Polarized symplectic structures

A.Awane

Abstract. We study various aspects and properties of polarized symplectic manifolds. We give a new proof of the Darboux theorem for symplectic manifolds equipped with Lagrangian foliation using only quadratures.

We give a special interest to the study of Poisson structures subordinate to a real polarization, in this case, the polarized Hamiltonians are locally affine mappings with respect to the affine structure of the Lagrangian foliation. Also, we show that polarized Hamiltonians consist of all real smooth mappings whose the associated vector Hamiltonian field preserves the Lagrangian foliation. We give some examples and properties of these objects.

The study of the integrability of the almost polarized symplectic manifolds are given in this work.

M.S.C. 2010: 37J05, 53D05, 53D12, 53D12, 70H17, 70H05.
Key words: Symplectic manifolds; Polarization; Hamiltonian; foliate vector fields; polarized Poisson manifold; $G-$structures.

1 Introduction

A symplectic manifold (M, θ) is said to be polarized if it is equipped with a Lagrangian foliation $\mathfrak{F}$. The notion of polarized symplectic manifold plays an important role in the geometric quantization of Kostant-Souriau ([11],[17]). Interesting properties of the geometry of Lagrangian foliations are given by A. Weinstein [16] and P. Dazord [5]. The natural model of polarized symplectic manifold is the cotangent bundle T^*M (phase space), equipped with the Liouville form and the real polarization defined by the vertical foliation of the fibration $\pi_M : T^*M \longrightarrow M$.

In symplectic geometry, the Darboux's theorem plays a fundamental role;

of course, this theorem is first proved by induction by G.Darboux himself. An other proof based on the Moser Lemma is given by A. Weinstein (in 1977).

The Darboux theorem for symplectic manifolds equipped with Lagrangian foliations, is given by I. Vaisman in (1989) in the context of Poisson structures on foliated manifolds. In this work I will reproduce the proof of Darboux's theorem for the

Balkan Journal of Geometry and Its Applications, Vol.25, No.1, 2020, pp. 1-18.

$k-$symplectic structures ([1], [2]), in the case of $k = 1$, i.e., for symplectic manifolds equipped with a Lagrangian foliation, using exclusively quadratures, i.e., operations of integration, elimination (application of the implicit function theorem) and partial differentiation.

A Hamiltonian vector field X of (M, θ) is said to be polarized if, in addition, X preserve the foliation $\mathfrak{F}$. A Hamiltonian mapping H, (a smooth real function on M) is said to be polarized if the associated Hamiltonian vector field X_H is polarized.

The set of all polarized Hamiltonian mappings, denoted by $\mathfrak{H}(M, \mathfrak{F})$, is a proper linear subspace of $\mathscr{C}^\infty(M)$ and it admits a natural law of Lie algebra $\{,\}$ satisfying in addition, the Leibniz identity with respect to polarized Hamiltonian mappings. The pair $(\mathfrak{H}(M, \mathfrak{F}), \{,\})$ is called polarized Poisson structure subordinate to the polarized symplectic structure $(\theta, \mathfrak{F})$. In this paper, we show that, the Lagrangian foliation $\mathfrak{F}$ is affine and each polarized Hamiltonian mapping H is a locally affine function on this foliation. Also, we show that the set $\mathfrak{H}(M, \mathfrak{F})$ consists of all smooth real functions H on M so that X_H preserve the foliation $\mathfrak{F}$.

In this work, we give some properties and examples of polarized Hamiltonians; and we study the polarized Poisson structure subordinate to a polarized symplectic structure and we give a natural polarized symplectic structure on the space hom $\left(\mathcal{G}, \mathbb{R}^2\right)$, for an arbitrary real Lie algebra $\mathcal{G}$.

In the last part, we study the almost symplectic polarized structures and we prove that such a structure is integrable if and only if its the Bernard's tensor of the corresponding $G-$structure vanishes identically.

2 Polarized linear spaces

Let E be an $\mathbb{R}-$linear space of dimension $2n$, θ be an exterior $2-$form on E and let F be a linear subspace E of codimension n.

Definition 2.1. We say that (θ, F) is a polarized symplectic structure on the space E if: (i) θ is nondegenerate. (ii) $\forall x, y \in F; \theta(x, y) = 0_{\mathbb{R}}$.

The following theorem gives the classification of linear polarized symplectic structures.

Theorem 2.1. *If (θ, F) is a polarized symplectic structure on E, then there is a basis $(e_i, e'_i)_{1\leq i\leq n}$ of E^* such that*

$$\theta = \sum_{j=1}^{n} \omega^j \wedge {\omega'}^j, \;\; F = \ker {\omega'}^1 \cap \cdots \cap \ker \omega'^k,$$

where $(\omega^i, \omega'^i)_{1\leq i\leq n}$ is the dual basis of $(e_i, e'_i)_{1\leq i\leq n}$.

$(e_i, e'_i)_{1\leq i\leq n}$ is called a polarized symplectic basis.

Proof. Let $f_1, \cdots, f_n, g_1, \cdots, g_n$ be a basis of E such that F is spanned by the vectors $f_1, \cdots, f_n$, and let $\left\{\gamma^1, \cdots, \gamma^n, \omega'^1, \cdots, \omega'^n\right\}$ be its dual basis. The second condition of the definition of a polarized symplectic structure allows us to see that the bilinear form θ takes the following form:

$$\theta = \sum_{i,j=1}^{n} \left(b_i^j \gamma^i + c_i^j \omega'^i \right) \wedge \omega'^j,$$

where $b_i^j, c_i^j \in \mathbb{R}$. For all $j = 1, \cdots, n$ we take

$$\omega^j = \sum_{i=1}^{n} \left(b_i^j \gamma^i + c_i^j \omega'^i \right), \ \ \mu^j = \sum_{i=1}^{n} b_i^j \gamma^i.$$

The set $\{\gamma^j \mid j = 1, ..., n\}$ is a basis of the dual space F^* of F. In fact, if an element $x \in F$ satisfies $\gamma^j(x) = 0$, for all j, then the linear forms $i(x)\theta$ vanish identically, and it follows from the non-degeneracy of θ that $x = 0$. The linear forms μ^j are independent in F^*, and consequently they form a basis of F^*. The linear forms ω^j are independent in E^*, the system

$$\left(\omega^j, \omega'^j\right)_{1 \leq j \leq n}$$

is a basis of E^*, and we have:

$$\theta = \sum_{j=1}^{n} \omega^j \wedge {\omega'}^j.$$

Let $(e_i, e'_i)_{1 \leq i \leq n}$ be a basis of E having $\left(\omega^j, \omega'^j\right)_{1 \leq j \leq n}$ for a dual basis. The bases $f_1, \cdots, f_n, g_1, \cdots, g_n$ and $(e_i, e'_i)_{1 \leq i \leq n}$ are related by:

$$f_i = b_i^j e_j \ \ ; \ \ g_i = c_i^j e_j + e'_i.$$

This proves, in particular, that the vectors e_j belong to F, and that $F = \ker {\omega'}^1 \cap \cdot \cdot \cdot \cap \ker \omega'^n$. ∎

Let E be a linear space of dimension $2n$ equipped with a polarized symplectic structure (θ, F).

The automorphisms of E which preserve (θ, F) is a Lie group, denoted by $Sp(1, n; E)$, and called a polarized symplectic group of E.

Let $Sp(1, n; \mathbb{R})$ be the group of matrices of polarized symplectic automorphisms of E expressed in the polarized symplectic basis $(\mathrm{e}_i, e'_i)_{1 \leq i \leq n}$ of E. The group $Sp(1, n; \mathbb{R})$ consists of all matrices of the type

$$\begin{pmatrix} A & C \\ 0 & \left(A^{-1}\right)^T \end{pmatrix}$$

where A, C are matrices $n \times n$ with entries in $\mathbb{R}$, A is invertible and $A\,C^T = CA^T$.

Proof. Let $f \in Sp(1, n; E)$, and $i.i' = 1, \cdot \cdot \cdot, n$. Write

$$f\left(e_i\right) = a_i^l e_l \ ; \ f(e'_i) = \ b_i^l e_l + c_i^m e'_m,$$

where, $a_i^j, b_i^j,\ c_i^j \in \mathbb{R}$, because each element of $Sp(k, n; E)$ leaves invariant the linear subspace F.

The relationship $\theta\left(f\left(e_i\right), f\left(e_j'\right)\right) = \theta\left(e_i, e_j'\right) = \delta_{ij}$ implies that

$$\theta\left(a_i^l e_l, c_j^m e_m'\right) = a_i^l c_j^m \delta_{lm} = \delta_{ij},$$

so,

$$A^T C = I_n,$$

where $A = \left(a_j^i\right),\ B = \left(b_j^i\right)$ and $C = \left(c_j^i\right)$. Then the matrices A and C are invertible and

$$C = \left(A^{-1}\right)^T.$$

Also, we have

$$\theta\left(f\left(e_i'\right), f\left(e_j'\right)\right) = \theta\left(e_i', e_j'\right) = 0,$$

then

$$\theta\left(b_i^l e_l + c_i^m e_m', b_j^{l'} e_{l'} + c_j^{m'} e_{m'}'\right) = b_i^{m'} c_j^{m'} - c_i^m b_j^m,$$

so,

$$B^T C = C^T B.$$

Then, the matrix of f with respect to the polarized symplectic basis is of the the expected form. □

We denote by $\mathfrak{sp}(1, n; E)$ the Lie algebra of the polarized symplectic group $Sp(1, n; E)$. $\mathfrak{sp}(1, n; E)$ is identified with the tangent space of the Lie group $Sp(1, n; E)$ in the identity mapping Id_E of E; it consists of all endomorphisms u of E satisfying the relation

$$(\forall x, y \in E)\ \ (u(F) \subseteq F,\ \ \theta(u(x), y) + \theta(x, u(y)) = 0).$$

In terms of sets of matrices, we denote by $\mathfrak{sp}(1, n; \mathbb{R})$ the Lie algebra of the polarized symplectic group $Sp(1, n; \mathbb{R})$.

The Lie algebra $\mathfrak{sp}(1, n; \mathbb{R})$ consists of all matrices of the type

$$\begin{pmatrix} A & S \\ 0 & -A^T \end{pmatrix}$$

where A, S are $n \times n$ real matrices with S symmetric.

Proof. Let $u \in \mathfrak{sp}(1, n; E)$, and $i = 1, ..., n$. Write

$$u\left(e_i\right) = a_i^l e_l$$

and

$$u\left(e_i'\right) = b_i^l e_l + c_i^m e_m'.$$

The relationship $\theta\left(u(x), y\right) + \theta\left(x, u(y)\right) = 0$, for all $x, y \in E$, implies that: $C = -A^T$ and $B^T = B$. Then the matrix of u with respect to the polarized symplectic basis is in the desired form. □

We observe that $Sp(1, n; \mathbb{R})$ is of dimension $\frac{n(3n+1)}{2}$.

3 Polarized symplectic manifolds

All manifolds considered in this work are supposed to be smooth and connected.

Let M be a differentiable manifold of even dimension $2n$ equipped with a foliation $\mathfrak{F}$ of codimension n and let θ be a differential $2-$form on M. We denote by E, the sub-bundle of TM defined by the tangent vectors to the leaves of $\mathfrak{F}$.

Definition 3.1. We say that (θ, E) is a polarized symplectic structure on M, if: (i) θ is closed. ($d\theta = 0$); (ii) θ is nondegenerate. (iii) $\theta(X,Y) = 0$ for all $X, Y \in \Gamma(E)$.

3.1 Examples

1. Polarized symplectic structure on the cotangent bundle. Let T^*M be the cotangent bundle of an $n-$dimensional manifold M and let $\pi_M : T^*M \longrightarrow M$ the natural projection of this fibration. It is well known that the total space of this fibration, provided with the form $\theta = d\lambda$ is a symplectic manifold, λ being the Liouville form on the cotangent bundle. Of course λ is defined by

$$\langle X_u, \lambda_u \rangle = \langle (\pi_M)_* (X_u), \omega_x \rangle$$

for all $u = (x, \omega_x) \in T^*M$, $X \in \Gamma(T(T^*M))$. With respect to a local coordinate system $\left(\overline{U} = (q^1, \ldots, q^n, p^1, \ldots, p^n)\right)$ of T^*M over $\left(U, \varphi = (q^1, \ldots, q^n)\right)$, we have

$$\lambda_{\overline{U}} = \sum_{i=1}^{n} p^i dq^i \ , \ \ \theta_{\overline{U}} = \sum_{i=1}^{n} dp^i \wedge dq^i.$$

The pair $(\theta; \mathfrak{F})$, defined by the differential $2-$form θ and the vertical foliation given by the fibration π_M, is a polarized symplectic structure on the cotangent bundle T^*M.

2. The spheres S^{2n} don't admit polarized symplectic structures for all $n \geq 1$.

3. Symplectic polarized structure on hom $\left(\mathcal{G}, \mathbb{R}^2\right)$. Let $\mathcal{G}$ be a real Lie algebra of dimension n. Let $(e_i)_{1 \leq i \leq n}$ be a basis of $\mathcal{G}$ and $\left(\omega^i\right)_{1 \leq i \leq n}$ be its dual basis and let hom $\left(\mathcal{G}, \mathbb{R}^2\right) = \mathcal{G}^* \otimes \mathbb{R}^2$ be the linear space of linear mappings from $\mathcal{G}$ with values in $\mathbb{R}^2$. The space hom $\left(\mathcal{G}, \mathbb{R}^2\right)$ is generated by linear mappings $\omega^i \otimes \overline{e}$, $\omega^i \otimes \overline{f}$ $(1 \leq i \leq n)$, where $\left(\overline{e}, \overline{f}\right)$ is the canonical basis of $\mathbb{R}^2$. Any element u of hom $\left(\mathcal{G}, \mathbb{R}^2\right)$ is written in a unique form $u = \sum_{i=1}^{n} \left(x_i \omega^i \otimes \overline{e} + y_i \omega^i \otimes \overline{f}\right)$ and can be represented by the matrix

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

where x_i, y_i are real numbers. We equip the space hom $\left(\mathcal{G}, \mathbb{R}^2\right)$ with the coordinate system $(x_i, y_i)_{1 \leq i \leq n}$. It is clear that hom $\left(\mathcal{G}, \mathbb{R}^2\right)$ is a differentiable manifold of dimension $2n$. We endow naturally this space with the polarized symplectic structure $(\theta, \mathfrak{F})$, where

$$\theta = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

and the foliation $\mathfrak{F}$ is given by the equations $dy_1 = 0, \cdots, dy_n = 0$.

Note that this structure does not depend on the Lie algebra law of $\mathcal{G}$. The law of $\mathcal{G}$ will appear in the study of polarized Poisson manifolds on hom $\left(\mathcal{G},\mathbb{R}^2\right)$.

3.2 Local model of a polarized symplectic structure

We have the Darboux's theorem concerning polarized symplectic manifolds ([1],[3]):

Theorem 3.1. *Let $(M, \theta, \mathfrak{F})$ be a polarized manifold of dimension $2n$. Then, for every point p of M there is an open U of M containing p equipped with local coordinates $(x^i, y^i)_{1\leq i\leq n}$ such that the differential forms θ is represented on U by*

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i$$

and the foliation $\mathfrak{F}$ is defined by the equations

$$dy^1 = 0, ..., dy^n = 0.$$

Proof. It follows from the Frobenius theorem that there exists a system of local coordinates $(x, y) = \left(x_1, \cdots, x_n, y^1, \cdots, y^n\right)$ defined on an open neighbourhood U of M containing p such that the derivatives

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

generate the tangent space of the leaves at every point of U. The problem is of a local nature, therefore we can assume that U is an open neighborhood of $\mathbb{R}^{2n}$ and $p = 0$. The two form θ is locally exact (Poincare's lemma), then we can assume that the differential forms θ can be written on the open set U in the form

$$\theta = d\left(\sum_{u=1}^{n} f^u dx_u + \sum_{s=1}^{n} g_s dy^s\right)$$

where f^u and g_s are smooth functions on U; thus

$$\begin{array}{rcl}\theta & = & \sum_{u,v} \frac{\partial f^u}{\partial x_v} dx_v \wedge dx_u + \sum_{u,t=1}^{n} \frac{\partial f^u}{\partial y^t} dy^t \wedge dx_u \\ & + & \sum_{v,s=1}^{n} \frac{\partial g_s}{\partial x_v} dx_v \wedge dy^s + \sum_{t,s=1}^{n} \frac{\partial g_s}{\partial y^t} dy^t \wedge dy^s,\end{array}$$

so,

$$\begin{array}{rcl}\theta & = & \sum_{u<v} \left(\frac{\partial f^v}{\partial x_u} - \frac{\partial f^u}{\partial x_v}\right) dx_u \wedge dx_v + \\ & + & \sum_{u,s=1}^{n} \left(\frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s}\right) dx_u \wedge dy^s + \sum_{t<s} \left(\frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s}\right) dy^t \wedge dy^s,\end{array}$$

$\mathfrak{F}$ is Lagrangian, then

$$\frac{\partial f^u}{\partial x_v} = \frac{\partial f^v}{\partial x_u},$$

for all $u, v = 1; \cdots, n$. For each $i = 1, \cdots, n$, we put:

$$x^i = g_i - \sum_{u=1}^{n} \int_0^{x_u} \frac{\partial f^u}{\partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi.$$

So,

$$\begin{array}{rcl} x^i & = & g_i - \boldsymbol{\int_0^{x_1} \frac{\partial f^1}{\partial y^i} (\xi, x_2, \cdots, x_n, y)\, d\xi} \\ & - & \int_0^{x_2} \frac{\partial f^2}{\partial y^i} (0, \xi, x_3, \cdots, x_n, y)\, d\xi \\ & - & \int_0^{x_3} \frac{\partial f^3}{\partial y^i} (0, 0, \xi, x_4, \cdots, x_n, y)\, d\xi \\ & - & \cdots \\ & - & \int_0^{x_n} \frac{\partial f^n}{\partial y^i} (0, 0, \cdots, 0, \xi, y)\, d\xi. \end{array}$$

Then we have

$$\begin{array}{rcl} \frac{\partial x^i}{\partial x_v} & = & \frac{\partial g_i}{\partial x_v} - \sum_{u=1}^{v-1} \frac{\partial}{\partial x_v} \int_0^{x_u} \frac{\partial f^u}{\partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \\ & & - \frac{\partial}{\partial x_v} \int_0^{x_v} \frac{\partial f^v}{\partial y^i} (0, \cdots, 0, \xi, x_{v+1}, \cdots, x_n, y)\, d\xi. \end{array}$$

But

$$\begin{array}{rcl} & & \frac{\partial}{\partial x_v} \int_0^{x_u} \frac{\partial f^u}{\partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \\ & = & \int_0^{x_u} \frac{\partial^2 f^u}{\partial x_v \partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \\ & = & \int_0^{x_u} \frac{\partial^2 f^u}{\partial y^i \partial x_v} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \\ & = & \int_0^{x_u} \frac{\partial}{\partial x_u} \frac{\partial f^v}{\partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \\ & = & \left[\frac{\partial f^v}{\partial y^i} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y) \right]_0^{x_u} \\ & = & \left[\frac{\partial f^v}{\partial y^i} (0, \cdots, 0, x_u, \cdots, x_n, y) - \frac{\partial f^v}{\partial y^i} (0, \cdots, 0, x_{u+1}, \cdots, x_n, y) \right]. \end{array}$$

Then

$$\begin{array}{rcl} \frac{\partial x^i}{\partial x_v} & = & \frac{\partial g_i}{\partial x_v} - \sum_{u=1}^{v-1} \left[\frac{\partial f^v}{\partial y^i} (0, \cdots, 0, x_u, \cdots, x_n, y) - \frac{\partial f^v}{\partial y^i} (0, \cdots, 0, x_{u+1}, \cdots, x_n, y) \right] \\ & & - \frac{\partial f^v}{\partial y^i} (0, \cdots, 0, x_v, x_{v+1}, \cdots, x_n, y) \\ & = & \frac{\partial g_i}{\partial x_v} - \frac{\partial f^v}{\partial y^i} (x_1, \cdots, x_n, y) \\ & = & \frac{\partial g_i}{\partial x_v} (x, y) - \frac{\partial f^v}{\partial y^i} (x, y). \end{array}$$

On the other hand, we have

$$\begin{array}{rcl} \frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s} & = & \frac{\partial g_s}{\partial y^t} - \left(\sum_{u=1}^{n} \int_0^{x_u} \frac{\partial^2 f^u}{\partial y^t \partial y^s} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \right) \\ & & - \frac{\partial g_t}{\partial y^s} + \left(\sum_{u=1}^{n} \int_0^{x_u} \frac{\partial^2 f^u}{\partial y^s \partial y^t} (0, \cdots, 0, \xi, x_{u+1}, \cdots, x_n, y)\, d\xi \right) \\ & = & \frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s}. \end{array}$$

By the relationship

$$\theta = \sum_{u,s}^{n} \left(\frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s} \right) dx_u \wedge dy^s + \sum_{t<s} \left(\frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s} \right) dy^t \wedge dy^s$$

and

$$\frac{\partial x^i}{\partial x_v} = \frac{\partial g_i}{\partial x_v} (x, y) - \frac{\partial f^v}{\partial y^i} (x, y);\, \frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s} = \frac{\partial g_s}{\partial y^t} - \frac{\partial g_t}{\partial y^s},$$

we deduce that

$$\begin{array}{ll}\theta & = \sum_{v,i}^{n} \left(\frac{\partial g_i}{\partial x_v} - \frac{\partial f^v}{\partial y^i}\right) dx_v \wedge dy^i + \sum_{t<s}^{n} \left(\frac{\partial x^s}{\partial y^t} - \frac{\partial x^t}{\partial y^s}\right) dy^t \wedge dy^s \\ & = \sum_{i=1}^{n} \sum_{v=1}^{n} \frac{\partial x^i}{\partial x_v} dx_v \wedge dy^i + \sum_{s,t}^{n} \frac{\partial x^s}{\partial y^t} dy^t \wedge dy^s \\ & = \sum_{s=1}^{n} \left(\frac{\partial x^s}{\partial x_v} dx_v + \frac{\partial x^s}{\partial y^t} dy^t\right) \wedge dy^s \\ & = \sum_{s=1}^{n} dx^s \wedge dy^s, \end{array}$$

this proves that

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

It remains to show that the local Pfaffian forms dx^i and dy^i are linearly independent at each point of U. For this, it suffices to show that the Pfaffian forms

$$\omega_s = \sum_{u=1}^{n} \left(\frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s}\right) dx_u$$

$(s = 1, \cdots, n)$ are linearly independent at each point of U. Let us show for this purpose that the matrix $B = (b_s^u)$ is invertible where $b_s^u = \frac{\partial g_s}{\partial x_u} - \frac{\partial f^u}{\partial y^s}$. Let $X = (X_1, \cdots, X_n) \in \mathbb{R}^n$ such that $BX^t = 0$. Then the local vector field

$$\overline{X} = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n}$$

belongs to the characteristic subspace $C_x(\theta)$ at each point of U, i.e. $i(\overline{X})\theta_x = 0$; the non degeneracy of θ proves that $\overline{X} = 0$, consequently, $X = (0, \cdots, 0)$; and we deduce that the matrix B is invertible. $\square$

Definition 3.2. The local coordinates systems $(x^i, y^i)_{1 \le i \le n}$ constitute an atlas of M, and the local coordinates systems $(x^i, y^i)_{1 \le i \le n}$ are called adapted coordinates systems.

Let $(x^i, y^i)_{1 \le i \le n}$ and $(\overline{x}^i, \overline{y}^i)_{1 \le i \le n}$ be two local adapted coordinate systems defined on an open neighbourhood W of M such that

$$\theta_W = \sum_{j=1}^{n} dx^j \wedge dy^j = \sum_{i=1}^{n} d\overline{x}^i \wedge d\overline{y}^i.$$

We have:

$$\overline{x}^i(x, y), \overline{y}^i(y)$$

for all i, because these charts are foliated with respect to the foliation $\mathfrak{F}$.

Therefore,

$$\begin{array}{rcl}\theta_W & = & \sum_{i=1}^{n} d\overline{x}^i \wedge d\overline{y}^i \\ & = & \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} dx^j + \frac{\partial \overline{x}^i}{\partial y^j} dy^j\right) \wedge \sum_{r=1}^{n} \frac{\partial \overline{y}^i}{\partial y^r} dy^r \\ & = & \sum_{i=1}^{n} \sum_{j,r=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j dy^r \\ & + & \sum_{i=1}^{n} \sum_{j,r=1}^{n} \left(\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r}\right) dy^j dy^r \\ & = & \sum_{r=1}^{n} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j\right) dy^r \\ & + & \sum_{i=1}^{n} \sum_{j<r} \left(\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r} - \frac{\partial \overline{x}^i}{\partial y^r} \frac{\partial \overline{y}^i}{\partial y^j}\right) dy^j dy^r \\ & = & \sum_{r=1}^{n} dx^r \wedge dy^r.\end{array}$$

Then

$$\frac{\partial \overline{x}^i}{\partial y^j} \frac{\partial \overline{y}^i}{\partial y^r} = \frac{\partial \overline{x}^i}{\partial y^r} \frac{\partial \overline{y}^i}{\partial y^j}; \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} dx^j = dx^r,$$

so

$$\begin{array}{rcl}\sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{y}^i}{\partial y^r} & = & 1 \\ \sum_{i=1}^{n} \frac{\partial \overline{x}^i}{\partial x^j} \frac{\partial \overline{y}^i}{\partial y^r} & = & 0 \quad \text{for } j \neq r.\end{array}$$

Then,

$$\begin{array}{rcl}\frac{\partial}{\partial x^r} \left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) & = & 1 \\ \frac{\partial}{\partial x^j} \left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) & = & 0 \ \text{ for } \ j \neq r,\end{array}$$

so,

$$\begin{array}{rcl}\left(\sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r}\right) & = & x^r + \varphi^r(y) \\ \sum_{i=1}^{n} \overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r} & = & \psi(x^r, y).\end{array}$$

But

$$\overline{x}^i \frac{\partial \overline{y}^i}{\partial y^r} \quad \frac{\partial y^r}{\partial \overline{y}^s} = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r(y),$$

then

$$\overline{x}^i \frac{\partial \overline{y}^i}{\partial \overline{y}^s} = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r(y)$$

so,

$$\overline{x}^i \delta \quad {}^{i}_{s} = \frac{\partial y^r}{\partial \overline{y}^s} \quad x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r(y)$$

consequently,

$$\begin{cases}\overline{x}^s = & \frac{\partial y^r}{\partial \overline{y}^s} x^r + \frac{\partial y^r}{\partial \overline{y}^s} \varphi^r(y) = \frac{\partial y^r}{\partial \overline{y}^s} x^r + \phi^r(y) \\ \overline{y}^s = & \overline{y}^s(y).\end{cases}$$

The expressions of these changes of coordinates in this atlas allow to deduce the following theorem:

Proposition 3.2. *The Lagrangian foliation $\mathfrak{F}$ is affine.*

This means that any leaf of the foliation $\mathfrak{F}$ is equipped with a structure of locally affine manifold. This theorem has been proved by several authors through the connection of R. Bott ([11] [5]). The non degeneracy of θ allows us to see that the mapping

$$\zeta : TM \longrightarrow T^*M, \ \ v \longmapsto i(v)\theta$$

is an isomorphism of vector bundles over M, and consequently, ζ defines an isomorphism from $\mathfrak{X}(M)$ onto $\mathscr{A}^1(M)$.

We denote by $\mu : \mathscr{A}^1(M) \longrightarrow \mathfrak{X}(M)$ the inverse isomorphism of ζ, and for each $\alpha \in \mathscr{A}^1(M)$, we denote by X_α, the vector field on M associated with α by this isomorphism: $\mu(\alpha) = X_\alpha$.

Let TM/E be the quotient bundle

$$TM/E = \bigcup_{x\in M} T_xM/E_x \quad , \quad \nu : TM \longrightarrow TM/E = \nu E$$

and let ν^*E be the dual bundle of νE:

$$\nu^*E = \bigcup_{x\in M} \nu^*E_x = \bigcup_{x\in M} (T_xM/E_x)^*.$$

The mapping ζ induces an isomorphism of vector bundles from E onto ν^*E.

In terms of local coordinates, $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, ν^*E is spanned by the Pfaffian forms $dy^1, \cdots, dy^n$ and ζ expresses the duality $\frac{\partial}{\partial x^i} \longmapsto dy^i$ between the geometry along the leaves and the transverse geometry of $\mathfrak{F}$.

Recall that, [12], a real function $f \in \mathscr{C}^\infty(M)$ is said to be basic, if for any vector field Y tangent to $\mathfrak{F}$, the function $Y(f)$ is identically zero. We denote by $\mathscr{A}_b^0(M, \mathfrak{F})$ the subring of $\mathscr{C}^\infty(M)$ which consists of all basic functions.

We recall also, that a vector field $X \in \mathfrak{X}(M)$ is said to be foliate, or that it is an infinitesimal automorphism of $\mathfrak{F}$, if in the neighborhood of any point of M, the local one parameter group associated to X leaves the foliation $\mathfrak{F}$ invariant. We denote by $\mathscr{I}(M, \mathfrak{F})$ the space of all foliate vector fields.

For each vector field X tangent to $\mathfrak{F}$, the Pfaffian form $\alpha = \zeta(X)$ belongs to the annihilator $Ann(E)$ of E.

4 Polarized Hamiltonian Vector Fields

Definition 4.1. A vector field $X \in \mathfrak{X}(M)$ is said to be locally polarized Hamiltonian if: (i) X is foliate; (ii) the Pfaffian form $\zeta(X)$ is closed.

We denote by $H^0(M, \mathfrak{F})$ the real linear space of locally polarized Hamiltonian vector fields

$$H^0(M, \mathfrak{F}) = \{X \in \mathscr{I}(M, \mathfrak{F}) \mid d(\zeta(X)) = 0\}.$$

An element $X \in \mathscr{I}(M, \mathfrak{F})$ is called a polarized Hamiltonian vector field if the Pfaffian form $\zeta(X)$ is exact. We denote by $H(M, \mathfrak{F})$ the real linear space which consists of all polarized Hamiltonian vector fields.

The image $\zeta(H(M, \mathfrak{F}))$ is a linear subspace of $\mathscr{A}_1(M)$. We take

$$\mathfrak{H}(M, \mathfrak{F}) = d^{-1}(\zeta(H(M, \mathfrak{F}))),$$

where d is the exterior differentiation operator.

Proposition 4.1. *For all $H \in \mathscr{C}^\infty(M)$, the following are equivalent:*

1. $H \in \mathfrak{H}(M, \mathfrak{F})$.

2. *there is a unique polarized vector field* $X_H \in H(M, \mathfrak{F})$ *such that:* $i\,(X_H)\,\theta = \zeta(X_H) = -dH$.

Let X be a locally polarized Hamiltonian vector field. In a neighborhood U of an arbitrary point of M, equipped with a local coordinate system $(x^i, y^i)_{1\leq i\leq n}$, there is a mapping $H \in C^\infty(U)$ such that $\zeta(X) = -dH$. And consequently, the equations of the motion of X are given by the following differential system, called the Hamilton's equations of X:

$$\begin{cases} \frac{dx^i}{dt} = -\frac{\partial H}{\partial y^i} \\ \frac{dy^i}{dt} = \frac{\partial H}{\partial x^i} \\ \frac{\partial H}{\partial x^i} \in \mathscr{A}_b^0(M). \end{cases}$$

Locally, the expressions of H and X are

$$H = \sum_{j=1}^{n} a_j(y^1, ..., y^n)x^j + b(y^1, ..., y^n)$$

and

$$X = -\sum_{s=1}^{n}\left(\sum_{j=1}^{n} x^j \frac{\partial a_j}{\partial y^s} + \frac{\partial b}{\partial y^s}\right)\frac{\partial}{\partial x^s} + \sum_{j=1}^{n} a_j \frac{\partial}{\partial y^j}$$

respectively, where $a_j, b \in \mathscr{A}_b^0\,(U, \mathfrak{F}_U)$.

A real function on M is said to be locally affine on the foliation $\mathfrak{F}$ if its restriction on each leaf of $\mathfrak{F}$ is locally affine function.

From the Hamilton equations we deduce the following proposition:

Proposition 4.2. *For each* $H \in C^\infty(M)$*, the following properties are equivalent:*

1. $H \in \mathfrak{H}(M; \mathfrak{F})$*;*

2. H *is locally affine function on the foliation* $\mathfrak{F}$.

Corollary 4.3. $\mathfrak{H}(M; \mathfrak{F})$ *is the set* $\mathfrak{a}(M; \mathfrak{F})$ *of all smooth real functions on* M *which are locally affine functions on the foliation* $\mathfrak{F}$*:*

$$\mathfrak{H}(M; \mathfrak{F}) = \mathfrak{a}(M; \mathfrak{F}).$$

Each element of $\mathfrak{H}(M, \mathfrak{F})$ is called a polarized Hamiltonian mapping and X_H is called the polarized Hamiltonian vector field associated with the polarized Hamiltonian H.

So, we have a mapping,

$$\rho : \mathfrak{H}(M, \mathfrak{F}) \longrightarrow H(M, \mathfrak{F});\; H \longmapsto X_H,$$

between polarized Hamiltonians and associated polarized Hamiltonian vector fields, and also the following commutative diagram:

$$\begin{array}{ccc} H(M, \mathfrak{F}) & \stackrel{\zeta}{\longrightarrow} & \mathscr{A}^1(M) \\ \nwarrow & & \nearrow \\ \rho & \mathfrak{H}(M, \mathfrak{F}) & -d \end{array}$$

5 Polarized Poisson structures subordinate to a polarized symplectic structure

The hypotheses and notations are those of the preceding paragraph.

Let $H, K \in \mathfrak{H}(M,\mathfrak{F})$ and X_H , X_K the associated polarized Hamiltonian vector fields. Then the Lie bracket $[X_H, X_K]$ is a polarized Hamiltonian vector field and it is associated with $\{K,H\} = \theta\,(X_H, X_K)$ i.e. $[X_H, X_K] = X_{\{K,H\}}$.

The mapping $(H,K) \longmapsto \{H,K\}$ from $\mathfrak{H}(M,\mathfrak{F}) \times \mathfrak{H}(M,\mathfrak{F})$ into $\mathfrak{H}(M,\mathfrak{F})$, defines a real Lie algebra structure on $\mathfrak{H}(M,\mathfrak{F})$, and satisfies in addition the Leibniz identity with respect to polarized Hamiltonian mappings. $\{H,K\}$ is called the polarized Poisson bracket of the polarized Hamiltonians H and K and the Lie algebra $(\mathfrak{H}(M,\mathfrak{F}), \{,\})$ is called polarized Poisson structure subordinate to the polarized symplectic structure (θ, E).

Proposition 5.1. *We have the following properties:*

1. $\mathscr{A}_b^0\,(M)$ *is an abelian Lie subalgebra of* $\mathfrak{H}(M,\mathfrak{F})$.
2. $H\,(M,\mathfrak{F})$ *is a real Lie algebra.*
3. $\left[H^0\,(M,\mathfrak{F})\,, H^0\,(M,\mathfrak{F})\right] \subset H\,(M,\mathfrak{F})$.
4. $H\,(M,\mathfrak{F})$ *is an ideal of* $H^0\,(M,\mathfrak{F})$.
5. *The sequence of Lie algebras:*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{H}(M,\mathfrak{F}) \overset{-\rho}{\longrightarrow} H\,(M,\mathfrak{F}) \hookrightarrow H^0(M,\mathfrak{F}) \longrightarrow H^0\,(M,\mathfrak{F})\,/H(M,\mathfrak{F}) \longrightarrow 0$$

is exact.

Let $(\mathfrak{H}(M,\mathfrak{F}), \{,\})$ be the polarized Poisson structure subordinate to the polarized symplectic structure (θ, E). Let P be the natural Poisson tensor associated with symplectic form θ:

$$P\,(\alpha, \beta) = -\theta\,(X_\alpha, X_\beta)$$

for all $\alpha, \beta \in \mathscr{A}^1(M)$.

Proposition 5.2. *We have the following properties:*

1. $P(dH, dK) = \{H, K\}\,, \forall H, K \in \mathfrak{H}(M,\mathfrak{F})$.
2. $P\,(dH, dK) = -X_H(K)$, $\forall H, K \in \mathfrak{H}(M,\mathfrak{F})$.
3. P *vanishes on the annihilator of* E *in the space* $\mathscr{A}^1(M)$.
4. P *is nondegenerate.*

With respect to a Darboux's local coordinates system $(x^i, y^i)_{1\leq i\leq n}$, the bracket $\{H,K\}$ is given by

$$\begin{aligned}\{H,K\}_U &= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial y^i}\frac{\partial K}{\partial x^i}-\frac{\partial H}{\partial x^i}\frac{\partial K}{\partial y^i}\right)\\ &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)\wedge\left(\frac{\partial}{\partial x^i}\right)(dH,dK)\,.\end{aligned}$$

So, we see that at every point of U we have

$$P=-\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial y^i}.$$

With respect to a local coordinate system $(x^i,y^i)_{1\leq i\leq n}$, we have

$$P=-\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial y^i}.$$

Let $H\in C^{\infty}(M)$ such that $X_H\in\mathscr{I}(M,\mathfrak{F})$ then with respect to the coordinates $(x^i,y^i)_{1\leq i\leq n}$ we have:

$$\begin{array}{lcc} X_H &=& X^i(x,y)\frac{\partial}{\partial x^i}+Y^i(y)\frac{\partial}{\partial y^i}\\ &=& P(dH,.)\\ &=& P\left(\frac{\partial H}{\partial x^i}dx^i+\frac{\partial H}{\partial y^i}dy^i;.\right)\\ &=& \frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i}-\frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i},\end{array}$$

then

$$\frac{\partial H}{\partial y^i}=X^i(x,y)\ \text{ and }\ \frac{\partial H}{\partial x^i}=-Y^i(y)$$

so,

$$H=-\sum Y^j(y)\,x^j,$$

and

$$X_H=-\sum x^j\frac{\partial Y^j(y)}{\partial y^i}_j+Y^i(y)\frac{\partial}{\partial y^i}.$$

We deduce the following result:

Proposition 5.3. *$\mathfrak{H}(M,\mathfrak{F})$ is the set of differentiable mappings $H\in C^{\infty}(M)$ such that the associated vector field X_H preserve the foliation:*

$$\mathfrak{H}(M,\mathfrak{F})=\{H\in C^{\infty}(M)\mid X_H\in\mathscr{I}(M,\mathfrak{F})\}\,.$$

6 Associated Poisson structures with the polarized symplectic structure on $\hom\left(\mathcal{G},\mathbb{R}^2\right)$

Let $(\mathcal{G},[,])$ be a real Lie algebra of dimension n endowed with a basis $(e_i)_{1\leq i\leq n}$. Let $\left(\omega^i\right)_{1\leq i\leq n}$ its dual basis.

We denote by C_{ij}^k the structural constants of $\mathcal{G}$: $[e_i, e_j] = C_{ij}^k e_k$.

We endow $\hom\left(\mathcal{G},\mathbb{R}^2\right)$ with the natural polarized symplectic structure $(\theta, \mathfrak{F})$ defined by the differential $2-$form $\theta = \sum_{i=1}^n dx^i \wedge dy^i$ and the foliation $\mathfrak{F}$ defined by the equations $dy^1 = 0, \cdots, dy^n = 0$.

Every element X of $\hom\left(\mathcal{G},\mathbb{R}^2\right)$ can be written in the following form:

$$X = \sum_{i=1}^n \left(x^i\omega^i \otimes \overline{e} + y^i\omega^i \otimes \overline{f}\right) = \begin{pmatrix} x^1 & \dots & x^n \\ y^1 & \dots & y^n \end{pmatrix}.$$

The linear mapping $X : \mathcal{G}\longrightarrow\mathbb{R}^2$ transforms $u = \sum_{i=1}^n \left(u^j e_j\right)$ into

$$X(u) = \sum_{i=1}^n \left(x^i u_i\right)\overline{e} + \sum_{i=1}^n \left(y^i u_i\right)\overline{f}.$$

In terms of matrices we have

$$X(u) = \begin{pmatrix} x^1 & \dots & x^n \\ y^1 & \dots & y^n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

The polarized Hamiltonians of the polarized symplectic structure are the differentiable functions $H \in \mathscr{C}^\infty\left(\hom\left(\mathcal{G},\mathbb{R}^2\right)\right)$ defined at X by expressions of the type

$$H(X) = \sum_{j=1}^n a_j(y^1, ..., y^n)x^j + b(y^1, ..., y^n),$$

where $a_1, \dots, a_n, b$ are basic functions.

The Polarized Poisson bracket of Polarized Hamiltonians

$$H = \sum_{j=1}^n a_j(y^1, ..., y^n)x^j + b(y^1, ..., y^n) \ ; \ K = \sum_{j=1}^n a'_j(y^1, ..., y^n)x^j + b'(y^1, ..., y^n),$$

is given by

$$\begin{aligned} \{H, K\} &= \sum_{i=1}^n \left(\frac{\partial H}{\partial y^i}\frac{\partial H}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial K}{\partial y^i}\right) \\ &= \left(x^j\frac{\partial a_j}{\partial y^i} + \frac{\partial b}{\partial y^i}\right)a'_i - a_i\left(x^j\frac{\partial a'_j}{\partial y^i} + \frac{\partial b'}{\partial y^i}\right) \\ &= \left(a'_i\frac{\partial a_j}{\partial y_i} - a_i\frac{\partial a'_j}{\partial y^i}\right)x^j + a'_i\frac{\partial b}{\partial y^i} - a_i\frac{\partial b'}{\partial y^i}. \end{aligned}$$

We use here the Einstein summation convention. The bracket, so defined, allows to provide $\mathfrak{H}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right), \mathfrak{F}\right)$ with a polarized Poisson structure subordinate to the real polarization $(\theta, \mathfrak{F})$.

7 The linear polarized Poisson structure of $\hom\left(\mathcal{G},\mathbb{R}^2\right)$

In addition to the Poisson structure subordinate to the real natural polarization on $\hom\left(\mathcal{G},\mathbb{R}^2\right)$, we can define another polarized Poisson structure $\left(\mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right), \mathfrak{F}\right); \{,\}^L\right)$, so-called the linear polarized Poisson structure of $(\mathcal{G}, [,])$.

Let $H \in \mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right)$, $X \in \hom\left(\mathcal{G},\mathbb{R}^2\right)$ and $j_1 : \mathcal{G}^* \longrightarrow \hom\left(\mathcal{G},\mathbb{R}^2\right)$ be the mapping defined by

$$j_1(\omega^i) = \omega^i \otimes \overline{e}.$$

The compososed mappings

$$\mathcal{G}^* \overset{j_1}{\longrightarrow} Hom\left(\mathcal{G},\mathbb{R}^2\right) \overset{dH_X}{\longrightarrow} \mathbb{R}$$

is completely defined by

$$(dH_X \circ j_1)\left(\omega^i\right) = dH_X\left(\omega^i \otimes \overline{e}\right) = \frac{\partial H}{\partial x^i}(X) = a_i,$$

consequently, $dH_X \circ j_1 = \sum_{i=1}^n a_i e_i$. We define

$$\{H,K\}^L(X) = pr_1\left\langle [dH_X \circ j_1, dK_X \circ j_1], X\right\rangle$$

pr_1 being the first projection $(x,y) \longmapsto x$, $\mathbb{R}^2 \longrightarrow \mathbb{R}$. Then,

$$\begin{array}{rcl}
\{H,K\}^L(X) & = & pr_1\left\langle [dH_X \circ j_1, dK_X \circ j_1], X\right\rangle \\
& = & \sum_{i,j=1}^n pr_1\left\langle \left[a^i e_i, a'^j e_j\right], X\right\rangle \\
& = & \sum_{i,j=1}^n pr_1\left\langle a_i a'_j C_{ij}^k e_k, X\right\rangle \\
& = & \sum_{i,j=1}^n a_i a'_j \sum_{m=1}^n C_{ij}^m x^m \\
& = & \sum_{1\le i<j\le n} \sum_{m=1}^n C_{ij}^m \left(a^i a'^j - a^j a'^i\right) x^m.
\end{array}$$

Proposition 7.1. *$\left(\mathfrak{H}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right);\{,\}^L\right)$ is a polarized Poisson structure on the foliated manifold $\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right)$, called the linear polarized Poisson structure of the Lie algebra $\mathcal{G}$.*

We give here the linear polarized Poisson structures corresponding to simple examples.

1. $\mathcal{G}$ is abelian Lie algebra. In this case $\{,\}^L = 0$; Consequently, $\left(\mathfrak{a}\left(\hom\left(\mathcal{G},\mathbb{R}^2\right),\mathfrak{F}\right);\{,\}^L\right)$ is the abelian polarized Poisson structure.

2. $\mathcal{G}$ is the Heisenberg's Lie algebra $\mathcal{H}_1$ of dimension 3. The Lie algebra law of $\mathcal{H}_1$ is given by $[e_1,e_2] = e_3$. And so for all $H,K \in \mathfrak{a}\left(\hom\left(\mathcal{H}_1,\mathbb{R}^2\right),\mathfrak{F}\right)$ $X \in \hom\left(\mathcal{H}_1,\mathbb{R}^2\right)$ where, $H(X) = a_i\left(y^1,y^2,y^3\right)x^i + b\left(y^1,y^2,y^3\right)$ and $K(X) = a'_i\left(y^1,y^2,y^3\right)x^i + b'\left(y^1,y^2,y^3\right)$, we have

$$\{H,K\}^L(X) = \left(a_1 a'_2 - a_2 a'_1\right)x^3.$$

8 Almost polarized symplectic structures

Let M be a differentiable manifold of dimension $2n$. We say that M is an almost polarized symplectic manifold if for every $x \in M$ the tangent space T_xM is equipped with a polarized symplectic structure of linear spaces

$$(\theta_x, F_x).$$

Of course, we assume that this structure is smooth, i.e., for every $x_0 \in M$ there exist an open neighborhood U_o of x_0 in M and a smooth cross-section $(\omega^i, \omega'^i)_{1\leq i\leq n}$: $U_0 \longrightarrow \mathscr{R}^*(U_0)$ of the bundle of coframes of M such that

$$\theta^P_{|U_0} = \sum_{i=1}^n \omega^i \wedge \omega'^i, \;\; F_{|U_0} = \ker \omega'^1 \cap ... \cap \ker \omega'^n.$$

A linear connection π on an almost polarized symplectic manifold is adapted to the almost polarized symplectic structure if, with respect to an adapted cross-section of the bundle of coframes of M, the connection form takes its values in the polarized symplectic Lie algebra $\mathfrak{s}_p(1, n; \mathbb{R})$; in other words, the components of the connection $(\pi^u_v) = \left(\alpha^i_j, \beta^i_j, \sigma^i_j, \gamma^i_j\right)$ with respect to an adapted cross-section satisfy

$$\beta^i_j = 0 \;\; ; \;\; \sigma^i_j = \sigma^j_i \;\; , \;\; \gamma^i_j = -\alpha^j_i$$

Let ω be the fundamental form of frames bundle $\mathscr{R}(M)$.

The components T^a of the torsion T of the linear connection are related to those of the connection form (π^u_v) and the fundamental form ω of the frames bundle by the relation

$$T^u = d\omega^u + \pi^u_v \wedge \omega^v.$$

An almost polarized symplectic structure on a $2n-$dimensional manifold M is equivalent to a given $G-$structure with $G = Sp(1, n; \mathbb{R})$. Such a $Sp(1, n; \mathbb{R})-$structure is integrable if this almost polarized symplectic structure corresponds to a polarized symplectic structure. Consequently we can return to the calculation of the Bernard's tensor in order to integrate this $G-$structure. But the vanishing of this tensor is equivalent to the existence of an adapted connection without torsion. Therefore we are going to study the problem of the existence of such a connection. Recall the following theorem:

Theorem 8.1. *([4],[15]) If a $G-$structure on a differentiable manifold M is integrable then the Bernard's tensor vanishes identically.*

The reverse is false in the general case. In the case of an almost polarized symplectic structure there is an equivalence between the integrability and the vanishing of this tensor (or the existence of an adapted connection without torsion).

The almost polarized symplectic structure is integrable, if and only if, about every point of M we can find a coordinate neighborhood U with a local coordinate system $(x^i, y^i)_{1\leq i\leq n}$, such that

$$\theta_{|U} = \sum_{i=1}^n dx^i \wedge dy^i \;\text{ and }\; F_{|U} = \ker dy^1 \cap ... \cap \ker dy^n.$$

Theorem 8.2. *Let M be a $2n-$dimensional manifold equipped with an almost polarized symplectic structure such that the distribution $x \longmapsto F(x)$ is integrable. Then the almost polarized symplectic structure is integrable if and only if the manifold M admits an adapted connection without torsion.*

Proof. Let π be an adapted connection without torsion. Then for any adapted cross-section $(\omega^i, \omega'^i)_{1\leq i\leq n}$, we have

$$d\omega^j = -(\alpha_s^j \wedge \omega^s + \sigma_s^j \wedge \omega'^s) \text{ and } d\omega'^j = -\gamma_s^j \wedge \omega'^s.$$

The differential of θ vanishes, in fact we have

$$d\theta = -\alpha_s^j \wedge \omega^s \wedge \omega'^j - \sigma_s^j \wedge \omega'^s \wedge \omega'^j + \omega^j \wedge \gamma_s^j \wedge \omega'^s = 0.$$

It results from the Darboux's theorem that every point of M has a coordinate neighborhood U with coordinate system $(x^i, y^i)_{1\leq i\leq n}$, such that

$$\theta_{|U} = \sum_{i=1}^{n} dx^i \wedge dy^i \text{ and } F_{|U} = \ker dy^1 \cap ... \cap \ker dy^n.$$

□

Remark 8.1. An integrable $Sp(1,n;\mathbb{R})-$structure is of infinite type because the Lie algebra $\mathfrak{s}_p(1,n;\mathbb{R})$ contains a matrix of rank 1.

References

[1] A.Awane. *Sur une généralisation des structures symplectiques*, Thèse Strasbourg, IRMA, 1984.

[2] A. Awane, *k-symplectic structures*, Journal of Mathematical Physics, 33 (1992), 4046-4052.

[3] A. Awane, M. Goze, *Pfaffian systems, k-symplectic systems*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2000.

[4] D. Bernard, *Sur la géométrie différentielle des G-structures*, Ann. de l'Institut Fourier, 10 (1960), 151-270.

[5] P. Dazord, *Sur la géométrie des sous fibrés et des feuilletages lagrangiens*, Ann. Scient. E.N.S. 4e Série, 13 (1981), 465-480.

[6] V. Guillemin, S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, 1984.

[7] P. Libermann, C.M. Marle, *Symplectic Geometry and Analytical Mechanics*, U.E.R. de Mathématiques, L.A. 212 et E.R.A. 944, 1020, 1021 du C.N.R.S.

[8] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées* J. Differential Geometry 13(1977), 253- 300.

[9] C.M. Marle, *Hommage à Jean Marie Souriau* SMF Gazette 133, juillet 2012.

[10] C.M. Marle, *Aux sources de la géométrie symplectique: les mémoires de Lagrange et de Poisson sur la méthode de variation des constantes*, Académie des Sciences. Sciences Mécaniques et Informatiques, 29 Janvier 2008.

[11] P. Molino, *Géométrie de Polarisation*, Travaux en cours Hermann (1984), 37-53.

[12] P. Molino, *Riemannian foliations*, Progress in Mathematics, 73, Birkhauser, Boston, 1988.

[13] J.M. Morvan, *Connexions et feuilletages en géométrie symplectique*, C.R.A.S. Série I, 296(1983), 765-768.

[14] J.M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

[15] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Inc., Englewood, New Jersey, 1964.

[16] A. Weinstein, *Lecture on symplectic manifolds*, Conferences board of Mathematical Science AMS no 29, 1977.

[17] N. Woodhouse, *Geometric quantization*, Clarendon Press, Oxford, 1980.

Author's address:

Azzouz Awane
LAMS. Department of Mathematics and Computer Science,
Ben M'sik's Faculty of Sciences. B.P.7955,
Bd Driss Harti, Casablanca,
Hassan II University of Casablanca, Morocco.
E-mail: awane.awane@gmail.com