.35) that

$$0 = g\left(\xi_{\alpha}, \sum_{\beta=1}^{r} \rho_{\beta}\xi_{\beta}\right) = \rho_{a}\left(1 + \lambda_a - \lambda_a^2\right) \tag{2.36}$$

for any $\alpha \in \{1, \ldots, r\}$. At the same time, because of the fact that $\lambda_a \in (1-\phi, \phi)$, it is clear that

$$1 + \lambda_a - \lambda_a^2 \neq 0. \tag{2.37}$$

Hence, it results from (2.36) and (2.37) that $\rho_a = 0$ for any $\alpha \in \{1, \ldots, r\}$. In other words, the tangent vector fields $\xi_1, \ldots, \xi_r$ are linearly independent. □

Lemma 2.2. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional locally decomposable golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the following expressions are equivalent:*

- **(a)** *For any $\alpha, \beta \in \{1, \ldots, r\}$, $a_{\alpha\beta} = \delta_{\alpha\beta}$.*
- **(b)** *For any $\alpha \in \{1, \ldots, r\}$, $\overline{\Phi}^{-1}(N_\alpha) \in \Gamma(TM)$.*

Proof. If $a_{\alpha\beta} = \delta_{\alpha\beta}$ for any $\alpha, \beta \in \{1, \ldots, r\}$, then we derive by a direct calculation from (2.7) that

$$\overline{\Phi}^{-1}(N_\alpha) = \varepsilon i_*(\xi_\alpha), \tag{2.38}$$

which implies that $\overline{\Phi}^{-1}(N_\alpha) \in \Gamma(TM)$. That is, we get (a)⇒(b). Conversely, we suppose that $\overline{\Phi}^{-1}(N_\alpha) \in \Gamma(TM)$ for any $\alpha \in \{1, \ldots, r\}$. By means of (2.7), we have

$$\sum_{\beta=1}^{r} (a_{\alpha\beta} - \delta_{\alpha\beta}) N_\beta = 0. \tag{2.39}$$

Thus, as $\{N_1, \ldots, N_r\}$ is a local orthonormal frame of the normal bundle $TM^\perp$, we obtain

$$a_{\alpha\beta} = \delta_{\alpha\beta}, \tag{2.40}$$

which shows (b)⇒(a). Consequently, the proof has been completed. □

3 Anti-Invariant Submanifolds of Golden Riemannian Manifolds

This section deals with an investigation regarding anti-invariant submanifolds in golden Riemannian manifolds.

To begin with, we remember the concept of an anti-invariant submanifold in golden Riemannian manifolds. Any anti-invariant submanifold M of a golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is submanifold such that the golden structure $\overline{\Phi}$ of the ambient manifold $\overline{M}$ carries each tangent vector of the submanifold M into its corresponding normal space in the ambient manifold $\overline{M}$, that is,

$$\overline{\Phi}(T_pM) \subseteq T_pM^\perp \tag{3.1}$$

for any point $p \in M$.

Let M be an n-dimensional anti-invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then we have $\Phi = 0$. Hence, (2.6) is given by

$$\overline{\Phi}(i_*X) = \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha \tag{3.2}$$

for any vector field $X \in \Gamma(TM)$.

Proposition 3.1. *Let M be an n-dimensional anti-invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the induced structure $\left(\Phi = 0, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_{r\times r}\right)$ on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$ satisfies the following relations:*

$$X = \varepsilon \sum_{\alpha=1}^{r} u_\alpha (X) \xi_\alpha, \text{ or } I = \varepsilon \sum_{\alpha=1}^{r} u_\alpha \otimes \xi_\alpha, \tag{3.3}$$

$$(1 - a_{\alpha\alpha}) u_\alpha (X) = 0, \tag{3.4}$$

$$\sum_{\beta=1}^{r} (\delta_{\alpha\beta} - a_{\alpha\beta}) \xi_\beta = 0 \tag{3.5}$$

and

$$g (X, Y) = \sum_{\alpha=1}^{r} u_\alpha (X) u_\alpha (Y) \tag{3.6}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Proof. Taking account of that $\Phi = 0$, the proof is obvious from (2.16), (2.17), (2.20) and (2.23). □

Proposition 3.2. *Let M be an n-dimensional anti-invariant submanifold of codimension r, isometrically immersed in an m-dimensional locally decomposable golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the induced structure $\left(\Phi = 0, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_{r\times r}\right)$ on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$ verifies the following relations:*

$$\varepsilon \sum_{\alpha=1}^{r} h_\alpha (X, Y) \xi_\alpha + \sum_{\alpha=1}^{r} u_\alpha (Y) A_\alpha X = 0, \tag{3.7}$$

$$(\nabla_X u_\alpha) Y = \sum_{\beta=1}^{r} u_\beta (Y) l_{\alpha\beta} (X) + \sum_{\beta=1}^{r} h_\beta (X, Y) a_{\alpha\beta} \tag{3.8}$$

and

$$\nabla_X \xi_\alpha = \varepsilon \sum_{\beta=1}^{r} a_{\alpha\beta} A_\beta X + \sum_{\beta=1}^{r} l_{\alpha\beta} (X) \xi_\beta \tag{3.9}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Proof. Using the fact that $\Phi = 0$, the proof can be easily seen from (2.24), (2.25) and (2.26). □

Let us consider the matrix $U = (\xi_1 \cdots \xi_r)$ of type $r \times r$. Then in order that the non-trivial solution of the system of equations $u_\alpha(X) = 0$ for any $\alpha \in \{1, \ldots, r\}$ does not exist, it is a necessary and sufficient condition that $rankU = n$. Thus, we have $r \geq n$.

Theorem 3.3. *Let M be an n-dimensional submanifold, isometrically immersed in a $2n$-dimensional locally decomposable golden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. If $a_{\alpha\beta} = \delta_{\alpha\beta}$ for any $\alpha, \beta \in \{1, \ldots, n\}$, then M is an anti-invariant submanifold. Moreover, the submanifold M is totally geodesic.*

Proof. We firstly note that $r = n = \dim M$, where r is the codimension of the submanifold M. We assume that $a_{\alpha\beta} = \delta_{\alpha\beta}$ for any $\alpha, \beta \in \{1, \ldots, n\}$. In this case, it follows from (2.20) that

$$(\Phi)\, U = 0, \tag{3.10}$$

where (Φ) is the corresponding matrix to the induced structure Φ. Also, we deduce from Lemma 2.1 that if $a_{\alpha\beta} = \delta_{\alpha\beta}$ for any $\alpha, \beta \in \{1, \ldots, n\}$, the tangent vector fields ξ_α's are linearly independent, or equivalently the 1-forms u_α's are linearly independent. Then there exists the inverse U^{-1} of the matrix U. Hence, we obtain from (3.10) that

$$\Phi = 0. \tag{3.11}$$

In consequence of (2.6), it seems from (3.11) that M is an anti-invariant submanifold. Now, we show that the submanifold M is totally geodesic. Using the anti-invariance of the submanifold M, then it results from (2.12), (2.21) and (3.7) that

$$\sum_{\alpha=1}^{n} u_\alpha(Y)\, h_\alpha(X, Z) = -\sum_{\alpha=1}^{n} u_\alpha(Z)\, h_\alpha(X, Y) \tag{3.12}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Applying (2.11) to (3.12), we get

$$\sum_{\alpha=1}^{n} u_\alpha(Y)\, h_\alpha(X, Z) = \sum_{\alpha=1}^{n} u_\alpha(Z)\, h_\alpha(X, Y) \tag{3.13}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Hence, by means of (3.12) and (3.13), we have

$$\sum_{\alpha=1}^{n} u_\alpha(Y)\, h_\alpha(X, Z) = 0, \tag{3.14}$$

which implies that $h_\alpha = 0$ for any $\alpha \in \{1, \ldots, n\}$ because of the linear independence of the 1-forms u_α's. That is, the second fundamental form h is identically zero. Therefore, M is a totally geodesic submanifold. □

Theorem 3.4. *Let M be an n-dimensional submanifold, isometrically immersed in a $2n$-dimensional locally decomposable golden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. If $\overline{\Phi}^{-1}(N_\alpha) \in \Gamma(TM)$ for any $\alpha \in \{1, \ldots, n\}$, then M is an anti-invariant submanifold. Furthermore, the submanifold M is totally geodesic.*

Proof. Taking into consideration Lemma 2.2, the proof can be shown in a method similar to that of Theorem 3.3. □

We also note that if M is an n-dimensional anti-invariant submanifold of a $2n$-dimensional locally decomposable golden Riemannian manifold, then the tangent vector fields ξ_α's have to be linearly independent and we have $a_{\alpha\beta} = \delta_{\alpha\beta}$, or equivalently $\overline{\Phi}^{-1}(N_\alpha) \in \Gamma(TM)$ for any $\alpha, \beta \in \{1, \ldots, n\}$. In addition, M is a totally geodesic submanifold.

Theorem 3.5. *Let M be an n-dimensional anti-invariant submanifold of codimension r, isometrically immersed in an m-dimensional locally decomposable golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. If $r > n$, then there exists a local orthonormal frame $\{N_1, \ldots, N_r\}$ of the normal bundle $TM^\perp$ such that*

$$N_i = \overline{\Phi} i_* E_i,\ i = 1, \ldots, n \tag{3.15}$$

and

$$\overline{\Phi}(N_A) = \lambda_A N_A,\ A = n+1, \ldots, r, \tag{3.16}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame of the tangent bundle TM and λ_A's are the eigenvalues of the golden structure $\overline{\Phi}$.

Proof. We recall that if $r > n = \dim M$, then the tangent vector fields ξ_α's are linearly dependent. Let $\{N_1, \ldots, N_r\}$ be a local orthonormal frame of the normal bundle $TM^\perp$ such that $a_{\alpha\beta} = \lambda_a \delta_{\alpha\beta}$, where λ_a's are the eigenvalues of the matrix $(a_{\alpha\beta})_{r\times r}$ for any $\alpha, \beta \in \{1, \ldots, r\}$. Considering (3.5) from the point of view of the tangent vector fields ξ'_α's, we obtain

$$(1 - \lambda_a)\, \xi'_\alpha = 0,\ \alpha = 1, \ldots, r. \tag{3.17}$$

Also, we remark from (2.19) and (2.21) that $\|\xi'_\alpha\|^2 = 1 + \lambda_a - \lambda_a^2$ for any $\alpha \in \{1, \ldots, r\}$. Therefore, we can suppose that the tangent vector fields ξ'_i's are linearly independent, $\xi'_A = 0$, $\lambda_i = 1$ and $\lambda_A^2 = \lambda_A + 1$ for any $i \in \{1, \ldots, n\}$ and $A \in \{n+1, \ldots, r\}$. In addition, from (2.19), we have

$$u'_j(\xi'_i) = \varepsilon \delta_{ij},\ i, j = 1, \ldots, n, \tag{3.18}$$

which tells us that the tangent vector fields ξ'_i's are unit and mutually orthogonal. Thus, the set $\{\xi'_1, \ldots, \xi'_n\}$ is a local orthonormal frame for the tangent bunde TM. For any $i, j \in \{1, \ldots, n\}$, we put

$$N_i^\star = \overline{\Phi} i_* \xi'_i \tag{3.19}$$

and

$$N_j^\star = \overline{\Phi} i_* \xi'_j. \tag{3.20}$$

Then taking into account that the submanifold M is anti-invariant and the Riemannian metric $\overline{g}$ is $\overline{\Phi}$-compatible, we get

$$\overline{g}\left(N_i^\star, N_j^\star\right) = \delta_{ij}. \tag{3.21}$$

Hence, we can choose the normal vector fields N_i''s such that $N_i' = \overline{\Phi} i_* E_i$ for any $i \in \{1, \ldots, n\}$. At the same time, we see from (2.33) that

$$\overline{\Phi}\left(N_A'\right) = \lambda_A N_A',\ A = n+1, \ldots, r. \tag{3.22}$$

In other words, the normal vector fields N_A''s are the eigenvectors of the golden structure $\overline{\Phi}$ corresponding to the eigenvalues λ_A's for any $A \in \{n+1, \ldots, r\}$. Consequently, the proof has been finished. □

Theorem 3.6. *Let M be an n-dimensional anti-invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. If $r > n$, then there exist unit and mutually orthogonal normal vector fields N_i's of the normal bundle $TM^{\perp}$ such that*

$$h_i = 0 \tag{3.23}$$

for any $i \in \{1, \ldots, n\}$.

Proof. Because of the fact that the submanifold M is anti-invariant, we get from (3.7) that

$$\sum_{j=1}^{n} u_j'(Y)\, h_j(X, Z) = -\sum_{j=1}^{n} u_j'(Z)\, h_j(X, Y) \tag{3.24}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Using (2.11) in (3.24)

$$\sum_{j=1}^{n} u_j'(Y)\, h_j(X, Z) = \sum_{j=1}^{n} u_j'(Z)\, h_j(X, Y) \tag{3.25}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Hence, it follows from (3.24) and (3.25) that

$$\sum_{j=1}^{n} u_j'(Y)\, h_j(X, Z) = 0. \tag{3.26}$$

On the other hand, if $\{E_1, \ldots, E_n\}$ is a local orthonormal frame for the tangent bundle TM, it is possible from Theorem 3.5 to choose the normal vector fields N_i's such that

$$N_i = \overline{\Phi} i_* E_i \tag{3.27}$$

for any $i \in \{1, \ldots, n\}$. Hence, by virtue of (3.2), we obtain

$$\sum_{j=1}^{n} \delta_{ij} N_j = \sum_{j=1}^{n} u_j'(E_i)\, N_j, \tag{3.28}$$

which implies from the linear independence of the normal vector fields N_j's that $\delta_{ij} = u_j'(E_i)$ for any $i, j \in \{1, \ldots, n\}$. Thus, putting $Y = E_i$ in (3.26), we have

$$h_i = 0 \tag{3.29}$$

for any $i \in \{1, \ldots, n\}$. As a result, the proof has been demonstrated. □

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