Root system and double extension of semisimple Lie groups

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Abstract. We consider geometric properties of semi-simple Lie groups by root system and try to use the relation between roots in generated Euclidean space and sectional curvature using Killing form. Then we investigate the behavior of Cartan decomposition with Lie algebra automorphisms and show how such automorphims connect the roots and subsequently sectional curvature. Finally, by adding a Killing form to dual pairing we create an Abelian metric Lie algebra and show that how primitive features of metric can present relations between the roots.

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1 Introduction

Assume G is a Lie group and $\mathfrak{g} = T_eG$ denote its Lie algebra. A Lie group H of a Lie group G is a subgroup which is also a submanifold, also, Lie algebra of H which we present it by $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. We define $ad : \mathfrak{g} \to \mathfrak{g}$ with

$$adX(Y) = [X, Y],$$

for all $X, Y \in \mathfrak{g}$. Whenever

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

we may regard ∇ as a bilinear mapping from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ and $\nabla_X Y$ is an invariant vector field if X and Y are invariant vector fields. One of the most known Lie subalgebras is center which defined as follow

Definition 1.1. The subgroup $Z(G) = \{x \in G : xy = yx, \forall y \in G\}$, is called the center of G. It is a Lie subgroup with corresponding Lie subalgebra

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0, \forall Y \in \mathfrak{g}\}.$$

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Definition 1.2. [3] Let $\mathfrak{g}$ be any Lie algebra. If $X, Y \in \mathfrak{g}$, define $K(X,Y) = Tr(adXadY)$. Then K is a symmetric bilinear form on $\mathfrak{g}$, called the Killing form.

K is associative, in the sense that $K([X,Y],Z) = K(X,[Y,Z])$, for all $X,Y,Z \in \mathfrak{g}$. The Killing form is invariant under all automorphisms of $\mathfrak{g}$ and it's also attributed to G and is in particular Ad(G)-invariant. Let G be a connected Lie group, if G is compact then Killing form is negative definite; if Killing form is negative definite then G is compact and has finite fundamental group; and G is compact and semi-simple if and only if Killing form is negative definite [5].

A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. The Lie algebra of a normal Lie subgroup of G is necessarily an ideal. If I_1 and I_2 are two ideals in a Lie algebra $\mathfrak{g}$ with zero intersection, then I_1 and I_2 are orthogonal subspaces with respect to the Killing form. The orthogonal complement with respect to Killing form of an ideal is again ideal.

A Lie algebra $\mathfrak{g}$ is called simple if it has no nontrivial ideals (that is 0 and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$). It is called semi-simple if it is a direct sum of simple Lie algebras or contains no nonzero solvable ideals. Note that this in particular implies that the center $Z(\mathfrak{g}) = 0$. A Lie group is called simple (respectively semi-simple) if its Lie algebra is simple (respectively semi-simple). The Cartan criterion states that a Lie algebra is semi-simple if and only if the Killing form is non-degenerate and for this reason it is called regular quadratic, in the other word the Killing form is a scalar product (nondegenerate symmetric bilinear form).

2 Semisimple Lie algebra

In this section we study differential geometry of complex semi-simple Lie groups and we will show that how the sectional curvature can play a key role. From definition of Killing form one obtains routinely

$$K(X,[Y,Z]) = K(Y,[Z,X]) = K(Z,[X,Y]),$$

for all $X,Y,Z \in \mathfrak{g}$. Then $\nabla_X Y = \frac{1}{2}[X,Y]$. Since Killing form is nondegenarate on any semi-simple Lie group G, actually it's a bi-invariant scalar product. Let H be a Lie subgroup of G contains identity element such that $\mathfrak{h}$ and $\mathfrak{g}$ denotes their Lie algebras, respectively. Then

$$\mathfrak{h}^{\perp} = \{X \in \mathfrak{h} \mid K(X,Y) = 0; \forall Y \in \mathfrak{g}\},$$

where $\mathfrak{h}^{\perp}$ is the orthogonal complement of h with respect to K.

Proposition 2.1. *[2] If $\mathfrak{g}$ is any finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{t}$ is a nilpotent subalgebra, then there is finite subset $\Phi \in \mathfrak{t}^*$ such that*
(i) $\mathfrak{g} = \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ *where*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | (adH - \alpha(H)I)^n X = 0\},$$

for all $H \in \mathfrak{t}$ and some n.
(ii) $\mathfrak{t} \subseteq \mathfrak{g}_0$.
(iii) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Any $\mathfrak{g}_\alpha$ are called generalized weight spaces of $\mathfrak{g}$ relative to $ad\mathfrak{t}$ with generalized weights α. The elements of $\mathfrak{g}_\alpha$ are called generalized weight vectors. The decomposition statement (1) holds for any representation of a nilpotent Lie algebra over $\mathbb{C}$ on a finite-dimensional complex vector space. Statement (2) is clear since $ad\mathfrak{t}$ is nilpotent on $\mathfrak{t}$. As a consequence $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. A nilpotent Lie subalgebra $\mathfrak{t}$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is a Cartan subalgebra if $\mathfrak{t} = \mathfrak{g}_0$. It's trivial $\mathfrak{t}$ is a Cartan subalgebra if and only if $\mathfrak{t} = N_{\mathfrak{g}}(\mathfrak{t})$. If $\mathfrak{g}$ is semisimple a Cartan algebra $\mathfrak{t}$ is maximal abelian.
These are two important theorems concerning Cartan subalgebras of finite-dimensional complex Lie algebras [7]:

Theorem 2.2. *Any finite-dimensional complex Lie algebra g has a Cartan subalgebra.*

Theorem 2.3. *If $\mathfrak{t}$ and $\mathfrak{t}^{'}$ are Cartan subalgebras of a finite-dimensional complex Lie algebra $\mathfrak{g}$, then there exists an inner auto-morphism $a \in Int\mathfrak{g}$ such that $a(\mathfrak{t}) = \mathfrak{t}^{'}$.*

The $\mathfrak{g}_\alpha$ are 1-dimensional and are therefore given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [V, X] = \alpha(V)X\}, \tag{2.1}$$

for all $V \in \mathfrak{t}$, $\alpha \in \mathfrak{t}^*$. Notice that $\mathfrak{g}_0$ is simply $C_{\mathfrak{g}}(\mathfrak{t})$, the centralizer of $\mathfrak{t}$; it includes $\mathfrak{t}$. The set of all nonzero $\alpha \in \mathfrak{t}^*$ for which $\mathfrak{g}_\alpha \neq 0$ is denoted by Φ; the elements of Φ are called the roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ and are finite in number. With this notation we have a root space decomposition (or Cartan decomposition) [3]:

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{t}) \oplus \coprod_{\alpha\in\Phi} \mathfrak{g}_\alpha, \tag{2.2}$$

such that $\Phi = \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$. It's trivial $C_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. We know that a subalgebra $\mathfrak{t}$ of a semi-simple Lie algebra $\mathfrak{g}$ is a CSA (Cartan subalgebra) if and only if it is a maximal toral subalgebra. Therefore the next theorem is valid and the hypothesis is compatible.

Theorem 2.4. *[3] Let G be a semi-simple Lie group. If $\mathfrak{g}$ is its Lie algebra and $\mathfrak{t}$ is a toral maximal subalgebra of $\mathfrak{g}$, then*
(i) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for any $\alpha, \beta \in \mathfrak{t}^*$.
(ii) If $\alpha \in \Phi$ and $X \in \mathfrak{g}_\alpha$, then X is nilpotent(ad−nilpotent).
(iii) If $\alpha, \beta \in \mathfrak{t}^*$, and $\alpha + \beta \neq 0$, then $K(X, Y) = 0$ for any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$.

Proof. The proof of (i) is trivial. To prove (ii) it's obvious that $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] = 0$ if $\gamma+\delta = 0$, and we know that Φ is finite set; therefore there exists an integer number n such that $[Y, ...[Y, X]]...] = ad^n(Y)X = 0$, for any $Y \in \beta$, $\beta \in \Phi$. In the other words $n\beta + \alpha$ cannot be a root for any n. Suppose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ and $V \in \mathfrak{t}$ are arbitrary elements; then

$$\alpha(V)K(X, Y) = K([V, X], Y) = -K(X, [V, Y]) = -\beta(V)K(X, Y). \tag{2.3}$$

Thus $(\alpha(V) + \beta(V))K(X, Y) = 0$ and the proof is complete. □

The curvature tensor If $\zeta_1, \zeta_2, \zeta_3$ are vector fields on $\mathfrak{g}$ then recall that the curvature tensor is given by

$$R(\zeta_1, \zeta_2)\zeta_3 = \frac{1}{4}[[\zeta_1, \zeta_2], \zeta_3].$$

We may regard R as a multilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$. Let $\alpha, \beta, \gamma \in \mathfrak{t}^*$ such that $\alpha + \beta \neq 0$ and $X \in \mathfrak{g}_\alpha$, $P \in \mathfrak{g}_{-\alpha}, Y \in \mathfrak{g}_\beta$, $Z \in \mathfrak{g}_\gamma$ and $V, W, L \in \mathfrak{t}$ are arbitrary vector fields. One computes

$$\begin{aligned} R(V,X)P &= \frac{1}{4}\alpha(V)[X,P], \\ R(X,Y)V &= \frac{1}{4}(\alpha+\beta)(V)[Y,X], \\ R(V,W)X &= R(V,W)L = R(X,P)V = 0. \end{aligned} \tag{2.4}$$

It is trivial that $K(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$.
The Ricci tensor For all $X, Y \in \mathfrak{g}$, the Ricci tensor of $\mathfrak{g}$ is given by

$$Ric(X,Y) = -\frac{1}{4}K(X,Y)$$

We express the following important theorem which can be proved in same way using the geometrical approach. We note first that for any $\alpha \in \mathfrak{t}^*$, there is unique bi-invariant vector field $W_\alpha \in \mathfrak{t}$ such that

$$K(W_\alpha, V) = \alpha(V), \tag{2.5}$$

for any $V \in \mathfrak{t}$[3].

Theorem 2.5 ([3])**.** *(a)* Φ *spans* $\mathfrak{t}^*$ *.*
(b) If $\alpha \in \Phi$*, then* $-\alpha \in \Phi$*.*
(c) Let $\alpha \in \Phi, X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$*, then* $[X,Y] = K(X,Y)V_\alpha$*.*
(d) If $\alpha \in \Phi$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is one dimensional, with basis* $\{V_\alpha\}$*.*
(e) $K(V_\alpha, V_\alpha) = \alpha(V_\alpha) \neq 0$*, for* $\alpha \in \Phi$*.*
(f) If $\alpha \in \Phi$ *and* X *is any nonzero element of* $\mathfrak{g}_\alpha$*, then there exists* $Y \in \mathfrak{g}_{-\alpha}$ *such that* X, Y*,* $H_\alpha = [X,Y]$ *span a three dimensional simple subalgebra of* $\mathfrak{g}$ *isomorphic to sl(2,F).*
(g) $H_\alpha = \dfrac{2V_\alpha}{K(V_\alpha, V_\alpha)}$*;* $\quad H_\alpha = -H_{-\alpha}$*.*

Notation From Proposition 8.4 in [3] we notice that if $\alpha \in \Phi$, the only scalar multiples of α which are roots are α and $-\alpha$.
We define the positive definite symmetric bilinear form $(\,,) : \mathfrak{t}^* \times \mathfrak{t}^* \longrightarrow F$ by $(\alpha, \beta) = K(V_\alpha, V_\beta) = \alpha(V_\beta) = \beta(V_\alpha)$, where $\alpha, \beta \in \mathfrak{t}^*$, $(\alpha, \beta) = \|\alpha\|\|\beta\| \cos\theta$ [8].

Theorem 2.6 ([3])**.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $\mathfrak{t}$ *be a maximal toral subalgebra. If* Φ *is a root system and* E *an Euclidean space, then*
(a) Φ *spans* E*, and 0 does not belong to* Φ*.*
(b) If $\alpha \in \Phi$ *then* $-\alpha \in \Phi$*, but no other scalar multiple of* α *is a root.*
(c) If $\alpha, \beta \in \Phi$*, then* $\beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \Phi$*.*
(d) If $\alpha, \beta \in \Phi$*, then* $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z}$*.*

According to the assertions of the theorem, we consider Φ as a root system in the Euclidean space E. If $\alpha \in \Phi$ and $\beta \in \Phi \cup 0$, then $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \{0, \pm 1, \pm 2, \pm 3\}$. Therefore

$(\alpha, \alpha) \neq 0$. Also, the possible values for θ, which is the angle between α and β as follows

$$\theta \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}\}. \tag{2.6}$$

Since the statement from Theorem 2.6 will be used frequently, we present it by $\langle \beta, \alpha \rangle$ briefly, and we shall use this form throughout this paper. If Φ is the root system of a semi-simple Lie algebra $\mathfrak{g}$, we also refer to $l = dim\mathfrak{t}$ as the rank of $\mathfrak{g}$.

Theorem 2.7. *Let G be a semisimple Lie group with a Killing form; then $\mathfrak{K} \leq 0$, where $\mathfrak{K}$ is the sectional curvature.*

Proof. From the definition of sectional curvature, we have

$$\begin{aligned} \mathfrak{K}(X, Y) &= \frac{R(X, Y, Y, X)}{K(X, X)K(Y, Y) - K(X, Y)^2} = \frac{1}{4} \frac{K([X, Y], [X, Y])}{K(X, X)K(Y, Y) - K(X, Y)^2} \\ &= \frac{1}{4} \frac{K(X, Y)^2 K(V_\alpha, V_\alpha)}{-K(X, Y)^2} = -\frac{1}{4} K(V_\alpha, V_\alpha), \end{aligned} \tag{2.7}$$

for any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha} (\alpha \in \Phi)$. Since $(\alpha, \alpha) > 0$, the proof is complete. □

Lemma 2.8. *Let G be a semisimple Lie group with a Killing form. Then*

$$\beta(V_\alpha) = 4\sqrt{c_\alpha c_\beta} \cos \theta,$$

for any $\alpha, \beta \in \Phi$ and $V \in \mathfrak{t}$.

Proof. From Theorem 2.7 we have

$$K(V_\alpha, V_\alpha) = -4c_\alpha = (\alpha, \alpha) = \parallel \alpha \parallel^2,$$

and hence $\parallel \alpha \parallel = 2\sqrt{-c_\alpha}$; also similar calculations show that $\parallel \beta \parallel = 2\sqrt{-c_\beta}$, then

$$\beta(V_\alpha) = g(V_\alpha, V_\beta) = (\alpha, \beta) = \|\alpha\| \|\beta\| \cos \theta = 4\sqrt{c_\alpha c_\beta} \cos \theta.$$

The proof is complete. □

Using Theorem 2.6 and straightforward calculations, one obtains

$$\langle \beta, \alpha \rangle = 2\sqrt{\frac{c_\beta}{c_\alpha}} \cos \theta, \tag{2.8}$$

where θ is the angle between vectors of α and β in Euclidean space. The following states are valid

If $\theta = \frac{\pi}{3}, 2\frac{\pi}{3}$, then $c_\beta = c_\alpha$,
If $\theta = \frac{\pi}{4}, 3\frac{\pi}{4}$, then $c_\beta = \sqrt{2} c_\alpha$,
If $\theta = \frac{\pi}{6}, 5\frac{\pi}{6}$, then $c_\beta = \sqrt{3} c_\alpha$.

The relation between curvature and sectional curvature is given by

$$R(V_\beta, X)Y = 4(\sqrt{c_\beta c_\alpha}\cos\theta)Ric(X,Y)V_\alpha,$$

for any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$. Ricci curvature is obtained as follow

$$Ric(V_\alpha, V_\beta) = -\sqrt{c_\alpha c_\beta}\cos\theta.$$

Using 2, we have

$$R(V_\beta, X)Y = -4Ric(V_\alpha, V_\beta)Ric(X,Y)V_\alpha,$$

and for $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ we can obtain

$$R(X,Y)V_\beta = -(Ric(V_\alpha, V_\gamma) + Ric(V_\beta, V_\gamma))[X,Y].$$

Lemma 2.9 ([3]). *Let α and β be nonproportional roots. If $(\alpha,\beta) > 0$ (i.e., if the angle between α and β is strictly acute), then $\alpha - \beta$ is a root. If $(\alpha,\beta) < 0$, then $\alpha + \beta$ is a root.*

Suppose $\alpha, \beta \in \Phi$ and $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, $Z \in \mathfrak{g}_\beta$, $P \in \mathfrak{g}_{-\beta}$ are arbitrary elements. Using theorem 2.5 we have

$$K([X,Y],[Z,P]) = K(X,Y)K(Z,P)K(V_\alpha, V_\beta). \tag{2.9}$$

From definition of Killing form we have

$$\begin{aligned} K([X,Y],[Z,P]) &= K(Z,[P,[X,Y]]) = K(Z,[[P,X],Y]) + K(Z,[X,[P,Y]]), \\ &= K([P,X],[Y,Z]) + K([P,Y],[Z,X]). \end{aligned} \tag{2.10}$$

From the Lemma 2.9 one can realize that in any case one of the last two terms of (2.10) must be zero. As a result of Theorem 2.4, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin \Phi$. As indicated in Theorem 2.5 part b, if $\alpha - \beta$ is a root then $\alpha + \beta$ cannot be a root and vice versa. Using Lemma 2.9 we review both of the following cases.

(1)If $(\alpha,\beta) > 0$, and $\alpha - \beta \in \Phi$, therefore $\alpha + \beta \notin \Phi$ then $[Z,X] = [P,Y] = 0$, so from (2.9) and (2.10) we have

$$K([P,X],[Y,Z]) = K(X,Y)K(Z,P)K(V_\alpha,V_\beta) = 4K(X,Y)K(Z,P)\sqrt{c_\alpha c_\beta}\cos\theta.$$

In this case $\cos\theta > 0$ and the possible values for θ are $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.

(2)If $(\alpha,\beta) < 0$, then $\alpha + \beta \in \Phi$. Since $\alpha - \beta \notin \Phi$ then $[P,X] = [Y,Z] = 0$, so from (2.9) and (2.10) we have

$$K([P,Y],[Z,X]) = K(X,Y)gK(Z,P)gK(V_\alpha,V_\beta) = 4K(X,Y)K(Z,P)\sqrt{c_\alpha c_\beta}\cos\theta.$$

Because $\cos\theta < 0$ and the possible values for θ are $\frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$. Furtheremore, if $(\alpha,\beta) < 0$, then

$$K(V_{\alpha+\beta}, V_{\alpha+\beta}) = -4c_\alpha - 4c_\beta + 8\sqrt{c_\alpha c_\beta}\cos\theta, \tag{2.11}$$

and if $(\alpha, \beta) > 0$, then

$$K(V_{\alpha-\beta}, V_{\alpha-\beta}) = -4c_\alpha - 4c_\beta - 8\sqrt{c_\alpha c_\beta}\cos\theta. \tag{2.12}$$

Suppose $\alpha + \beta$ is a root; then for $\|\alpha + \beta\| = 2\sqrt{-c_{\alpha+\beta}}$, and also from (2.11), one obtains

$$c_{\alpha+\beta} = c_\alpha + c_\beta - 2\sqrt{c_\alpha c_\beta}\cos\theta. \tag{2.13}$$

Also, since $\alpha + \beta$ is a root if $\cos\theta < 0$ and from (2.13), we get

$$\cos\theta = \frac{c_{\alpha+\beta} - c_\alpha - c_\beta}{-2\sqrt{c_\alpha c_\beta}} < 0, \tag{2.14}$$

where θ is the angle between α and β. Therefore $c_{\alpha+\beta} < c_\alpha + c_\beta$. In the case of $\alpha - \beta \in \Phi$ it's straightforward that $\cos\theta > 0$, and $c_{\alpha-\beta} > c_\alpha + c_\beta$. Let $\alpha, \beta, \lambda \in \Phi$, so $(\alpha, \beta) < 0$, thus $\alpha + \beta \in \Phi$; assume $[V_\lambda, [X, Y]] = 0$, for all $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, and then $(\alpha + \beta)(V_\lambda) = \alpha(V_\lambda) + \beta(V_\lambda) = 0$ and we have $(\alpha + \beta, \lambda) = (\alpha, \lambda) + (\beta, \lambda) = 0$. In this case we conclude $(\alpha, \lambda) = (\beta, \lambda) = 0$ or $(\alpha, \lambda) = -(\beta, \lambda)$, and we obtain

$$\|\alpha\| = -\frac{\cos\theta_1}{\cos\theta}\|\beta\|,$$

where θ and θ_1 are the angles between α, λ and β, λ respectively. Therefore the relationship between the measures of any pair of roots can be expressed by another root which should be not perpendicular to the other two roots. We should stress that $\cos\theta_1$ and $\cos\theta$ have opposite signs. If θ_2 is the angle between α and β, then

$$\langle\beta, \alpha\rangle = -2\frac{\cos\theta\cos\theta_2}{\cos\theta_1},$$

and from (2.8) we get

$$\frac{c_\beta}{c_\alpha} = \frac{\cos^2\theta}{\cos^2\theta_1}.$$

Lemma 2.10. *Let α and β be nonproportional roots. There exist $\gamma \in \Phi$, where θ_1, θ_2 and θ_3 are the angles between γ, α; β, γ and $\alpha + \beta, \gamma$, respectively. If $(\alpha, \beta) < 0$, then $\cos\theta_1 \leq \cos\theta_3$ and $\cos\theta_2 \leq \cos\theta_3$.*

Proof. First note that $(\gamma, \alpha + \beta) = (\gamma, \alpha) + (\gamma, \beta)$ implies

$$\|\alpha\|\cos\theta_1 + \|\beta\|\cos\theta_2 = \|\alpha + \beta\|\cos\theta_3.$$

Assuming that $\cos\theta_3 < 0$, the possible amounts for θ_3 are $\frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$; for $\mathfrak{l} = \cos\theta_3$, we infer

$$-\|\alpha\|\cos\theta_1 - \|\beta\|\cos\theta_2 = -\mathfrak{l}\|\alpha + \beta\| \leq -\mathfrak{l}\left(\|\alpha\| + \|\beta\|\right),$$

and the last equation can be written as

$$\frac{-\cos\theta_1}{-\mathfrak{l}}\|\alpha\| + \frac{-\cos\theta_2}{-\mathfrak{l}}\|\beta\| \leq \|\alpha\| + \|\beta\|.$$

The claim follows. $\square$

Any nonzero vector α in the generated Euclidean space can potentially define a reflection σ_α, with reflecting hyperplane $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. This can explicitly be written as [3]:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

Suppose $\langle \beta, \alpha\rangle = \langle \sigma_\alpha(\beta), \sigma_\beta(\alpha)\rangle$, where σ is a reflection; then

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{(\sigma_\alpha(\beta), \sigma_\beta(\alpha))}{(\sigma_\beta(\alpha), \sigma_\beta(\alpha))} = \frac{(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \alpha - \frac{(\alpha,\beta)}{(\beta,\beta)}\beta)}{(\alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta, \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta)}.$$

Assuming $(\alpha,\beta) \neq 0$, by straightforward calculations we get

$$\cos\theta = \|\beta\|\|\alpha\|, \tag{2.15}$$

where θ is the angle between α and β. Assume $K(V_\alpha, V_\delta) = K(V_\beta, V_\delta) \neq 0$, where $\alpha, \delta, \beta \in \Phi$ and $(\alpha, \beta) \neq 0$. Then $(\alpha,\delta) = (\beta,\delta)$ leads to

$$\frac{\|\alpha\|}{\|\beta\|} = \frac{\cos\theta_2}{\cos\theta_1}, \tag{2.16}$$

where θ_1 and θ_2 are the angles between α, δ and β, δ respectively. From Table 1. in [3], it can be seen that $\frac{\cos\theta_2}{\cos\theta_1} = 1, \sqrt{2}, \sqrt{3}$. Let θ_3 be the angle between α and β; then

$$\begin{aligned}
&(i) If \frac{\|\alpha\|}{\|\beta\|} = \frac{\cos\theta_2}{\cos\theta_1} = 1 \Rightarrow \theta_2 = \theta_1, \theta_3 = \frac{\pi}{3}, \frac{2\pi}{3},\\
&(ii) If \frac{\|\alpha\|}{\|\beta\|} = \frac{\cos\theta_2}{\cos\theta_1} = \sqrt{2} \Rightarrow \theta_1 = \frac{\pi}{3}, \theta_2 = \frac{\pi}{4}; \theta_1 = \frac{2\pi}{3}, \theta_2 = \frac{3\pi}{4}; \theta_3 = \frac{\pi}{4}, \frac{3\pi}{4},\\
&(iii) If \frac{\|\alpha\|}{\|\beta\|} = \frac{\cos\theta_2}{\cos\theta_1} = \sqrt{3} \Rightarrow \theta_1 = \frac{\pi}{3}, \theta_2 = \frac{\pi}{6}; \theta_1 = \frac{2\pi}{3}, \theta_2 = \frac{5\pi}{6}; \theta_3 = \frac{\pi}{6}, \frac{5\pi}{6}.
\end{aligned} \tag{2.17}$$

Now let $\langle \alpha, \delta\rangle = \langle \sigma_\delta(\alpha), \sigma_\alpha(\delta)\rangle$ and $\langle \beta, \delta\rangle = \langle \sigma_\delta(\beta), \sigma_\beta(\delta)\rangle$, then $\cos\theta_1 = \|\alpha\|\|\delta\|$, $\cos\theta_2 = \|\beta\|\|\delta\|$ and hence $\frac{\|\alpha\|}{\|\beta\|} = \frac{\cos\theta_1}{\cos\theta_2}$, using assertion 1 in 2.17 it can be concluded $\cos\theta_2 = \cos\theta_1$.

3 Automorphism of Lie algebras

Definition 3.1. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is called a homomorphism of Lie algebras if it preserves the bracket in the following manner

$$\varphi([X,Y]) = [\varphi X, \varphi Y],$$

for any $X, Y \in \mathfrak{g}$.

Any homomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ is called automorphism. Let φ be a Lie algebra automorphism; for all $V, W \in \mathfrak{t}$, we have

$$\varphi([V, W]) = [\varphi(V), \varphi(W)] = 0,$$

and thus $\mathfrak{t}$ is invariant under φ. From theorem 2.5 we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin \Phi \cup \{0\}$ then

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = 0,$$

for any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ such that $\alpha+\beta \notin \Phi \cup \{0\}$; therefore if $\varphi(\mathfrak{g}_{\alpha+\beta})\mathfrak{g}_\delta =$, then $\delta \notin \Phi \cup \{0\}$. Furthermore, if $X \in \mathfrak{g}_\alpha$ $(\alpha \in \Phi)$ is an arbitrary element and $\varphi(X) = V$, $(V \in \mathfrak{g}_0)$, such that $\alpha(V) \neq 0$, then

$$\varphi([V, X]) = \varphi(\alpha(V)X) = \alpha(V)\varphi(X), \tag{3.1}$$

and

$$\varphi([V, X]) = [\varphi(V), \varphi(X)] = 0. \tag{3.2}$$

Comparing (3.1) and (3.2), it follows that $\varphi([V, X]) = 0$ and hence $[V, X] = \alpha(V) = 0$, which is impossible; then such automorphisms and $\mathfrak{g}_\alpha$, $(\alpha \in \Phi)$ cannot exist such that $\varphi(\mathfrak{g}_\alpha) \subseteq \mathfrak{g}_0$.

Theorem 3.1. *Let G be a semisimple Lie group with a Killing form. Then:*
(1)If $\varphi : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$, then $\varphi : \mathfrak{g}_{-\alpha} \to \mathfrak{g}_{-\beta}$.
(2) $\varphi : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$, then $\varphi(V_\alpha) = V_\beta$, for any $\alpha \in \Phi$ and $V \in \mathfrak{t}$

Proof. (1) Let $\varphi : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$ and $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ be arbitrary elements. From theorem 2.5, we have

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = K(X, Y)\varphi(V_\alpha) \in \mathfrak{t}, \tag{3.3}$$

Assume $\varphi(\mathfrak{g}_{-\alpha}) \subseteq \mathfrak{g}_\gamma$, $\gamma \in \Phi$, since $[\varphi(X), \varphi(Y)] \in \mathfrak{t}$; from Theorem 2.4 we get $\beta + \gamma = 0$ and the proof of (1) is complete.
(2) Killing form being invariant under all automorphisms, we infer

$$\varphi([X, Y]) = K(X, Y)\varphi(V_\alpha), \tag{3.4}$$

and

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = [\tilde{X}, \tilde{Y}] = K(\tilde{X}, \tilde{Y})V_\beta, \tag{3.5}$$

where $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, $\tilde{X} \in \mathfrak{g}_\beta$, $\tilde{Y} \in \mathfrak{g}_{-\beta}$. Comparing 3.4 and 3.5 complete the proof. □

4 Creating a metric Lie algebra by Cartan subalgebra

In this section we define a symmetric bilinear form $\{.,.\}$ on $\mathfrak{t}_{ad^*} \times \mathfrak{t}^*$ by adding Killing form to the dual pairing of $\mathfrak{t}$ and $\mathfrak{t}^*$, that is by

$$\{V + \alpha, W + \beta\} = \alpha(W) + \beta(V) + K(V, W), \tag{4.1}$$

for $V, W \in \mathfrak{t}$, $\alpha, \beta \in \mathfrak{t}^*$. The coadjoint representation ad $ad^* : \mathfrak{t} \to End(\mathfrak{t}^*)$ is given by

$$V.\alpha(W) = ad^*(V)\alpha(W) = -\alpha \circ ad(V)(W),$$

for all $\alpha \in \mathfrak{t}^*$ and $V, W \in \mathfrak{t}$. On the vector space $\mathfrak{h}_{ad^*} \times \mathfrak{h}^*$ we define a Lie bracket $[,]$ by [4]

$$[V + \alpha, W + \beta] = [V, W] + ad^*_{\mathfrak{h}}(V)\beta - ad^*_{\mathfrak{h}}(W)\alpha + \theta(V, W). \tag{4.2}$$

It's trivial that the metric Lie algebra $\mathfrak{t}_{ad^*} \times \mathfrak{t}^*$ is an Abelian Lie algebra [6]. It is not hard to prove that $\{.,.\}$ is invariant and nondegenrate, and its signature equals $(dim\mathfrak{t}, dim\mathfrak{t})$. Hence $(\mathfrak{t}_{ad^*} \times \mathfrak{t}^*, \{.,.\})$ is a metric Lie algebra [1].

Lemma 4.1. *Let $\mathfrak{t}_{ad^*} \times \mathfrak{t}^*$ be a metric extended Lie algebra with $\mathfrak{t}$ a maximal toral subalgebra of a semi-simple Lie algebra. For any $V_\alpha + \beta \in \mathfrak{t}_{ad^*} \times \mathfrak{t}^*$, the possible amounts for $\dfrac{\|\beta\|}{\|\alpha\|}$ are $1, \sqrt{2}, \sqrt{3}$ and undetermined, for $\alpha, \beta \in \Phi$.*

Proof. Since the extended Lie algebra is a metric one, using (4.1) one can obtain

$$\{V_\alpha + \beta, V_\alpha + \beta\} = 2\beta(V_\alpha) + K(V_\alpha, V_\alpha) > 0.$$

Therefore, $2(\beta, \alpha) > -(\alpha, \alpha)$ and

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} > -1, \tag{4.3}$$

from Theorem 2.6, one can find 4.3; taking positive integer numbers, we present it as $\langle \beta, \alpha \rangle$; then $\langle \beta, \alpha \rangle = 2\dfrac{\|\beta\|}{\|\alpha\|} \cos\theta$, where θ is the angle between β and α. From the definition of the positive definite symmetric bilinear form $(.,.)$ it's trivial θ restricted to $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$, and $\langle \beta, \alpha \rangle = 0, 1, 2, 3$. From Section 9.4 and Table 1. in [3] it ca be seen that

$$\frac{\|\beta\|}{\|\alpha\|} = 1, \sqrt{2}, \sqrt{3}, undetermined. \tag{4.4}$$

The proof is complete. □

Corollary 4.2. *Let $(\mathfrak{t}_{ad^*} \times \mathfrak{t}^*, \{.,.\})$ be a metric Lie algebra where $\mathfrak{t}$ is a maximal toral subalgebra of a semi-simple Lie algebra and $V_\alpha + \beta \in \mathfrak{t}_{ad^*} \times \mathfrak{t}^*$ is an arbitrary element; then $\alpha - \beta \in \Phi$ if $(\alpha, \beta) \neq 0$.*

Proof. Using Lemma 4.1 one can see that $\cos\theta \geq 0$; if $(\alpha, \beta) \neq 0$, then $\cos\theta > 0$, from Lemma 2.9; the proof is complete. □

Lemma 4.3. *Let $(\mathfrak{t}_{ad^*} \times \mathfrak{t}^*, \{.,.\})$ be a metric Lie algebra which $\mathfrak{t}$ is a maximal toral subalgebra of a semisimple Lie algebra and $V_\alpha + \beta, V_\beta + \alpha \in \mathfrak{t}_{ad^*} \times \mathfrak{t}^*$, then*

$$\{V_\alpha + \beta, V_\beta + \alpha\} \neq 0,$$

where $\alpha, \beta \in \Phi$ and $V \in \mathfrak{t}$.

Proof. Since both $V_\alpha+\beta, V_\beta+\alpha\in\mathfrak{t}_{ad^*}\times\mathfrak{t}^*$, both $\frac{\|\beta\|}{\|\alpha\|}$ and $\frac{\|\alpha\|}{\|\beta\|}$ satisfy 4.4 and the only possible angle between α and β is $\frac{\pi}{3}$ and $\|\beta\|=\|\alpha\|$. Assume $V_\alpha+\beta, V_\beta+\alpha$ are orthogonal; then

$$\{V_\beta+\alpha, V_\beta+\alpha\}=2\|\alpha\|\|\beta\|\cos\theta+\|\beta\|^2=\|\alpha\|\|\beta\|+\|\beta\|^2=0.$$

But this is impossible and the proof is complete. □

Theorem 4.4. *Let $(\mathfrak{t}_{ad^*}\times\mathfrak{t}^*,\{.,.\})$ be a metric Lie algebra and the arbitrary elements $V_\alpha+\beta, V_\gamma+\delta\in\mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ are orthogonal. θ_1,θ_2 and θ_3, are the angles between β, γ; δ, α and α,γ respectively. Assume that $\frac{\|\beta\|}{\|\alpha\|}$ and $\frac{\|\delta\|}{\|\gamma\|}$ are determinable and equal; then one of the θ_1, θ_2 must be equal to $\frac{\pi}{2}$, where $\alpha,\beta,\gamma,\delta\in\Phi$.*

Proof. From (4.1) we have

$$\begin{aligned}\{V_\alpha+\beta, V_\gamma+\delta\}&=\beta(V_\gamma)+\delta(V_\alpha)+K(V_\gamma,V_\alpha)=(\beta,\gamma)+(\delta,\alpha)+(\alpha,\gamma)\\&=\|\beta\|\|\gamma\|\cos\theta_1+\|\delta\|\|\alpha\|\cos\theta_2+\|\alpha\|\|\gamma\|\cos\theta_3.\end{aligned}\tag{4.5}$$

Assume $V_\alpha+\beta, V_\gamma+\delta$ are orthogonal; then

$$\|\beta\|\|\gamma\|\cos\theta_1+\|\delta\|\|\alpha\|\cos\theta_2+\|\alpha\|\|\gamma\|\cos\theta_3=0.$$

Thus

$$\frac{\|\beta\|}{\|\alpha\|}\cos\theta_1+\frac{\|\delta\|}{\|\gamma\|}\cos\theta_2+\cos\theta_3=0.\tag{4.6}$$

Assume $V=\frac{\|\beta\|}{\|\alpha\|}=\frac{\|\delta\|}{\|\gamma\|}$. Since V is determinable, then $V=1,\sqrt{2},\sqrt{3}$ and $V(\cos\theta_1+\cos\theta_2)+\cos\theta_3=0$ or

$$V=\frac{-\cos\theta_3}{\cos\theta_1+\cos\theta_2};$$

using 2.6, it can be concluded that possible amounts for cos of angles between roots are given by $\{0,\pm\frac{1}{2},\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{3}}{2}\}$. Now let $V=1$; in this case $-\cos\theta_3=\cos\theta_1+\cos\theta_2\neq 0$, and all the possible cases are as follows:

- If $\theta_1=\frac{\pi}{2}$, then $\theta_2=\pi-\theta_3$,
- If $\theta_2=\frac{\pi}{2}$, then $\theta_1=\pi-\theta_3$.

Let $V=\sqrt{2}$, then $-\frac{\sqrt{2}}{2}cos\theta_3=\cos\theta_1+\cos\theta_2$, and we have

- If $\theta_3=\frac{\pi}{4}$, then $(\theta_1=\frac{\pi}{2};\theta_2=\frac{2\pi}{3})$ or $(\theta_1=\frac{2\pi}{3};\theta_2=\frac{\pi}{2})$,

- If $\theta_3 = \dfrac{3\pi}{4}$, then $(\theta_1 = \dfrac{2\pi}{3}; \theta_2 = \dfrac{\pi}{2})$ or $(\theta_1 = \dfrac{\pi}{2}; \theta_2 = \dfrac{2\pi}{3})$.

Let $V = \sqrt{3}$, then $-\dfrac{\sqrt{3}}{3} cos\theta_3 = \cos\theta_1 + \cos\theta_2$ and we have

- If $\theta_3 = \dfrac{\pi}{6}$, then $(\theta_1 = \dfrac{2\pi}{3}; \theta_2 = \dfrac{\pi}{2})$ or $(\theta_1 = \dfrac{\pi}{2}; \theta_2 = \dfrac{2\pi}{3})$,
- If $\theta_3 = \dfrac{5\pi}{6}$, then $(\theta_1 = \dfrac{2\pi}{3}; \theta_2 = \dfrac{\pi}{2})$ or $(\theta_1 = \dfrac{\pi}{2}; \theta_2 = \dfrac{2\pi}{3})$.

The proof is trivial. □

Theorem 4.5. *Let $V_\alpha + \beta, V_\gamma + \delta \in \mathfrak{t}_{ad^*} \times \mathfrak{t}^*$ be arbitrary elements such that $\langle \beta, \alpha \rangle = \langle \sigma_\alpha(\beta), \sigma_\beta(\alpha) \rangle$, $\langle \delta, \gamma \rangle = \langle \sigma_\gamma(\delta), \sigma_\delta(\gamma) \rangle$; then*

$$\{V_\alpha + \beta, V_\gamma + \delta\} \leq 2$$

where σ is reflection and $\alpha, \beta, \gamma, \delta \in \Phi$ and $V \in \mathfrak{t}$.

Proof. Note first that by Lemma 4.1, $\dfrac{\|\beta\|}{\|\alpha\|}, \dfrac{\|\delta\|}{\|\gamma\|} = 1, \sqrt{2}, \sqrt{3}$ and assume θ is the angle between the α and β, from 2.15, one can see always $\theta \in \{\dfrac{\pi}{3}, \dfrac{\pi}{4}, \dfrac{\pi}{6}\}$ and the following cases are included

- If $\theta = \dfrac{\pi}{3} \Rightarrow \dfrac{\|\beta\|}{\|\alpha\|} = 1$, $\|\beta\|\|\alpha\| = \dfrac{1}{2}$, and finally $\|\beta\| = \|\alpha\| = \dfrac{\sqrt{2}}{2}$,
- If $\theta = \dfrac{\pi}{4} \Rightarrow \dfrac{\|\beta\|}{\|\alpha\|} = \sqrt{2}$, $\|\beta\|\|\alpha\| = \dfrac{\sqrt{2}}{2}$, and finally $\|\beta\| = 1$, $\|\alpha\| = \dfrac{\sqrt{2}}{2}$,
- If $\theta = \dfrac{\pi}{6} \Rightarrow \dfrac{\|\beta\|}{\|\alpha\|} = \sqrt{3}$, $\|\beta\|\|\alpha\| = \dfrac{\sqrt{3}}{2}$, and finally $\|\beta\| = \dfrac{\sqrt{3}}{\sqrt{2}}$, $\|\alpha\| = \dfrac{\sqrt{2}}{2}$.

Therefore $\|\alpha\|$ is constant and equal to $\dfrac{\sqrt{2}}{2}$, hence

$$\{V_\alpha + \beta, V_\alpha + \beta\} = \|\alpha\|^2 + 2\|\alpha\|\|\beta\| \cos\theta = \frac{1}{2} + \sqrt{2}\|\beta\| \cos\theta,$$

and one can easily obtain that:

- If $\theta = \dfrac{\pi}{3} \Rightarrow \|V_\alpha + \beta\|^2 = 1$,
- If $\theta = \dfrac{\pi}{4} \Rightarrow \|V_\alpha + \beta\|^2 = \dfrac{3}{2}$,
- If $\theta = \dfrac{\pi}{6} \Rightarrow \|V_\alpha + \beta\|^2 = 2$.

Thus $\|V_\alpha+\beta\| = 1, \frac{\sqrt{3}}{\sqrt{2}}, \sqrt{2}$, same calculations are valid for $V_\gamma+\delta$. Now it can be seen that the greater amount for both elements is $\sqrt{2}$ and the fundamental properties of metric

$$\{V_\alpha+\beta, V_\gamma+\delta\} \leq \|V_\alpha+\beta\|\|V_\gamma+\delta\|,$$

complete the proof. □

Corollary 4.6. *Let $V_\alpha+\beta \in \mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ be an element for which $\langle\beta,\alpha\rangle = \langle\sigma_\alpha(\beta),\sigma_\beta(\alpha)\rangle$; then $V_\alpha+\beta$ is an unit if and only if $\theta=\frac{\pi}{3}$, where θ is the angle between α and β and σ is reflection.*

Proof. Due to the proof of Theorem 4.5 one can see that $\|\alpha\| = \frac{\sqrt{2}}{2}$; then

$$1 = \{V_\alpha+\beta, V_\alpha+\beta\} = \|\alpha\|^2 + 2\|\alpha\|\|\beta\|\cos\theta = \frac{1}{2} + \sqrt{2}\|\beta\|\cos\theta.$$

A review of all possible cases shows that:

1. If $\theta = \frac{\pi}{3} \Rightarrow 1 = \frac{1}{2} + 1.\frac{1}{2}$,
2. If $\theta = \frac{\pi}{4} \Rightarrow 1 \neq \frac{1}{2} + \sqrt{2}.\frac{\sqrt{2}}{2}$,
3. If $\theta = \frac{\pi}{6} \Rightarrow 1 \neq \frac{1}{2} + \sqrt{2}.\frac{\sqrt{3}}{\sqrt{2}}.\frac{\sqrt{3}}{2}$.

Now the proof is trivial. □

Theorem 4.7. *Let $V_\alpha+\beta \in \mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ be an element for which $\langle\beta,\alpha\rangle = \langle\sigma_\alpha(\beta),\sigma_\beta(\alpha)\rangle$. Since $(\alpha,\beta)\neq 0$, then $\mathfrak{K}(X,Y) = -\frac{1}{8}$, for all $X\in\mathfrak{g}_{(\alpha-\beta)}$, $Y\in\mathfrak{g}_{-(\alpha-\beta)}$.*

Proof. Because $V_\alpha+\beta \in \mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ and it's invariant under reflections, using Lemma 4.1 and 2.15 leads to $(\alpha,\beta)>0$, and also from Lemma 2.9 one can see $\alpha-\beta\in\Phi$. The relation between sectional curvatures is given by 2.12 as follows

$$\mathfrak{K}(X,Y) = -\frac{1}{4}K(V_{\alpha-\beta},V_{\alpha-\beta}) = c_\alpha + c_\beta + 2\sqrt{c_\alpha c_\beta}\cos\theta. \tag{4.7}$$

By straightforward calculations for all $\theta = \frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{6}$ one can obtain $c_{\alpha-\beta} = -\frac{1}{8}$ and the proof is complete. □

Theorem 4.8. *Let $V_\alpha+\beta \in \mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ be an element which $\langle\beta,\alpha\rangle = \langle\sigma_\alpha(\beta),\sigma_\beta(\alpha)\rangle$ and there exists $\delta\in\Phi$ such that $K(V_\alpha,V_\delta) = K(V_\beta,V_\delta)$. Then $V_\beta+\alpha\in\mathfrak{t}_{ad^*}\times\mathfrak{t}^*$ can exist, and $V_\alpha+\beta$ is a unit vector.*

Proof. The assumptions illustrate that $\cos\theta_3 = \|\alpha\|\|\beta\|$ and $\cos\theta_3 > 0$, where θ_3 is the angle between α and β. Also, from Theorem 4.5 it can be seen that $\|\alpha\| = \frac{\sqrt{2}}{2}$, and since $(\alpha, \delta) = (\beta, \delta)$ one obtains 2.16. Now $\|\beta\| = \frac{\cos\theta_3}{\|\alpha\|}$ and 2.16 imply that

$$\|\beta\|^2 = \frac{\cos\theta_1 \cos\theta_3}{\cos\theta_2}. \tag{4.8}$$

By 2.17 one can realize that only (i) is possible, that is, $\theta_2 = \theta_1$ and $\theta_3 = \frac{\pi}{3}$, then $\|\alpha\| = \|\beta\| = \frac{\sqrt{2}}{2}$, then from Lemma 4.1 and the proof of Lemma 4.3 it can be seen that $V_\beta + \alpha$ can exist and

$$\{V_\alpha + \beta, V_\alpha + \beta\} = \|\alpha\|^2 + 2\|\alpha\|\|\beta\| \cos\theta_3 = 1.$$

The proof is complete. □

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