On the Orlicz-Brunn-Minkowski theory

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Abstract. Recently, Gardner, Hug and Weil developed an Orlicz-Brunn-Minkowski theory. Following this, in the paper we further consider the Orlicz-Brunn-Minkowski theory. The fundamental notions of mixed quermassintegrals, mixed $p$-quermassintegrals and inequalities are extended to an Orlicz setting. Inequalities of Orlicz Minkowski and Brunn-Minkowski type for Orlicz mixed quermassintegrals are obtained. One of these has connections with the conjectured log-Brunn-Minkowski inequality and we prove a new log-Minkowski-type inequality. A new version of Orlicz Minkowski’s inequality is proved. Finally, we show Simon’s characterization of relative spheres for the Orlicz mixed quermassintegrals.

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Key words: $L^p$ addition; Orlicz addition; Orlicz mixed volume; mixed quermassintegrals; mixed $p$-quermassintegrals; Orlicz mixed quermassintegrals; Orlicz-Minkowski inequality; Orlicz-Brunn-Minkowski inequality.

1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets $K$ and $L$, defined by

$$K + L = \{x + y \mid x \in K, y \in L\},$$

it is usually called Minkowski addition and combine volume play an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to $L^p$-Brunn-Minkowski theory: The first, a set called as $L^p$ addition, introduced by Firey in [6] and [7]. Denoted by $+_p$, for $1 \leq p \leq \infty$, defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p,$$

for all $x \in \mathbb{R}^n$ and compact convex sets $K$ and $L$ in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (1.1) is interpreted as $h(K +\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If $K$ is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y \mid y \in K\},$$
for \( x \in \mathbb{R}^n \), defines the support function \( h(K, x) \) of \( K \). A nonempty closed convex set is uniquely determined by its support function. \( L_p \) addition and inequalities are the fundamental and core content in the \( L_p \)-Brunn-Minkowski theory. For recent important results and more information from this theory, we refer to [12], [13], [14], [15], [20], [22], [23], [24], [25], [26], [27], [30], [31], [35], [36], [37] and the references therein. In recent years, a new extension of \( L_p \)-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [28] and [29]. In these papers the notions of \( L_p \)-centroid body and \( L_p \)-projection body were extended to an Orlicz setting. The Orlicz centroid inequality for star bodies was introduced in [39] which is an extension from convex to star bodies. The other articles advance the theory can be found in literatures [11], [17], [18] and [32]. Very recently, Gardner, Hug and Weil ([9]) constructed a general framework for the Orlicz-Brunn-Minkowski theory, and made clear for the first time the relation to Orlicz spaces and norms.

They introduced the Orlicz addition \( K + \varphi \) of compact convex sets \( K \) and \( L \) in \( \mathbb{R}^n \) containing the origin, implicitly, by

\[
\varphi \left( \frac{h(K, x)}{h(K + \varphi L, x)}, \frac{h(L, x)}{h(K + \varphi L, x)} \right) = 1,
\]

for \( x \in \mathbb{R}^n \), if \( h(K, x) + h(L, x) > 0 \), and by \( h(K + \varphi L, x) = 0 \), if \( h(K, x) = h(L, x) = 0 \).

Here \( \varphi \in \Phi_2 \), the set of convex functions \( \varphi : [0, \infty)^2 \to [0, \infty) \) that are increasing in each variable and satisfy \( \varphi(0, 0) = 0 \) and \( \varphi(1, 0) = \varphi(0, 1) = 1 \).

Unlike the \( L_p \) case, an Orlicz scalar multiplication cannot generally be considered separately. The particular instance of interest corresponds to using (1.2) with \( \varphi(x_1, x_2) = \varphi_1(x_1) + \varphi \varphi_2(x_2) \) for \( \varepsilon > 0 \) and some \( \varphi_1, \varphi_2 \in \Phi \), in which case we write \( K + \varphi_{\varepsilon} \) \( L \) instead of \( K + \varphi \) \( L \), where \( \varphi_{\varepsilon} : [0, \infty) \to (0, \infty) \) that are increasing and satisfy \( \varphi_i(1) = 1 \) and \( \varphi_i(0) = 0 \), where \( i = 1, 2 \). Orlicz addition reduces to \( L_p \) addition, \( 1 \leq p < \infty \), when \( \varphi(x_1, x_2) = x_1^p + x_2^p, \) or \( L_\infty \) addition, \( \varphi(x_1, x_2) = \max\{x_1, x_2\} \). Moreover, Gardner, Hug and Weil ([9]) introduced the Orlicz mixed volume, obtaining the equation

\[
\frac{(\varphi_1);(1)}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varphi_{\varepsilon} L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \varphi_2 \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS(K, u),
\]

where \( S(K, u) \) is the mixed surface area measure of \( K \) and \( \varphi \in \Phi_2, \varphi_1, \varphi_2 \in \Phi \).

Here \( K \) is a convex body containing the origin in its interior and \( L \) is a compact convex set containing the origin, assumptions we shall retain for the remainder of this introduction.

Denoting by \( V_\varphi(K, L) \), for any \( \varphi \in \Phi \), the integral on the right side of (1.3) with \( \varphi \varphi_2 \) replaced by \( \varphi \), we see that either side of the equation (1.3) is equal to \( V_\varphi(K, L) \) and therefore this new Orlicz mixed volume plays the same role as \( V_p(K, L) \) in the \( L_p \)-Brunn-Minkowski theory. In [9], Gardner, Hug and Weil obtained the Orlicz-Minkowski inequality.

\[
V_\varphi(K, L) \geq V(K) \cdot \varphi \left( \frac{V(L)}{V(K)} \right)^{1/n},
\]

for \( \varphi \in \Phi \). If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \).
In Section 3, we compute the Orlicz first variation of quermassintegrals, call as Orlicz mixed quermassintegrals, obtaining the equation
\begin{equation}
\frac{\varphi_1}{n-i} \lim_{\epsilon \to 0^+} W_i(K + \varphi, \epsilon L) - W_i(K) = \frac{1}{n} \int_{S^{n-1}} \varphi_2 \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u).
\end{equation}
for \(\varphi \in \Phi_2\), \(\varphi_1, \varphi_2 \in \Phi\) and \(1 \leq i \leq n\), and \(W_i\) denotes the usual quermassintegrals, and \(S_i(K, u)\) is the \(i\)-th mixed surface area measure of \(K\). Denoting by \(W_{\varphi, i}(K, L)\), for any \(\varphi \in \Phi\), the integral on the right side of (1.5) with \(\varphi_2\) replaced by \(\varphi\), we see that either side of the equation (1.5) is equal to \(W_{\varphi, i}(K, L)\) and therefore this new Orlicz mixed volume (Orlicz mixed quermassintegrals) plays the same role as \(W_{\varphi, i}(K, L)\) in the \(L_p\)-Brunn-Minkowski theory. Note that when \(i = 0\), (1.5) becomes (1.3). Hence we have the following definition of Orlicz mixed quermassintegrals.

\begin{equation}
W_{\varphi, i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u).
\end{equation}
In Section 4, we establish Orlicz-Minkowski inequality for the Orlicz mixed quermassintegrals.

\begin{equation}
W_{\varphi, i}(K, L) \geq W_i(K) \cdot \varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)},
\end{equation}
for \(\varphi \in \Phi\) and \(0 \leq i \leq n\). If \(\varphi\) is strictly convex, equality holds if and only if \(K \) and \(L\) are dilates or \(L = \{0\}\). Note that when \(i = 0\), (1.7) becomes to (1.4). In particularly, putting \(\varphi(t) = t^p\), \(1 \leq p < \infty\) in (1.7), (1.7) reduces to the following \(L_p\)-Minkowski inequality for mixed \(p\)-quermassintegrals established by Lutwak [21].

\begin{equation}
W_{p, i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p,
\end{equation}
for \(p > 1\) and \(0 \leq i \leq n\), with equality if and only if \(K \) and \(L\) are dilates or \(L = \{0\}\). Putting \(i = 0\), \(\varphi(t) = t^p\) and \(1 \leq p < \infty\) in (1.7), (1.7) reduces to the well-known \(L_p\)-Minkowski inequality established by Firey [7]. For \(p > 1\),

\begin{equation}
V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n},
\end{equation}
with equality if and only if \(K \) and \(L\) are dilates or \(L = \{0\}\).

In Section 5, we establish the following Orlicz-Brunn-Minkowski inequality for quermassintegrals of Orlicz addition.

\begin{equation}
1 \geq \varphi \left( \frac{W_i(K)}{W_i(K + \varphi L)} \right)^{1/(n-i)} \cdot \left( \frac{W_i(L)}{W_i(K + \varphi L)} \right)^{1/(n-i)},
\end{equation}
for \(\varphi \in \Phi_2\) and \(0 \leq i \leq n\). If \(\varphi\) is strictly convex, equality holds if and only if \(K \) and \(L\) are dilates or \(L = \{0\}\). Note that when \(\varphi(x_1, x_2) = x_1^p + x_2^p\), \(1 \leq p < \infty\) in (1.11), (1.11) reduces to the following \(L_p\)-Brunn-Minkowski inequality for quermassintegrals established by Lutwak [21]. If

\begin{equation}
W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)},
\end{equation}
with equality if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \), and where \( p \geq 1 \) and \( 0 \leq i < n \). Putting \( i = 0 \), \( \varphi(x_1, x_2) = x_1^p + x_2^p \) and \( 1 \leq p < \infty \) in (1.11), (1.11) reduces to the well-known \( L_p \)-Brunn-Minkowski inequality established by Firey [7].

\[
(1.12) \quad V(K + p L)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n},
\]

with equality if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \), and where \( p > 1 \). A special case of (1.10) was recently established by Gardner, Hug and Weil [9].

\[
(1.13) \quad 1 \geq \varphi \left( \left( \frac{V(K)}{V(K + \varphi \, \varepsilon \, L)} \right)^{1/n} \cdot \left( \frac{V(L)}{V(K + \varphi \, \varepsilon \, L)} \right)^{1/n} \right),
\]

for \( \varphi \in \Phi_2 \). If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \). When \( i = 0 \), (1.10) becomes to (1.12). Moreover, We prove also the Orlicz Minkowski inequality (1.4) and the Orlicz Brunn-Minkowski inequality (1.12) are equivalent, and (1.7) and (1.10) also are equivalent.

When we were about to submit our paper, we were informed that G. Xiong and D. Zou [38] had also obtained Orlicz Minowski and Brunn-Mingkowski inequalities for Orlicz mixed quermassintegrals. Please note that we use a completely different approach, although the two inequalities coincide with theirs.

In 2012, Böröczky, Lutwak, Yang, and Zhang [2] conjecture a log-Minkowski inequality for origin-symmetric convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \).

\[
(1.14) \quad \int_{S^{n-1}} \frac{h(L, u)}{h(K, u)} h(K, u) dS(K, u) \geq V(K) \log \left( \frac{V(L)}{V(K)} \right).
\]

In [2], (1.14) is proved by them only when \( n = 2 \). Very recently, Gardner, Hug and Weil [9] proved a new version of (1.14) for convex bodies, not origin-symmetric convex bodies.

\[
(1.15) \quad \int_{S^{n-1}} \log \left( 1 - \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS(K, u) \leq V(K) \log \left( 1 - \frac{V(L)^{1/n}}{V(K)^{1/n}} \right)^n,
\]

with equality if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \), and where \( L \subset \text{int} \, K \). They also shown that combining (1.14) and (1.15) may get the classical Brunn-Minkowski inequality. In Section 6, we give a new log-Minkowski-type inequality

\[
(1.16) \quad \int_{S^{n-1}} \log \left( 1 - \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u) \leq W_i(K) \log \left( 1 - \frac{W_i(L)^{1/(n-i)}}{W_i(K)^{1/(n-i)}} \right)^n,
\]

with equality if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \). When \( i = 0 \), (1.16) becomes (1.15). We also point out a conjecture which is an extension of the log Minkowski inequality as follows.

\[
(1.17) \quad \frac{1}{n} \int_{S^{n-1}} \log \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u) \geq \log \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]

When \( i = 0 \), (1.17) becomes the log-Minkowski inequality (1.14). Combining (1.16) and (1.17) together split the following classical Brunn-Minkowski inequality for quermassintegrals (see Section 6).

\[
W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},
\]
with equality if and only if $K$ and $L$ are dilates or $L = \{0\}$.

In 2010, the Orlicz projection body $\Pi_\varphi$ of $K$ ($K$ is a convex body containing the origin in its interior) defined by Lutwak, Yang and Zhang [28]

\[ h(\Pi_\varphi, u) = \inf \left\{ \lambda > 0 \mid \frac{1}{nV(K)} \int_{S^{n-1}} \varphi \left( \frac{|u \cdot v|}{\lambda h(K, v)} \right) h(K, v) dS(K, v) \leq 1 \right\}, \]

for $\varphi \in \Phi$ and $u \in S^{n-1}$. A different Orlicz version of Minkowski’s inequality (1.8) is presented in Section 7. This results from replacing the left side of (1.8) by the quantity

\[ \overline{W}_{\varphi, i}(K, L) = \inf \left\{ \lambda > 0 \mid \frac{1}{nW_i(K)} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{\lambda h(K, u)} \right) h(K, u) dS_i(K, u) \leq 1 \right\}, \]

for $\varphi \in \Phi$ and $0 \leq i < n$. We prove the following new Orlicz Minkowski type inequality.

\[ \overline{W}_{\varphi, i}(K, L) \geq \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}, \]

where $\varphi \in \Phi$ and $1 \leq i < n$. If $\varphi$ is strictly convex and $W_i(L) > 0$, equality holds if and only if $K$ and $L$ are dilates. A special version of (1.20) was recently established by Gardner, Hug and Weil [9].

\[ \overline{V}_{\varphi}(K, L) \geq \left( \frac{V(L)}{V(K)} \right)^{1/n}, \]

If $\varphi$ is strictly convex and $V(L) > 0$, then equality holds if and only if $K$ and $L$ are dilates and where

\[ \overline{V}_{\varphi}(K, L) = \inf \left\{ \lambda > 0 \mid \frac{1}{nV(K)} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{\lambda h(K, u)} \right) h(K, u) dS(K, u) \leq 1 \right\}, \]

for $\varphi \in \Phi$.

Finally, in Section 8, we show Simon’s characterization of relative spheres for the Orlicz mixed quermassintegrals.

### 2 Notations and preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ be the class of nonempty compact convex subsets of $\mathbb{R}^n$, let $\mathcal{K}_0^n$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}_{\text{in}}^n$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. A set $K \in \mathcal{K}^n$ is called a convex body if its interior is nonempty. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter $B$ for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. For a compact set $K$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of $K$ and call this the volume of $K$. If $K$ is a nonempty closed (not necessarily bounded) convex set, then

\[ h(K, x) = \sup \{ x \cdot y \mid y \in K \}, \]
for $x \in \mathbb{R}^n$, defines the support function of $K$, where $x \cdot y$ denotes the usual inner product $x$ and $y$ in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. Support function is homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x),$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$. Let $d$ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = \|h(K, u) - h(L, u)\|_{\infty},$$

where $\| \cdot \|_{\infty}$ denotes the sup-norm on the space of continuous functions $C(\mathbb{S}^{n-1})$.

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let $\Phi_n, n \in \mathbb{N}$, denote the set of convex functions $\varphi : [0, \infty)^n \to [0, \infty)$ that are strictly increasing in each variable and satisfy $\varphi(0) = 0$ and $\varphi(e_j) = 1 > 0$, $j = 1, \ldots, n$. When $n = 1$, we shall write $\Phi$ instead of $\Phi_1$. The left derivative and right derivative of a real-valued function $f$ are denoted by $(f)_l'$ and $(f)_r'$, respectively.

### 2.1 Mixed quermassintegrals

If $K_i \in \mathcal{K}^n$ ($i = 1, 2, \ldots, r$) and $\lambda_i$ ($i = 1, 2, \ldots, r$) are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by (see e.g. [3])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \cdots i_n},$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \cdots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.1), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V(K[n-i], L[i])$. If $K_1 = \cdots = K_{n-1} = K$, $K_{n-i+1} = \cdots = K_n = B$ The mixed volumes $V_i(K[n-i], B[i])$ is written as $W_i(K)$ and call as quermassintegrals (or $i$-th mixed quermassintegrals) of $K$. We write $W_i(K, L)$ for the mixed volume $V(K[n-i-1], B[i], L[1])$ and call as mixed quermassintegrals. Aleksandrov [1] and Fenchel and Jessen [5] (also see Busemann [4] and Schneider [33]) have shown that for $K \in \mathcal{K}_{\text{comp}}^n$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on $\mathbb{S}^{n-1}$, such that the mixed quermassintegrals $W_i(K, L)$ has the following representation:

$$W_i(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u)dS_i(K, u).$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_n, \cdot)$ on $\mathbb{S}^{n-1}$, called the mixed surface area measure of $K_1, \ldots, K_n$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)dS(K_1, \ldots, K_{n-1}, u).$$

In fact, the measure $S(K_1, \ldots, K_{n-1}, \cdot)$ can be defined by the proper that (2.3) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$. 

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When \( L = B, S(K[n-i], L[i], \cdot) \) is written as \( S_i(K, \cdot) \) and called as \( i \)-th mixed surface area measure. A fundamental inequality for mixed quermassintegrals stats that: For \( K, L \in \mathcal{K}_n \) and \( 0 \leq i < n-1 \),

\[
W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1}W_i(L),
\]

with equality if and only if \( K \) and \( L \) are homothetic and \( L = \{o\} \). Good general references for this material are [4] and [19].

### 2.2 Mixed \( p \)-quermassintegrals

Mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The mixed quermassintegrals \( W_{p,0}(K, L), W_{p,1}(K, L), \ldots, W_{p,n-1}(K, L) \), as the first variation of the ordinary quermassintegrals, with respect to Firey addition: For \( K, L \in \mathcal{K}_{oo}^n \), and real \( p \geq 1 \), defined by (see e.g. [21])

\[
W_{p,i}(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon \cdot L) - W_i(K)}{\varepsilon}.
\]

The mixed \( p \)-quermassintegrals \( W_{p,i}(K, L) \), for all \( K, L \in \mathcal{K}_{oo}^n \), has the following integral representation:

\[
W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u),
\]

where \( S_{p,i}(K, \cdot) \) denotes the Boel measure on \( S^{n-1} \). The measure \( S_{p,i}(K, \cdot) \) is absolutely continuous with respect to \( S_i(K, \cdot) \), and has Radon-Nikodym derivative

\[
\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p},
\]

where \( S_i(K, \cdot) \) is a regular Boel measure on \( S^{n-1} \). The measure \( S^{n-1}(K, \cdot) \) is independent of the body \( K \), and is just ordinary Lebesgue measure, \( S \), on \( S^{n-1} \). \( S_i(B, \cdot) \) denotes the \( i \)-th surface area measure of the unit ball in \( \mathbb{R}^n \). In fact, \( S_i(B, \cdot) = S \) for all \( i \). The surface area measure \( S_0(K, \cdot) \) just is \( S(K, \cdot) \). When \( i = 0 \), \( S_{p,0}(K, \cdot) \) is written as \( S_p(K, \cdot) \) (see [25, [26]). A fundamental inequality for mixed \( p \)-quermassintegrals stats that: For \( K, L \in \mathcal{K}_{oo}^n, p > 1 \) and \( 0 \leq i < n-1 \),

\[
W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-1}W_i(L)^p,
\]

with equality if and only if \( K \) and \( L \) are homothetic. \( L_p \)-Brun-Minkowski inequality for quermassintegrals established by Lutwak [21]. If \( K \in \mathcal{K}_{oo}^n, L \in \mathcal{K}_o^n \) and \( p \geq 1 \) and \( 0 \leq i \leq n \), then

\[
W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)},
\]

with equality if and only if \( K \) and \( L \) are dilates or \( L = \{o\} \). Obviously, putting \( i = 0 \) in (2.6), the mixed \( p \)-quermassintegrals \( W_{p,1}(K, L) \) become the well-known \( L_p \)-mixed volume \( V_p(K, L) \), defined by (see e.g. [25])

\[
V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u).
\]
2.3 The Orlicz mixed volume

For $\varphi \in \Phi$, $K \in K^n_{\infty}$, and $L \in K^n_{\infty}$, Gardner, Hug and Weil [9] defined the Orlicz mixed volumes, $V_\varphi(K, L)$ by

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS(K, u).$$

They obtained the Orlicz-Minkowski inequality.

$$V_\varphi(K, L) \geq V(K) \cdot \varphi \left( \left( \frac{V(L)}{V(K)} \right)^{1/n} \right),$$

for all $K \in K^n_{\infty}$, $L \in K^n_{\infty}$ and $\varphi \geq 2$. If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates or $L = \{a\}$.

Orlicz mixed quermassintegrals is defined in Section 3, by

$$W_\varphi;i(K, L) =: \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u),$$

for all $K \in K^n_{\infty}$, $L \in K^n_{\infty}$, $\varphi \geq 2$ and $0 < i < n$. Obviously, when $\varphi(t) = t^p$ and $p \geq 1$, Orlicz mixed quermassintegrals reduces to the mixed $p$-quermassintegrals $W_{p,i}(K, L)$ defined in (2.6). When $i = 0$, (2.13) reduces to (2.11).

2.4 Orlicz addition

Let $m \geq 2$, $\varphi \in \Phi_m$, $K_j \in K^n_{\infty}$ and $j = 1, \ldots, m$, we define the Orlicz addition of $K_1, \ldots, K_m$, denoted by $+_{\varphi}(K_1, \ldots, K_m)$, is defined by

$$h( +_{\varphi}(K_1, \ldots, K_m), x) = \inf \left\{ \lambda > 0 \mid \varphi \left( \frac{h(K_1, x)}{\lambda}, \ldots, \frac{h(K_m, x)}{\lambda} \right) \leq 1 \right\},$$

for $x \in \mathbb{R}^n$. Equivalently, the Orlicz addition $+_{\varphi}(K_1, \ldots, K_m)$ can be defined implicitly (and uniquely) by

$$\varphi \left( \frac{h(K_1, x)}{h(+_{\varphi}(K_1, \ldots, K_m), x)}, \ldots, \frac{h(K_m, x)}{h(+_{\varphi}(K_1, \ldots, K_m), x)} \right) = 1,$$

for all $x \in \mathbb{R}^n$. An important special case is obtained when

$$\varphi(x_1, \ldots, x_m) = \sum_{j=1}^{m} \varphi_j(x_j),$$

for some fixed $\varphi_j \in \Phi$ such that $\varphi_1(1) = \cdots = \varphi_m(1) = 1$. We then write $+_{\varphi}(K_1, \ldots, K_m) = K_1 +_{\varphi} \cdots +_{\varphi} K_m$. This means that $K_1 +_{\varphi} \cdots +_{\varphi} K_m$ is defined either by

$$h(K_1 +_{\varphi} \cdots +_{\varphi} K_m, u) = \sup \left\{ \lambda > 0 \mid \sum_{j=1}^{m} \varphi_j \left( \frac{h(K_j, x)}{\lambda} \right) \leq 1 \right\},$$
for all $x \in \mathbb{R}^n$, or by the corresponding special case of (2.15).

For real $p \geq 1$, $K, L \in K_0^n$, and $\alpha, \beta \geq 0$ (not both zero), the Firey linear combination $\alpha \cdot K + \beta \cdot L \in K_0^n$ can be defined by (see [6] and [7])

$$h(\alpha \cdot K + \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$ Obviously, Firey and Minkowski scalar multiplications are related by $\alpha \cdot K = \alpha^{1/p} K$.

In [9], Gardner, Hug and Weil define the Orlicz linear combination $+_{\varphi}(K, L, \alpha, \beta)$ for $K, L \in K_0^n$ and $\alpha, \beta \geq 0$, defined by

$$h(\varphi(K, L, \alpha, \beta), x) = \frac{h(K, x)}{h(+_{\varphi}(K, L, \alpha, \beta), x)},$$ if $\alpha h(K, x) + \beta h(L, x) > 0$, and by $h(+_{\varphi}(K, L, \alpha, \beta), x) = 0$ if $\alpha h(K, x) + \beta h(L, x) = 0$, for all $x \in \mathbb{R}^n$. It is easy to verify that when $\varphi_1(t) = \varphi_2(t) = t^p$, $p \geq 1$, the Orlicz linear combination $+_{\varphi}(K, L, \alpha, \beta)$ equals the Firey combination $\alpha \cdot K + \beta \cdot L$. Henceforth we shall write $K +_{\varphi, \varepsilon} L$ instead of $+_{\varphi}(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$, and assume throughout that this is defined by (2.17), where $\alpha = 1, \beta = \varepsilon$, and $\varphi_1, \varphi_2 \in \Phi$.

3 Orlicz mixed quermassintegrals

In order to define a new concept: Orlicz mixed quermassintegrals, we need Lemmas 3.1-3.4 and Theorem 3.5.

Lemma 3.1. ([9]) If $\varphi \in \Phi_m$, then Orlicz addition $+_{\varphi} : (K_0^n)^m \rightarrow K_0^n$ is continuous, GL(n) covariant, monotonic, projection covariant and has the identity property.

Lemma 3.2. ([9]) If $K, L \in K_0^n$, then

$$K +_{\varphi, \varepsilon} L \rightarrow K,$$

in the Hausdorff metric as $\varepsilon \rightarrow 0^+$.

Lemma 3.3. If $K, L \in K_0^n$ and $0 \leq i < n$, Then

$$\lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K +_{\varphi, \varepsilon} L) - W_i(K)}{\varepsilon} = \frac{n - i}{n} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{h(K +_{\varphi, \varepsilon} L, u) - h(K, u)}{\varepsilon} dS_i(K, u),$$

where, $\lim_{\varepsilon \rightarrow 0^+} \frac{h(K +_{\varphi, \varepsilon} L, u) - h(K, u)}{\varepsilon}$ uniformly for $u \in S^{n-1}$.

Proof. For brevity, we temporarily write $K_{\varepsilon} = K +_{\varphi, \varepsilon} L$. Starting with the decomposition

$$W_i(K_{\varepsilon}) - W_i(K) = \sum_{j=0}^{n-1} W_i(K_{\varepsilon}[j + 1], K[n - i - j - 1]) - W_i(K_{\varepsilon}[j], K[n - i - j]).$$

Notice that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K_{\varepsilon}[j + 1], K[n - i - j - 1]) - W_i(K_{\varepsilon}[j], K[n - i - j])}{\varepsilon}$$
\[
\begin{align*}
&= \frac{1}{n} \int_{S^{n-1}} \frac{h(K_{\varepsilon}, u) - h(K, u)}{\varepsilon} dS_i(K_{\varepsilon}[j], K[n - i - j - 1], u) \\
&= \frac{1}{n} \int_{S^{n-1}} \left( \frac{h(K_{\varepsilon}, u) - h(K, u)}{\varepsilon} - \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi_{\varepsilon} L, u) - h(K, u)}{\varepsilon} \right) \\
&\quad \times dS_i(K_{\varepsilon}[j], K[n - i - j - 1], u) \\
&\quad + \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi_{\varepsilon} L, u) - h(K, u)}{\varepsilon} dS_i(K_{\varepsilon}[j], K[n - i - j - 1], u).
\end{align*}
\]

By assumption, the integrand in (3.3) converges uniformly to zero for \( u \in S^{n-1} \).
Since \( K_{\varepsilon} \to K \) as \( \varepsilon \to 0^+ \), by Lemma 3.2, and the \( i \)-th mixed surface area measures \( S_i(K_{\varepsilon}[j], K[n - i - j - 1]) \) are uniformly bounded for \( \varepsilon \in (0, 1] \), the first integral in the previous sum converges to zero. Noting that \( S_i(K_{\varepsilon}[j], K[n - i - j - 1]) \to S_i(K, u) \) weakly as \( \varepsilon \to 0^+ \). Hence

\[
\lim_{\varepsilon \to 0^+} \frac{W_i(K + \varphi_{\varepsilon} L) - W_i(K)}{\varepsilon} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi_{\varepsilon} L, u) - h(K, u)}{\varepsilon} \\
\times dS_i(K_{\varepsilon}[j], K[n - i - j - 1], u) \\
= \frac{n}{n} \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi_{\varepsilon} L, u) - h(K, u)}{\varepsilon} dS_i(K, u).
\]

Lemma 3.4. For \( \varepsilon > 0 \) and \( u \in S^{n-1} \), let \( h_{\varepsilon} = h(K + \varphi_{\varepsilon} L, u) \). If \( K \in K^n_{\infty} \) and \( L \in K^n_0 \), then

\[
(3.4) \quad \frac{dh_{\varepsilon}}{d\varepsilon} = -h(K, u) \frac{d\varphi_{\varepsilon}^{-1}(y)}{dy} \varphi_{\varepsilon} \left( \frac{h(L, u)}{h_{\varepsilon}} \right) \\
\left( \varphi_{\varepsilon}^{-1} \left( 1 - \varepsilon \varphi_{\varepsilon} \left( \frac{h(L, u)}{h_{\varepsilon}} \right) \right) \right)^2 + \varepsilon \cdot \frac{h(L, u)h(L_n, u)}{h_{\varepsilon}^2} \frac{d\varphi_{\varepsilon}^{-1}(y)}{dy} \frac{d\varphi_{\varepsilon}(z)}{dz},
\]

where

\[
y = 1 - \varepsilon \varphi_{\varepsilon} \left( \frac{h(L, u)}{h_{\varepsilon}} \right),
\]

and

\[
z = \frac{h(L, u)}{h_{\varepsilon}}.
\]

Proof. Suppose \( \varepsilon > 0 \), \( L \in K^n_{\infty}, K \in K^n_{\infty} \) and \( u \in S^{n-1} \), and notice that

\[
h_{\varepsilon} = h(K + \varphi_{\varepsilon} L, u),
\]

we have

\[
\frac{h(K, u)}{h_{\varepsilon}} = \varphi_{\varepsilon}^{-1} \left( 1 - \varepsilon \varphi_{\varepsilon} \left( \frac{h(L, u)}{h_{\varepsilon}} \right) \right).
\]
On the other hand

\[
\frac{dh_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon} \left( \frac{h(K, u)}{\varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h_\varepsilon} \right) \right)} \right)
\]

\[
= \frac{h(K, u)}{h_\varepsilon} \frac{d\varphi_1^{-1}(y)}{dy} \left[ \varphi_2 \left( \frac{h(L, u)}{h_\varepsilon} \right) - \varepsilon \frac{d\varphi_2(z) h(L, u) dh_\varepsilon}{dz} \right]
\]

\[
\left( \varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h_\varepsilon} \right) \right) \right)^2
\]

where

\[
y = 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h_\varepsilon} \right),
\]

and

\[
z = \frac{h(L, u)}{h_\varepsilon}.
\]

By simplifying the equation from above, (3.4) easily follows. □

**Theorem 3.5.** Let \( \varphi \in \Phi_2 \), and \( \varphi_1, \varphi_2 \in \Phi \). If \( K \in \mathcal{K}_o^n, L \in \mathcal{K}_o^n \) and \( 1 \leq i \leq n \), then

\[
(3.5) \quad \left( \frac{\varphi_1}{\varphi_2} \right)_i(1) \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varphi, \varepsilon L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{S_{n-1}} \varphi_2 \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u).
\]

**Proof.** From Lemma 3.3, we obtain

\[
\lim_{\varepsilon \to 0^+} \frac{W_i(K + \varphi, \varepsilon L) - W_i(K)}{\varepsilon} = \frac{n - i}{n} \int_{S_{n-1}} \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi, \varepsilon L, u) - h(K, u)}{\varepsilon} dS_i(K, u)
\]

\[
= \frac{n - i}{n} \lim_{\varepsilon \to 0^+} \int_{S_{n-1}} \frac{dh_\varepsilon}{d\varepsilon} dS_i(K; u).
\]

From Lemmas 3.1-3.2 and Lemma 3.4, and noting that \( y \to 1^- \) as \( \varepsilon \to 0^+ \), we have

\[
\frac{d\varphi_1^{-1}(y)}{d\varepsilon} = \lim_{y \to 1^-} \frac{\varphi_1^{-1}(y) - \varphi_1^{-1}(1)}{y - 1} = \frac{1}{(\varphi_1)_i'(1)}
\]

the equation (3.5) easily follows. □

The theorem plays a central role in our deriving new concept of the Orlicz mixed quermassintegrals. Here, we give the another proof.

**Proof.** From the hypotheses, we have for \( \varepsilon > 0 \)

\[
h(K + \varphi, \varepsilon L, u) = \frac{h(K, u)}{\varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi, \varepsilon L, u)} \right) \right)}.
\]
Hence

\[
(3.6) \quad \lim_{\varepsilon \to 0^+} \frac{h(K + \varphi, \varepsilon L, u) - h(K, u)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{h(K, u)}{\varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi, \varepsilon L, u)} \right) \right)} - h(K, u)
\]

\[
= \lim_{\varepsilon \to 0^+} \frac{h(K, u) \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi, \varepsilon L, u)} \right)}{\varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi, \varepsilon L, u)} \right) \right)} \lim_{y \to 1^-} \frac{\varphi_1^{-1}(y) - \varphi_1^{-1}(1)}{y - 1},
\]

where

\[
y = 1 - \varepsilon \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi, \varepsilon L, u)} \right),
\]

and note that \( y \to 1^- \) as \( \varepsilon \to 0^+ \). Notice that

\[
\lim_{y \to 1^-} \frac{\varphi_1^{-1}(y) - \varphi_1^{-1}(1)}{y - 1} = \frac{1}{(\varphi_1)_1'(1)},
\]

and from (2.2),(3.6) and Lemmas 3.1-3.2, (3.5) easy follows.

Denoting by \( W_{\varphi, i}(K, L) \), for any \( \varphi \in \Phi \) and \( 1 \leq i < n \), the integral on the right-hand side of (3.5) with \( \varphi_2 \) replaced by \( \varphi \), we see that either side of the equation (3.5) is equal to \( W_{\varphi, i}(K, L) \) and therefore this new Orlicz mixed volume \( W_{\varphi, i}(K, L) \) (Orlicz mixed quermassintegrals) has been born.

**Definition 3.1.** (Orlicz mixed quermassintegrals) For \( \varphi \in \Phi \), Orlicz mixed quermassintegrals, \( W_{\varphi, i}(K, L) \), for \( 0 \leq i < n \), defined by

\[
W_{\varphi, i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{h(K, u)} \right) h(K, u) dS_i(K, u),
\]

for all \( K \in K_0^n \), \( L \in K_0^n \).

**Remark 3.2.** Let \( \varphi_1(t) = t + p \), \( p \geq 1 \) in (3.5), the Orlicz sum \( K + \varphi, L \) reduces to the \( L_p \) addition \( K + \varphi, L \), and the Orlicz mixed quermassintegrals \( W_{\varphi, i}(K, L) \) become the well-known mixed \( p \)-quermassintegrals \( W_{p, i}(K, L) \). Obviously, when \( i = 0 \), \( W_{\varphi, i}(K, L) \) reduces to Orlicz mixed volumes \( V_{\varphi}(K, L) \) defined by Gardner, Hug and Weil [9].

**Theorem 3.6.** If \( \varphi_1, \varphi_2 \in \Phi \), \( \varphi \in \Phi_2 \) and \( K \in K_0^n \), \( L \in K_0^n \), and \( 0 \leq i < n \), then

\[
W_{\varphi, i}(K, L) = \frac{(\varphi_1)_i'(1)}{n - i} \lim_{\varepsilon \to 0^+} \frac{W_{i}(K + \varphi, \varepsilon L) - W_{i}(K)}{\varepsilon},
\]

**Proof.** This follows immediately from Theorem 3.5 and (3.7).
4 Orlicz-Minkowski type inequality

In the Section, we need define a Borel measure in \( S^{n-1} \), \( \tilde{W}_{n,i}(K,v) \), called as \( i \)-th normalized cone measure.

**Definition 4.1.** If \( K \in K^n_\infty \), \( i \)-th normalized cone measure, \( \tilde{W}_{n,i}(K,v) \), defined by

\[
dW_{n,i}(K,v) = \frac{h(K,v)}{nW_i(K)}dS_i(K,v).
\]

When \( i = 0 \), \( W_{n,i}(K,v) \) becomes to the well-known normalized cone measure \( V_n(K,v) \), by

\[
\frac{1}{nW_i(K)}\int_{S^{n-1}} \varphi \left( \frac{h(L,u)}{h(K,u)} \right) h(K,u)dS_i(K,u) \geq \varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]

**Lemma 4.1.** (Jensen’s inequality) Suppose that \( \mu \) is a probability measure on a space \( X \) and \( g : X \to I \subset \mathbb{R} \) is a \( \mu \)-integrable function, where \( I \) is a possibly infinite interval. If \( \varphi : I \to \mathbb{R} \) is a convex function, then

\[
\int_X \varphi(g(x))d\mu(x) \geq \varphi \left( \int_X g(x)d\mu(x) \right).
\]

If \( \varphi \) is strictly convex, equality holds if and only if \( g(x) \) is constant for \( \mu \)-almost all \( x \in X \) (see [16]).

**Lemma 4.2.** Let \( 0 < a \leq \infty \) be an extended real number, and let \( I = [0,a) \) be a possibly infinite interval. Suppose that \( \varphi : I \to [0,\infty) \) is convex with \( \varphi(0) = 0 \). If \( K \in K^n_\infty \) and \( L \in K^n_o \) are such that \( L \subset \text{int}(aK) \), then

\[
\frac{1}{nW_i(K)}\int_{S^{n-1}} \varphi \left( \frac{h(L,u)}{h(K,u)} \right) h(K,u)dS_i(K,u) \geq \varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]

If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{0\} \).

**Proof.** In view of \( L \subset \text{int}(aK) \), so \( 0 \leq \frac{h(L,u)}{h(K,u)} < a \) for all \( u \in S^{n-1} \). By (4.1) and note that (2.2) with \( K = L \), it follows the \( i \)-th normalized cone measure \( \tilde{W}_{n,i}(K,u) \) is a probability measure on \( S^{n-1} \). Hence by using Jensen’s inequality (4.3), the Minkowski’s inequality (2.4), and the fact that \( \varphi \) is increasing, to obtain

\[
\frac{1}{nW_i(K)}\int_{S^{n-1}} \varphi \left( \frac{h(L,u)}{h(K,u)} \right) h(K,u)dS_i(K,u) = \int_{S^{n-1}} \varphi \left( \frac{h(L,u)}{h(K,u)} \right) d\tilde{W}_{n,i}(K,u)
\]

\[
\geq \varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]
\begin{align}
\varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\end{align}

In the following, we discuss the equal condition of (4.4). Suppose the equality holds in (4.4) and \( \varphi \) is strictly convex, so that \( \varphi > 0 \) on \((0, a)\). Moreover, notice the injectivity of \( \varphi \), we have equality in Minkowski inequality (2.4), so there are \( r \geq 0 \) and \( x \in \mathbb{R}^n \) such that \( L = rK + x \) and hence
\[
h(L, u) = rh(K, u) + x \cdot u
\]
for all \( u \in S^{n-1} \). Since equality must hold in Jensen’s inequality (4.3) as well, when \( \varphi \) is strictly convex we can conclude from the equality condition for Jensen’s inequality that
\[
\frac{1}{nW_i(K)} \int_{S^{n-1}} \frac{h(L, u)}{h(K, u)} h(K, u) dS_i(K, u) = \frac{h(L, v)}{h(K, v)}
\]
for \( S_i(K, \cdot) \)-almost all \( v \in S^{n-1} \). Hence
\[
\frac{1}{nW_i(K)} \int_{S^{n-1}} \left( r + \frac{x \cdot u}{h(K, u)} \right) h(K, u) dS_i(K, u) = r + \frac{x \cdot v}{h(K, v)}
\]
for \( S_i(K, \cdot) \)-almost all \( v \in S^{n-1} \). From this and the fact that the centroid of \( S_i(K, \cdot) \) is at the origin, we get
\[
0 = x \cdot \left( \frac{1}{nW_i(K)} \int_{S^{n-1}} u dS_i(K, u) \right) = \frac{1}{nW_i(K)} \int_{S^{n-1}} x \cdot u dS_i(K, u) = \frac{x \cdot v}{h(K, v)},
\]
that is, \( x \cdot v = 0 \), for \( S_i(K, \cdot) \)-almost all \( v \in S^{n-1} \). Hence \( x = 0 \), namely \( L = rK \).

**Theorem 4.3.** Let \( \varphi \in \Phi \). If \( K \in \mathcal{K}^n_\infty \), \( L \in \mathcal{K}^n_0 \) and \( 0 \leq i < n \), then
\[
W_{\varphi, i}(K, L) \geq W_i(K) \cdot \varphi \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]

If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{0\} \).

**Proof.** This follows immediately from (3.7) and Lemma 4.2, with \( a = \infty \).

**Corollary 4.4.** ([21]) If \( K \in \mathcal{K}^n_\infty \) and \( L \in \mathcal{K}^n_\infty \), and \( p > 1 \) and \( 0 \leq i < n \), then
\[
W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i} p W_i(L)^p,
\]
with equality if and only if \( K \) and \( L \) are dilates or \( L = \{0\} \).

**Proof.** This follows immediately from (4.7) with \( \varphi(t) = t^p \) and \( p > 1 \).

**Remark 4.2.** When \( a = \infty \), putting \( \varphi(t) = e^t - 1 \) in (4.4), we obtain
\[
\log \int_{S^{n-1}} \exp \left( \frac{h(L, u)}{h(K, u)} \right) dW_{n,i}(K, u) \geq \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.
\]
Similarly, $L_p$-Minkowski inequality (1.8) can be written as

$$
(4.9) \quad \left( \int_{S^{n-1}} \left( \frac{h(L, u)}{h(K, u)} \right)^p \, dW_{n,i}(K, u) \right)^{1/p} \geq \left( \frac{W_i(K)}{W_i(L)} \right)^{1/(n-i)}.
$$

When $p = 1$, (4.9) becomes to a new form of the Minkowski inequality (2.4). The left side of (4.9) is just the $p$th mean of the function $h(L, u)/h(K, u)$ with respect to $W_{n,i}(K, \cdot)$. Notice that $p$th means increase with $p > 1$, so we find that the Minkowski inequality (2.4) implies $L_p$-Minkowski inequality (2.8).

## 5 Orlicz-Brunn-Minkowski type inequality

In this section, we establish the Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals.

**Theorem 5.1.** Let $\varphi \in \Phi_2$. If $K \in \mathcal{K}_0^n$, $L \in \mathcal{K}_0^n$ and $1 \leq i < n$, then

$$
(5.1) \quad 1 \geq \varphi \left( \frac{W_i(K)^{1/(n-i)}}{W_i(K + \varphi L)^{1/(n-i)}}, \frac{W_i(L)^{1/(n-i)}}{W_i(K + \varphi L)^{1/(n-i)}} \right).
$$

If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates of $L = \{o\}$.

**Proof.** From the hypotheses and Theorem 4.3, we obtain

$$
(5.2) \quad W_i(K + \varphi L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h(K, u)}{h(K + \varphi L, u)} \right) h(K + \varphi L, u) \, dS_i(K + \varphi L, u)
$$

$$
= \frac{1}{n} \int_{S^{n-1}} \varphi_1 \left( \frac{h(K, u)}{h(K + \varphi L, u)} \right) + \varphi_2 \left( \frac{h(L, u)}{h(K + \varphi L, u)} \right) h(K + \varphi L, u) \, dS_i(K + \varphi L, u)
$$

$$
= W_{\varphi_1,i}(K + \varphi L, K) + W_{\varphi_2,i}(K + \varphi L, L)
$$

$$
\geq W_i(K + \varphi L) \varphi \left( \frac{W_i(K)^{1/(n-i)}}{W_i(K + \varphi L)^{1/(n-i)}}, \frac{W_i(L)^{1/(n-i)}}{W_i(K + \varphi L)^{1/(n-i)}} \right).
$$

This is just (5.1).

If equality holds in (5.2), then in (5.2), with $K$, $L$ and $\varphi$ replaced by $K + \varphi L$, $K$ and $\varphi_1$ (and by $K + \varphi L$, $L$ and $\varphi_2$), respectively. So if $\varphi$ is strictly convex, then $\varphi_1$ and $\varphi_2$ are also, so both $K$ and $L$ are multiples of $K + \varphi L$, and hence are dilates of each other or $L = \{o\}$. \hfill $\square$

**Corollary 5.2.** ([21]) If $p > 1$, $K \in \mathcal{K}_0^n$, $L \in \mathcal{K}_0^n$, while $0 \leq i < n$, then

$$
(5.3) \quad W_i(K + p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)},
$$

with equality if and only if $K$ and $L$ are dilates of $L = \{o\}$.

**Proof.** The result follows immediately from Theorem 5.1 with $\varphi(x_1, x_2) = x_1^p + x_2^p$ and $p > 1$. \hfill $\square$
Theorem 5.3. Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals implies Orlicz Minkowski inequality for Orlicz mixed quermassintegrals.

Proof. Since \( \varphi_1 \) is increasing, so \( \varphi_1^{-1} \) is also increasing and hence from (5.1), we obtain for \( \varepsilon > 0 \)

\[
W_i(K + \varphi, L) \geq \left( \frac{W_i(K)}{\varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \left( \frac{W_i(L)}{W_i(K + \varphi, L)} \right)^{1/(n-i)} \right) \right) \right)^{n-i}.
\]

From Theorem 3.6, we obtain

\[
W_{\varphi_2,i}(K, L) \geq \frac{(\varphi_1)'(1)}{n-i} W_i(K) \times \lim_{\varepsilon \to 0^+} \frac{\left( \varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \left( \frac{W_i(L)}{W_i(K + \varphi, L)} \right)^{1/(n-i)} \right) \right) \right)^{n-i} - W_i(K)}{\varepsilon}
\]

\[
= (\varphi_1)'(1) \lim_{\varepsilon \to 0^+} \frac{W_i(K)}{\left( \varphi_1^{-1} \left( 1 - \varepsilon \varphi_2 \left( \left( \frac{W_i(L)}{W_i(K + \varphi, L)} \right)^{1/(n-i)} \right) \right) \right)^{2(n-i)}} \times \left( \frac{W_i(L)}{W_i(K + \varphi, L)} \right)^{1/(n-i)} \lim_{z \to 1^-} \frac{\varphi_1^{-1}(z) - \varphi_1^{-1}(1)}{z - 1},
\]

where

\[
z = 1 - \varepsilon \varphi_2 \left( \left( \frac{W_i(L)}{W_i(K + \varphi, L)} \right)^{1/(n-i)} \right),
\]

and note that \( z \to 1^- \) as \( \varepsilon \to 0^+ \). On the other hand, in view of

\[
\lim_{z \to 0^+} \frac{\varphi_1^{-1}(z) - \varphi_1^{-1}(1)}{z - 1} = \frac{1}{(\varphi_1)'(1)},
\]

and from Lemma 3.2. Hence

(5.4) \( W_{\varphi_2,i}(K, L) \geq W_i(K) \varphi_2 \left( \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right). \)

Replace \( \varphi_2 \) by \( \varphi \), this yields the Orlicz Minkowski inequality in (4.7). The equality condition follows immediately from the equality of Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals. \( \square \)
From the proof of Theorem 5.1, we may see that Orlicz Minkowski inequality for Orlicz mixed quermassintegrals implies also Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals, and this combines Theorem 5.3, we found that

**Theorem 5.4.** Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals is equivalent to Orlicz Minkowski inequality for Orlicz mixed quermassintegrals. Namely: Let \( \varphi_2 \in \Phi \) and \( \varphi \in \Phi_2 \). If \( K \in \mathcal{K}^n_{\infty}, L \in \mathcal{K}^n_{\infty} \) and \( 1 \leq i < n \), then

\[
W_{\varphi_2, i}(K, L) \geq W_i(K) \varphi_2 \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}
\]

\[
\Leftrightarrow 1 \geq \varphi \left( \frac{W_i(K)^{1/(n-i)}}{W_i(K+\varphi L)^{1/(n-i)}}, \frac{W_i(L)^{1/(n-i)}}{W_i(K+\varphi L)^{1/(n-i)}} \right).
\]

If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{0\} \).

**Corollary 5.5.** Orlicz dual Brunn-Minkowski inequality is equivalent to Orlicz dual Minkowski inequality. Namely: Let \( \varphi_2 \in \Phi \) and \( \varphi \in \Phi_2 \). If \( K \in \mathcal{K}^n_{\infty} \) and \( L \in \mathcal{K}^n_{\infty} \), then

\[
V_{\varphi_2}(K, L) \geq V(K) \varphi_2 \left( \frac{V(L)}{V(K)} \right)^{1/n} \Leftrightarrow 1 \geq \varphi \left( \frac{V(K)^{1/n}}{V(K+\varphi L)^{1/n}}, \frac{V(L)^{1/n}}{V(K+\varphi L)^{1/n}} \right).
\]

If \( \varphi \) is strictly convex, equality holds if and only if \( K \) and \( L \) are dilates or \( L = \{0\} \).

**Proof.** The result follows immediately from Theorem 5.4 with \( i = 0 \). \( \square \)

### 6 The log-Minkowski type inequality

Assume that \( K, L \in \mathcal{K}^n_{\infty} \), then the log Minkowski combination, \( (1 - \lambda) \cdot K + \lambda \cdot L \), is defined by

\[
(1 - \lambda) \cdot K + \lambda \cdot L = \bigcap_{u \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n \mid x \cdot u \leq h(K, u)^{1-\lambda}h(L, u)^{\lambda} \},
\]

for all real \( \lambda \in [0, 1] \). Böröczky, Lutwak, Yang, and Zhang [2] conjecture that for origin-symmetric convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) and \( 0 \leq \lambda \leq 1 \),

\[
V((1 - \lambda) \cdot K + \lambda \cdot L) \geq V(K)^{1-\lambda}V(L)^{\lambda}.
\]

In [2], they proved (6.1) only when \( n = 2 \) and \( K, L \) are origin-symmetric convex bodies, and note that while it is not true for general convex bodies. Moreover, they also shown that (6.1), for all \( n \), is equivalent to the following log-Minkowski inequality

\[
\int_{\mathbb{S}^{n-1}} \log \left( \frac{h(L, u)}{h(K, u)} \right) dV_n(K, v) \geq \frac{1}{n} \log \left( \frac{V(L)}{V(K)} \right),
\]

where \( V_n(K, \cdot) \) is the normalized cone measure for \( K \). In fact, replacing \( K \) and \( L \) by \( K + L \) and \( K \), respectively, (6.2) becomes to the following

\[
\int_{\mathbb{S}^{n-1}} \log \left( \frac{h(K, u)}{h(K + L, u)} \right) dV_n(K + L, v) \geq \log \left( \frac{V(K)}{V(K + L)} \right)^{1/n}.
\]
In [9], Gardner, Hug and Weil gave a new version of (6.3) for the nonempty compact convex subsets $K$ and $L$, not origin-symmetric convex bodies, as follows. If $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_{oo}^n$, then

$$\log \left( \frac{W_i(K)_{1/(n-1)} - W_i(L)_{1/(n-1)}}{W_i(K)_{1/(n-1)}} \right) \geq \int_{S_{n-1}} \log \left( \frac{h(K, u)}{h(K + L, u)} \right) dW_{n,i}(K, u),$$

with equality if and only if $K$ and $L$ are dilates or $L = \{0\}$. They also showed that combining (6.3) and (6.4), may get the classical Brunn-Minkowski inequality.

$$V(K + L)_{1/n} - V(L)_{1/n} \geq V(K)_{1/n},$$

whenever $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_{oo}^n$ and (6.2) holds with $K$ and $L$ replaced by $K + L$ and $K$, respectively. In particular, if (6.2) holds (as it does, for origin-symmetric convex bodies when $n = 2$), then (6.2) and (6.4) together split the classical Brunn-Minkowski inequality.

In the following, we give a new version of (6.4).

**Lemma 6.1.** If $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_{oo}^n$ are such that $L \subset \text{int} K$ and $1 \leq i < n$, then

$$\log \left( \frac{W_i(K + L)_{1/(n-1)} - W_i(L)_{1/(n-1)}}{W_i(K)_{1/(n-1)}} \right) \geq \int_{S_{n-1}} \log \left( \frac{h(K, u)}{h(K + L, u)} \right) dW_{n,i}(K, u),$$

with equality if and only if $K$ and $L$ are dilates or $L = \{0\}$. 

**Proof.** Since $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_{oo}^n$ are such that $L \subset \text{int} K$. Let $\varphi(t) = -\log(1 - t)$, and notice that $\varphi(0) = 0$ and $\varphi$ is strictly increasing and strictly convex on $[0, 1]$ with $\varphi(t) \to \infty$ as $t \to 1^-$. Hence the inequality (6.5) is a direct consequence of Lemma 4.3 with this choice of $\varphi$ and $a = 1$. \hfill \Box

**Theorem 6.2.** If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_{oo}^n$ and $1 \leq i < n$, then

$$\log \left( \frac{W_i(K + L)_{1/(n-1)} - W_i(L)_{1/(n-1)}}{W_i(K + L)_{1/(n-1)}} \right) \geq \int_{S_{n-1}} \log \left( \frac{h(K, u)}{h(K + L, u)} \right) dW_{n,i}(K + L, u),$$

with equality if and only if $K$ and $L$ are dilates or $L = \{0\}$. 

**Proof.** If $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_{oo}^n$, then $K + L \in \mathcal{K}_{oo}^n$. In view of $L \subset \text{int}(K + L)$ and from Lemma 6.1 with $K$ replaced by $K + L$, (6.6) easy follows. \hfill \Box

Putting $i = 0$ in (6.6), (6.6) reduces to (6.4). Here, we point out a new conjecture which is an extension of the log Minkowski inequality (6.2): **Conjecture** If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_{oo}^n$ and $1 \leq i < n$, then

$$\int_{S_{n-1}} \log \left( \frac{h(L, u)}{h(K, u)} \right) dW_{n,i}(K, u) \geq \frac{1}{n-i} \log \left( \frac{W_i(L)}{W_i(K)} \right).$$

**Corollary 6.3.** If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_{oo}^n$ and $1 \leq i < n$, then

$$\int_{S_{n-1}} \log \left( \frac{h(K, u)}{h(K + L, u)} \right) dW_{n,i}(K + L, u) \geq \frac{1}{n-i} \log \left( \frac{W_i(K)}{W_i(K + L)} \right).$$
Proof. The result follows immediately from (6.7) with replacing $K$ and $L$ by $K + L$ and $K$, respectively.

It is easy that combine (6.6) and (6.8) together split the following classical Brunn-Minkowski inequality for quermassintegrals. If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_{oo}^n$ and $0 \leq i \leq n$, then

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},$$

with equality if and only if $K$ and $L$ are dilates or $L = \{0\}$.

7 A new version of Orlicz Minkowski’s inequality

In 2010, the Orlicz projection body $\Pi_\varphi$ of $K$ defined by Lutwak, Yang and Zhang [28]

$$h(\Pi_\varphi, u) = \inf \left\{ \lambda > 0 \mid \int_{S_{n-1}} \varphi \left( \frac{|u \cdot v|}{\lambda h(K,v)} \right) dV_n(K,v) \leq 1 \right\},$$

for $K \in \mathcal{K}_{oo}^n, u \in S^{n-1}$, where $V_n(K, \cdot)$ is the normalized cone measure for $K$. Here, we define the $i$-th Orlicz mixed projection body.

Definition 7.1. Let $K \in \mathcal{K}_{oo}^n, L \in \mathcal{K}_{oo}^n, \varphi \in \Phi$ and $0 \leq i < n$, the $i$-th Orlicz mixed projection body, $\Pi_{\varphi,i}$, define by

$$h(\Pi_{\varphi,i}, u) = \inf \left\{ \lambda > 0 \mid \int_{S_{n-1}} \varphi \left( \frac{|u \cdot v|}{\lambda h(K,v)} \right) dW_{n,i}(K,v) \leq 1 \right\},$$

for $u \in S^{n-1}$, where $W_{n,i}(K, \cdot)$ is the $i$-th normalized cone measure for $K$ defined in (4.1).

Obviously, when $i = 0$, (7.2) becomes (7.1). In the Section, definition 7.1 of the $i$-th Orlicz projection body suggests defining, by analogy,

$$\widetilde{W}_{\varphi,i}(K,L) = \inf \left\{ \lambda > 0 \mid \int_{S_{n-1}} \varphi \left( \frac{h(L,u)}{\lambda h(K,u)} \right) dW_{n,i}(K,u) \leq 1 \right\},$$

and call as $\widetilde{W}_{\varphi,i}(K,L)$ Orlicz type quermassintegrals.

Theorem 7.1. If $\varphi \in \Phi$ and $K \in \mathcal{K}_{oo}^n, L \in \mathcal{K}_{oo}^n$ and $1 \leq i < n$, then

$$\widetilde{W}_{\varphi,i}(K,L) \geq \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)}.$$ 

If $\varphi$ is strictly convex and $W_i(L) > 0$, equality holds if and only if $K$ and $L$ are dilates.

Proof. Replacing $K$ by $\lambda K$, $\lambda > 0$ in (4.4) with $a = \infty$, we have

$$\int_{S_{n-1}} \varphi \left( \frac{h(L,u)}{\lambda h(K,u)} \right) dW_{n,i}(K,u) \geq \varphi \left( \frac{1}{\lambda} \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right).$$
Let

$$
\int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{\lambda h(K, u)} \right) dW_{n,i}(K, u) \leq 1.
$$

Hence

$$
\varphi \left( \frac{1}{\lambda} \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right) \leq 1.
$$

In view of $\varphi$ is strictly increasing, we obtain

$$
(7.6) \quad \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \leq \lambda.
$$

From (7.3) and (7.6), (7.4) easy follows.

In the following, we discuss the equality condition of (7.4). Suppose that equality holds, $\varphi$ is strictly convex and $W_i(L) > 0$. From (7.3), the exist $\mu = \widehat{W}_{\varphi,i}(K, L) > 0$ satisfies

$$
\int_{S^{n-1}} \varphi \left( \frac{h(L, u)}{\mu h(K, u)} \right) dW_{n,i}(K, u) = 1.
$$

Hence

$$
\mu = \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)};
$$

namely:

$$
\varphi \left( \frac{1}{\mu} \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right) = 1.
$$

Therefore the equality in (7.5) holds for $\lambda = \mu$. From the equality condition of (4.4), it follows $\mu K$ and $L$ are dilates.

When $\varphi(t) = t^p$ and $p \geq 1$ in (7.3), it easy follows that

$$
\widehat{W}_{\varphi,i}(K, L) = \left( \frac{W_{\varphi,i}(K, L)}{W_i(K)} \right)^{1/p}.
$$

Putting $\varphi(t) = t^p$ and $p \geq 1$ in (7.4), (7.4) reduces to the classical $L_p$-Minkowski inequality (1.8) for mixed $p$-quermassintegrals.

There is no direct relationship between the Orlicz-Minkowski inequalities (4.7) and (7.4). Indeed, when $\varphi > 0$ on $(0, \infty)$, these can be written in the forms

$$
\frac{W_{\varphi,i}(K, L)}{W_i(K)} \geq \varphi \left( \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right), \quad (7.7)
$$

and

$$
\varphi \left( \frac{\widehat{W}_{\varphi,i}(K, L)}{W_i(K)} \right) \geq \varphi \left( \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right).
$$

respectively, and each of the two quantities on the left-hand sides can be larger than the other. This is very interesting.
8 Simon’s characterization of relative spheres

**Theorem 8.1.** Suppose $K \in \mathcal{K}_\infty^n$, $L \in \mathcal{K}_\infty^n$, and $\mathcal{S} \subset \mathcal{K}_\infty^n$ is a class of bodies such that $K, L \in \mathcal{S}$. If $0 \leq i < n - 1$ and $\varphi \in \Phi$, and

\[(8.1)\quad W_{\varphi, i}(Q, K) = W_{\varphi, i}(Q, L), \quad \text{for all } Q \in \mathcal{S},
\]

then $K = L$.

**Proof.** To see this take $Q = K$, and from (3.10) and Theorem 4.4, we have

\[W_i(K) = W_{\varphi, i}(K, K) = W_{\varphi, i}(K, L) \geq W_i(K) \varphi \left( \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right).\]

If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates or $L = \{o\}$. Hence

\[\varphi \left( \left( \frac{W_i(L)}{W_i(K)} \right)^{1/(n-i)} \right) \leq 1.\]

If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates or $L = \{o\}$. Note that $\varphi$ is increasing, we obtain

\[W_i(L) \leq W_i(K).\]

Take $Q = L$, we have

\[W_i(L) = W_{\varphi, i}(L, L) = W_{\varphi, i}(L, K) \geq W_i(L) \varphi \left( \left( \frac{W_i(K)}{W_i(L)} \right)^{1/(n-i)} \right).\]

If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates or $L = \{o\}$. Hence

\[\varphi \left( \left( \frac{W_i(K)}{W_i(L)} \right)^{1/(n-i)} \right) \leq 1.\]

If $\varphi$ is strictly convex, equality holds if and only if $K$ and $L$ are dilates or $L = \{o\}$. Hence

\[W_i(K) \leq W_i(L).\]

This yields $W_i(K) = W_i(L)$. Hence $K = L$. □

**Corollary 8.2.** Suppose $K \in \mathcal{K}_\infty^n$, $L \in \mathcal{K}_\infty^n$, and $\mathcal{S} \subset \mathcal{K}_\infty^n$ is a class of bodies such that $K, L \in \mathcal{S}$. If $\varphi \in \Phi$, and

\[(8.2)\quad V_{\varphi}(Q, K) = V_{\varphi}(Q, L), \quad \text{for all } Q \in \mathcal{S},
\]

then $K = L$.

**Proof.** The result follows immediately from Theorem 8.1 with $i = 0$. □

Putting $\varphi(t) = t^p$ and $p > 1$ in Theorem 8.1, we obtain the following result which was proved by Lutwak [21].
Corollary 8.3. Suppose $K \in \mathcal{K}_n^{\infty}$, $L \in \mathcal{K}_n^{\infty}$, and $S \subset \mathcal{K}_n^{\infty}$ is a class of bodies such that $K, L \in S$. If $p > 1$, $0 < i < n - 1$, and

$$W_{p,i}(Q, K) = W_{p,i}(Q, L), \quad \text{for all } Q \in S,$$

then $K = L$.

Theorem 8.4. Suppose $0 < i < n$ and $\varphi \in \Phi$. For $K \in \mathcal{K}_n^{\infty}$, the following statements are equivalent:

(i) The body $K$ is centered,

(ii) The measure $W_{n,i}(K, \cdot)$ is even.

(iii) $W_{\varphi,i}(K, Q) = W_{\varphi,i}(K, -Q)$, for all $Q \in \mathcal{K}_n^{\infty}$.

(iv) $W_{\varphi,i}(K, Q) = W_{\varphi,i}(K, -Q)$, for $Q = K$.

Proof. To see that (i) implies (ii), recall that if $K$ is centered, then $h(K, \cdot)$ is an even function, and $S_i(K)$ is an even measure. The implication is now a consequence of the fact that $dW_{n,i}(K, \cdot) = \frac{1}{W_i(K)} h(K, \cdot) dS_i(K, \cdot)$.

That (ii) yields (iii) is a consequence of the following integrable representation

$$W_{\varphi,i}(K, Q) = W_i(K) \int_{S^{n-1}} \varphi \left( \frac{h(Q, u)}{h(K, u)} \right) dW_{n,i}(K, u),$$

and the fact that, in general, $h(-Q, u) = h(Q, -u)$, for all $u \in S^{n-1}$. Obviously, (iv) follows directly from (iii).

To see that (iv) implies (i), notice that (iv), for $Q = K$, gives

$$W_i(K) = W_{\varphi,i}(K, -K).$$

The desired result follows from the fact that $W_i(-K) = W_i(K)$ and the equality conditions of the Orlicz-Minkowski inequality (4.7).

Corollary 8.5. Suppose $\varphi \in \Phi$. For $K \in \mathcal{K}_n^{\infty}$, the following statements are equivalent:

(i) The body $K$ is centered,

(ii) The measure $V_{n,i}(K, \cdot)$ is even.

(iii) $V_{\varphi,i}(K, Q) = V_{\varphi,i}(K, -Q)$, for all $Q \in \mathcal{K}_n^{\infty}$.

(iv) $V_{\varphi,i}(K, Q) = V_{\varphi,i}(K, -Q)$, for $Q = K$.

Proof. The results follow immediately from Theorem 8.5 with $i = 0$.

Corollary 8.6. Suppose $0 < i < n$ and $p > 1$. For $K \in \mathcal{K}_n^{\infty}$, the following statements are equivalent:

(i) The body $K$ is centered,

(ii) The measure $S_{p,i}(K, \cdot)$ is even.

(iii) $W_{p,i}(K, Q) = W_{p,i}(K, -Q)$, for all $Q \in \mathcal{K}_n^{\infty}$.

(iv) $W_{p,i}(K, Q) = W_{p,i}(K, -Q)$, for $Q = K$.

Proof. The results follow immediately from Theorem 8.5 with $\varphi(t) = t^p$ and $p > 1$.

This was proved by Lutwak [21]. That (iii) implies that $K$ is centrally symmetric, for the case $p = 1$ and $i = 0$, was shown (using other methods) by Goodey [10].

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On the Orlicz-Brunn-Minkowski theory


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