Covering properties and product spaces of bipolar fuzzy topology

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Abstract. In this paper, we introduce the concepts of countably compact, second countable and Lindelöf bipolar fuzzy topological spaces. Also, the $\varepsilon-$partition of a given cover of bipolar fuzzy topological space has been defined. We define the product bipolar fuzzy topological space of a family of bipolar fuzzy topological spaces and investigate some of its properties.

M.S.C. 2010: 03E72, 54A40.
Key words: Bipolar fuzzy set; bipolar fuzzy topology; countably compact bipolar fuzzy space; second countable bipolar fuzzy space; Lindelöf bipolar fuzzy spaces; $\varepsilon-$partition; product bipolar fuzzy topological space.

1 Introduction

Zadeh [17] introduced the notion of fuzzy sets. After that, fuzzy sets have been used to generalize almost every concept in all the branches of sciences and technology. Chang [3] used the fuzzy sets to define the fuzzy topological space as a generalization of the general topological space. Then almost all the concepts of general topology have been generalized in the fuzzy setting. Mixed fuzzy topological spaces were studied by Tripathy and Ray [13, 14, 15]. Different types of fuzzy topological spaces have been investigated by Tripathy and Debnath [11, 12], Dutta and Tripathy [5] and others. In 1994, Zhang [18] introduced the concept of bipolar fuzzy sets. After that, Lee [7, 8] defined basic operations on bipolar fuzzy sets. Moreover, Zhang [19] further investigated some properties of bipolar fuzzy sets, Akram and Dudek [1] studied regular bipolar fuzzy graphs and Azhagappan and Kamaraj [2] defined bipolar fuzzy topological spaces. Recently, Kim et al. [9] introduced: bipolar fuzzy point, neighborhood system, compactness and some other properties of bipolar fuzzy topological spaces. Coker [4] studied intuitionistic fuzzy topological spaces. Goguen [6] studied L-fuzzy sets. Lowen [10] and Wong [16] studied compactness, product and quotient in fuzzy topological spaces.

In this article, in Section 3 we introduce some covering properties of the bipolar fuzzy topological spaces, like countable compact, second countable and Lindelöf. Also we define the $\varepsilon-$partition of a given cover of bipolar fuzzy topological space and define

Applied Sciences, Vol. 24, 2022, pp. 261-269.

compactness, countable compact, Lindelöf spaces, with the help of the $\varepsilon-$partition. Moreover, we show that surjective and continuous image of a Lindelöf (countable compact) space is Lindelöf (countable compact). In Section 4, we define product the bipolar fuzzy topological space, and show that countable product of second countable bipolar fuzzy topological space is second countable.

2 Preliminaries

Let X be a nonempty set. Then a pair $A = (A^+, A^-)$ is called a bipolar fuzzy set in X, if $A^+ : X \to [0, 1]$ and $A^- : X \to [-1, 0]$ are mappings. For each $x \in X$, we use the positive membership degree A^+ to denote the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set A and the negative membership degree A^- to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bipolar fuzzy set A. The empty bipolar fuzzy set is denoted by $0_{bp} = (0^+, 0^-)$ and defined by $0^+(x) = 0 = 0^-(x)$, for all $x \in X$. The whole bipolar fuzzy set is denoted by $1_{bp} = (1^+, 1^-)$ and defined by $1^+(x) = 1$ and $1^-(x) = -1$, for all $x \in X$.

We provide some definitions and results from the literature listed in the references, which will be used in this article.

Definition 2.1. [8] Let X be a nonempty set and let A, B be two bipolar fuzzy sets in X.

(i) We say that A is subset of B, denoted by $A \subset B$, if for each $x \in X$,

$$A^+(x) \leq B^+(x) \text{ and } A^-(x) \geq B^-(x).$$

(ii) We say that A is equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of A, denoted by $A^c = ((A^c)^+, (A^c)^-)$, is a bipolar fuzzy set in X defined as: for each $xinX$, $A^c(x) = (1 - A^+(x), -1 - A^-(x))$, i.e.,

$$(A^c)^+(x) = 1 - A^+(x),\ (A^c)^-(x) = -1 - A^-(x).$$

(iv) The intersection of A and B, denoted by $A \cap B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cap B)(x) = (A^+(x) \wedge B^+(x), A^-(x) \vee B^-(x)).$$

(v) The union of A and B, denoted by $A \cup B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cup B)(x) = (A^+(x) \vee B^+(x), A^-(x) \wedge B^-(x)).$$

Definition 2.2. Let X be a nonempty set and let $\{A_i : i \in I\}$ be a family of subsets of X.

(i) The intersection of $\{A_i : i \in I\}$, denoted by $\bigcap_{i\in I} A_i$ is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcap_{i\in I} A_i)(x) = (\bigwedge_{i\in I} A_i^+(x), \bigvee_{i\in I} A_i^-(x)).$$

(ii) The union of $\{A_i : i \in I\}$, denoted by $\bigcup_{i\in I} A_i$ is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcup_{i\in I} A_i)(x) = (\bigvee_{i\in I} A_i^+(x), \bigwedge_{i\in I} A_i^-(x)).$$

Definition 2.3. [9] Let X be a nonempty set and let τ be a collection of bipolar fuzzy sets of X. Then τ is called a bipolar fuzzy topology on X, if it satisfies the following axioms:

(i) $0_{bp}, 1_{bp} \in \tau$.

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$.

(iii) If $\{A_i : i \in I\} \subset \tau$, then $\bigcup_{i\in I} A_i \in \tau$.

In this case, the pair (X, τ) is called a bipolar fuzzy topological space and each member of τ is called an open bipolar fuzzy set in X. The complement of an open bipolar fuzzy set is called closed bipolar fuzzy set.

Definition 2.4. [9] Let X and Y be two nonempty sets, let $A \subset X$ and $B \subset Y$ and let $f : X \to Y$ be a mapping. Then

(i) The image of A under f, denoted by $f(A) = (f(A^+), f(A^-))$, is a bipolar fuzzy set in Y defined as follows: for each $y \in Y$,

$$f(A^+)(y) = \begin{cases} \vee_{x\in f^{-1}(y)} A^+(x), & if \quad f^{-1}(y) \neq \varnothing\,; \\ 0, & otherwise. \end{cases}$$

and

$$f(A^-)(y) = \begin{cases} \wedge_{x\in f^{-1}(y)} A^-(x), & if \quad f^{-1}(y) \neq \varnothing\,; \\ 0, & otherwise. \end{cases}$$

(ii) The preimage of B under f, denoted by $f^{-1}(B) = (f^{-1}(B^+), f^{-1}(B^-))$, is a bipolar fuzzy set in X defined as follows: for each $x \in X$,

$$[f^{-1}(B^+)](x) = B^+ \circ f(x) \text{ and } [f^{-1}(B^-)](x) = B^- \circ f(x).$$

Definition 2.5. [9] Let (X, τ_1), (Y, τ_2) be two bipolar fuzzy topological spaces. Then a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous, if $f^{-1}(V) \in \tau_1$, for each $V \in \tau_2$.

3 Covering property

Definition 3.1. [9] Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{A}$ be a collection of bipolar fuzzy sets of X,

(i) $\mathcal{A}$ is called a cover of X, if $1_{bp} \subset \bigcup \mathcal{A}$, i.e.,

$$(\bigcup \mathcal{A})^+(x) = 1 \text{ and } (\bigcup \mathcal{A})^-(x) = -1 \ \forall \ x \in X.$$

(ii) Let $\mathcal{A}$ be a cover of X. Then $\mathcal{B} \subset \mathcal{A}$ is a subcover of $\mathcal{A}$, if $\mathcal{B}$ is a cover of X.

(iii) $\mathcal{A}$ is called an open cover of X, if $\mathcal{A}$ is a cover of X and $\mathcal{A} \subset \tau$.

Definition 3.2. [9] A bipolar fuzzy topological space (X, τ) is said to be compact, if each open cover of X has a finite subcover.

Definition 3.3. A bipolar fuzzy topological space (X, τ) is said to be countably compact if and only if every countable open cover of the space has a finite subcover.

Definition 3.4. A bipolar fuzzy topological space (X, τ) is said to be second countable, if there exists a countable subfamily $\mathcal{B}$ of τ such that each member of τ can be expressed as the union of some members of $\mathcal{B}$.

Theorem 3.1. *If a bipolar fuzzy topological space (X, τ) is second countable, then the following properties are equivalent:*

(i) X is compact.

(ii) X is countably compact.

Proof. $(i) \implies (ii)$ Let X be a compact bipolar fuzzy topological space, then clearly it is countable compact.
$(ii) \implies (i)$ Let $\mathcal{A} = \{A_i, i \in I\}$ be any open cover of X. Let $\mathcal{B} = \{B_n, n \in N\}$ be a countable base of (X, τ), since X is second countable. For each $A_i \in \mathcal{A}$,

$$A_i = \bigcup_{k=1}^{i_0} B_{i_k},$$

where i_0 may be ∞. Let $\mathcal{B}_0 = \{B_{i_k}, i \in I, 1 \leq k \leq i_0\}$ form a countable collection of open subsets of X. Also, $\mathcal{B}_0$ is open cover of X. There exists $\mathcal{B}_1 \subset \mathcal{B}_0$ a finite subcover of X, since X is countable compact. Every member of $\mathcal{B}_1$ is contained in some member A_i, these A_i's form a finite subfamily of $\mathcal{A}$ and is a cover of X. □

Given a cover $\mathcal{A} = \{A_i, i \in I\}$ of X, this implies that $sup_{i \in I}\{A_i^+(x)\} = 1$ and $inf_{i \in I}\{A_i^-(x)\} = -1$, for all $x \in X$. Therefore, for any $0 < \varepsilon < 1$, and for any $x \in X$, there exist two bipolar fuzzy sets A_p and A_q such that $A_p^+(x) \geq 1 - \varepsilon$ and $A_q^-(x) \leq -1 + \varepsilon$. At each point $x \in X$, select two such A_i's, say A_p and A_q. Let $\Gamma_{p,\varepsilon}$ and $\Gamma_{q,\varepsilon}$ be the sets of all points x such that $A_p^+(x) \geq 1 - \varepsilon$ and $A_q^-(x) \leq -1 + \varepsilon$ respectively. Let $\Gamma_{p,q,\varepsilon} = \Gamma_{p,\varepsilon} \cap \Gamma_{q,\varepsilon}$. For a fixed ε, $\{\Gamma_{p,q,\varepsilon}\}$ is a partition of space X, which is called ε-partition by $\mathcal{A}$. Note that every partition of the space X depends on the initial choice of A_i's.

Also, for any $x \in X$, there exist two bipolar fuzzy subsets A_p and A_q, such that $A_p^+(x) = 1$ and $A_q^-(x) = -1$; then the sets of all points x with the same A_p and A_q are denoted by $\Gamma_{p,0}$ and $\Gamma_{q,0}$ respectively and $\Gamma_{p,q,0} = \Gamma_{q,0} \cap \Gamma_{q,0}$; then $\{\Gamma_{p,q,0}\}$ forms a partition of X. $\mathcal{A}$ is then said to have a 0-partition of X.

Theorem 3.2. *A bipolar fuzzy topological space (X, τ) is compact if and only if there exists a finite 0-partition for each open cover of X.*

Proof. Let $\mathcal{A} = \{A_i, i \in I\}$ be an open cover of X. Then there exists a finite subcover $\mathcal{A}_0 = \{A_k\}, k = 1, 2, ...n$. Since $max(A_1^+(x), A_2^+(x), ...A_n^+(x)) = 1$ and $min(A_1^-(x), A_2^-(x), ...A_n^-(x)) = -1$, for all $x \in X$, then a 0-partition can be constructed from $\mathcal{A}_0$. This 0-partition is finite, because $\mathcal{A}_0$ is a finite subcover. The 0-partition by $\mathcal{A}_0$ is also a 0-partition by $\mathcal{A}$, because $\mathcal{A}_0$ is a subfamily of $\mathcal{A}$. So there exists a finite 0-partition for each open cover of X.

Conversely, let $\mathcal{A}$ has a finite 0-partition $\{\Gamma_{p,q,0}\}$, $p = 1, 2, ..., n$ and $q = 1, 2, ..., m$. Let A_p and A_q be the fuzzy sets defining $\{\Gamma_{p,q,0}\}$. Clearly, $\{A_p, A_q\}$, $p = 1, 2, ..., n, q = 1, 2, ..., m$ is a finite sub-cover of $\mathcal{A}$. □

In view of the above theorems, we state the following results, for which we omit the proof.

Theorem 3.3. *A bipolar fuzzy topological space (X, τ) is countably compact if and only if there exists a finite 0-partition for each countable open cover of X.*

A consequence of Theorem 3.2 and Theorem 3.3 is the following result.

Corollary 3.4. *If there exists a point $x \in X$ and an open cover (countable open cover) $\mathcal{A}$ of X such that $A_i^+(x) < 1$ or $A_i^-(x) > -1$, for all $A_i \in \mathcal{A}$, then (X, τ) is not compact (countably compact).*

Definition 3.5. A bipolar fuzzy topological space (X, τ) is Lindelöf if and only if there exists countable subcover for every open cover of X.

Theorem 3.5. *Every second countable bipolar fuzzy topological space (X, τ) is Lindelöf.*

Proof. Let $\mathcal{A} = \{A_i, i \in I\}$ be any open cover of X. Let $\mathcal{B} = \{B_n, n \in N\}$ be a countable basis of X. Since X is second countable, each A_i is the union of members of $\mathcal{B}$. Let

$$A_i = \bigcup_{k=1}^{i_0} B_{i_k},$$

where i_0 may be infinite. Let $\mathcal{B}_0 = \{B_{i_k}, i \in I, 1 \leq k \leq i_0\}$, $\mathcal{B}_0$, which forms an open cover for X. $\mathcal{B}_0$ is a subfamily of $\mathcal{B}$; then it is also countable. By the construction, every subset of $\mathcal{B}_0$ is contained in some subsets A_i of $\mathcal{A}$. Hence the collection of these A_i's is a countable subcover of $\mathcal{A}$. □

Theorem 3.6. *A bipolar fuzzy topological space (X, τ) is Lindelöf if and only if each open cover has a countable ε-partition of X, for all ε such that $0 < \varepsilon < 1$.*

Proof. Let $\mathcal{A}$ be an open cover of X. Then there exists a countable subcover $\mathcal{A}_0 = \{A_k, k = 1, 2, 3...\}$ of $\mathcal{A}$, since X is Lindelöf. For each $0 < \varepsilon < 1$, an ε-partition can be constructed from $\mathcal{A}_0$; this ε-partition is countable because $\mathcal{A}_0$ is a countable subcover. The ε-partition by $\mathcal{A}_0$ is also a ε-partition by $\mathcal{A}$, because $\mathcal{A}_0$ is a subfamily of $\mathcal{A}$. So, for all $0 < \varepsilon < 1$, there exists a countable ε-partition for each open cover of X.

Conversely, let $\mathcal{A}$ be an open cover of X. For each $0 < \varepsilon < 1$, suppose that $\{\Gamma_{p,q,\varepsilon}, i \in I(\varepsilon)\}$ is a countable ε-partition by $\mathcal{A}$ of X. Let $\Gamma_{p,q,\varepsilon} = \Gamma_{p,\varepsilon} \cap \Gamma_{q,\varepsilon}$ and $A_{p,\varepsilon}$ and $A_{q,\varepsilon}$ are members of $\mathcal{A}$. Let $\varepsilon = 1/n, n = 2, 3, ...$ Then the family $\{\{A_{p,\varepsilon}, A_{q,\varepsilon}\} : p, q \in I(\varepsilon); \varepsilon = 1/n, n = 2, 3, ...\}$ of bipolar fuzzy sets forms a countable subcover of $\mathcal{A}$. □

Theorem 3.7. *Let f be a continuous mapping from a Lindelöf bipolar fuzzy space X onto a bipolar fuzzy space Y. Then Y is Lindelöf.*

Proof. Let $\mathcal{B} = \{B_i : i \in I\}$ be an open cover of Y. Now $\mathcal{A} = \{f^{-1}(B_i) : B_i \in \mathcal{B}\}$ is a family of open sets of X, since f is continuous function. Then

$$(\bigcup \mathcal{A})^+(x) = \bigvee_{i\in I} f^{-1}(B_i)^+(x) = \bigvee_{i\in I} B_i^+(f(x)) = 1$$

and

$$(\bigcup \mathcal{A})^-(x) = \bigwedge_{i\in I} f^{-1}(B_i)^-(x) = \bigwedge_{i\in I} B_i^-(f(x)) = -1,$$

for each $x \in X$. Then clearly, $\mathcal{A}$ is an open cover of X. Since (X, τ) is Lindelöf, there exists a countable subcover $\mathcal{A}_0 = \{f^{-1}(B_i) : B_i \in \mathcal{B}, i = 1, 2, ...\}$ of X. But the fact that f is onto implies that $f[f^{-1}(B_i)] = B_i$. Hence the collection of images of members of $\mathcal{A}_0$ is a countable subfamily of $\mathcal{B}$ which also cover Y. Consequently, Y is Lindelöf. □

Theorem 3.8. *Let f be a continuous mapping from a countably compact bipolar fuzzy space X onto a bipolar fuzzy space Y. Then Y is countably compact.*

Proof. The proof is similar to the previous theorem. □

Now we discuss an example of bipolar fuzzy topological space, to illustrate the various concepts.

Example 3.6. Let X have a general topology μ. Let $A = (A^+, A^-)$ be a bipolar fuzzy set of X. Let $\Gamma_{A^+,\alpha} = \{x : A^+(x) > \alpha, \alpha \in [0, 1)\}$ and let $\Gamma_{A^-,\beta} = \{x : A^-(x) < \beta, \beta \in (-1, 0]\}$. Therefore, $\Gamma_{A^+,\alpha}$ and $\Gamma_{A^-,\beta}$ are ordinary subsets of X. Consider the collection τ of bipolar fuzzy sets $A = (A^+, A^-)$ of X such that for each $\alpha \in [0, 1)$ and $\beta \in (-1, 0]$, $\Gamma_{A^+,\alpha}$ and $\Gamma_{A^-,\beta}$ are open in the topology μ. So, (X, τ) is a bipolar fuzzy topological space. Obviously $0_{bp}, 1_{bp} \in \tau$. If $A = (A^+, A^-), B = (B^+, B^-) \in \tau$, then $C = A \cap B$, i.e., $C = (A^+ \cap B^+, A^- \cup B^-)$ is also a member of τ since $C^+(x) = min(A^+(x), B^+(x))$ and $C^-(x) = max(A^-(x), B^-(x))$ implies $\Gamma_{C^+,\alpha} = \Gamma_{A^+,\alpha} \cap \Gamma_{B^+,\alpha}$, for each $\alpha \in [0, 1)$ and $\Gamma_{C^-,\beta} = \Gamma_{A^-,\beta} \cap \Gamma_{B^-,\beta}$, for each $\beta \in (-1, 0]$, respectively.

Let $\{A_i : i \in I\}$ be a family of bipolar fuzzy open sets in (X, τ); then $\bigcup_{i \in I} A_i$ is a bipolar fuzzy open set in (X, τ), since

$$\left(\bigcup_{i \in I} A_i\right)^+ (x) = sup_{i \in I}\{A_i^+(x)\}$$

and

$$\left(\bigcup_{i \in I} A_i\right)^- (x) = inf_{i \in I}\{A_i^-(x)\}$$

imply

$$\Gamma_{\left(\bigcup_{i \in I} A_i\right)^+, \alpha} = \bigcup_{i \in I} \Gamma_{A_i^+, \alpha},$$

for each $\alpha \in [0, 1)$ and

$$\Gamma_{\left(\bigcup_{i \in I} A_i\right)^-, \beta} = \bigcup_{i \in I} \Gamma_{A_i^-, \beta},$$

for each $\beta \in (-1, 0]$ respectively. A bipolar fuzzy set $A = (A^+, A^-)$ in X is open if A^+ and A^- are continuous functions.

Let $A_x = (A_x^+, A_x^-)$ be a bipolar fuzzy set with continuous A_x^+ and A_x^-, such that

$$A_x^+ = \begin{cases} 1, & for \quad y = x; \\ < 1, & for \quad y \neq x. \end{cases}$$

and

$$A_x^- = \begin{cases} -1, & for \quad y = x; \\ > -1, & for \quad y \neq x. \end{cases}$$

Therefore, the collection $\{A_x : x \in X\}$ is an open cover of (X, τ). If X is not a finite set, then this collection contains no finite subcover. Hence (X, τ) is not compact.

Let $\{B_n : n \in N\}$ be a countable collection of disjoint subsets of X, which covers X. Consider the collection of bipolar fuzzy sets $\{A_n : n \in N\}$ with continuous mappings A_n^+ and A_n^- such that

$$A_n^+ = \begin{cases} 1, & for \quad y \in B_n; \\ < 1, & for \quad y \notin B_n. \end{cases}$$

and

$$A_n^- = \begin{cases} -1, & for \quad y \in B_n; \\ > -1, & for \quad y \notin B_n. \end{cases}$$

Clearly, the countable collection $\{A_n : n \in N\}$ is an open cover of (X, τ). If all B_n's are nonempty, then A_n has no finite subcover. Hence (X, τ) is not countably compact.

4 Product bipolar fuzzy topology

Definition 4.1. [9] Let (X, τ) be a bipolar fuzzy topological space. A collection $\mathcal{B} \subset \tau$ is called a base for τ if all the members of τ can be expressed as the union of some members of $\mathcal{B}$.

Definition 4.2. [9] Let (X, τ) be a bipolar fuzzy topological space. A collection $\mathcal{S} \subset \tau$ is called a subbase for τ, if the family of all finite intersections of members of $\mathcal{S}$ forms a base for τ.

Let $\{X_i : i \in I\}$ be a family of spaces, and let $X = \prod_{i\in I} X_i$ be the Cartesian product of this family; then X is the usual product space of this family. Let the projection mapping P_i from X onto X_i and let each X_i be a bipolar fuzzy topological spaces, with bipolar fuzzy topology τ_i; let $B_i \in \tau_i$. Then $P_i^{-1}[B_i]$ is a bipolar fuzzy set of X. Let $\mathcal{S}_i$ denote the collection $\mathcal{S}_i = \{P_i^{-1}(B_i) : B_i \in \tau_i\}$. Now the family $\mathcal{S} = \bigcup\limits_{i\in I} \mathcal{S}_i$ of bipolar fuzzy sets is used to generate a bipolar fuzzy topology τ for X in the following manner: let the family of all finite intersections of members of $\mathcal{S}$ is $\mathcal{B}$ and the family of all unions of members of $\mathcal{B}$ is τ. So τ is a bipolar fuzzy topology for X, with $\mathcal{B}$ as a base and $\mathcal{S}$ a subbase.

Definition 4.3. Given a family of bipolar fuzzy topological spaces $\{(X_i, \tau_i) : i \in I\}$, the fuzzy topology τ defined as above is a product bipolar fuzzy topology for $X = \prod_{i\in I} X_i$, and (X, τ) is product bipolar fuzzy topological spaces.

Theorem 4.1. *Let (X, τ) be the product bipolar fuzzy topological space of the family of bipolar fuzzy topological spaces $\{(X_i, \tau_i) : i \in I\}$. Then:*

(i) the projection P_i is continuous for each $i \in I$;

(ii) the product bipolar fuzzy topology is the smallest bipolar fuzzy topology for X such that (i) is true;

(iii) let f be a function from Y to X and let (Y, σ) be a bipolar fuzzy topological space; then f is continuous if and only if for every $i \in I, P_i \circ f$ is F-continuous.

Proof. The claims (i) and (ii) can be easily verified by using the definition of product bipolar fuzzy topology. We prove claim (iii). Let f be continuous; P_i be also continuous, since P_i is projection map from X onto X_i. Therefore for every $i \in I$, $P_i \circ f$ is continuous.

Conversely, let $B \in \tau_i$. Then $(P_i \circ f)^{-1}[B] = (f^{-1} \circ P_i^{-1})[B]$ is $\sigma-$open, since for every $i \in I$, $P_i \circ f$ is continuous. Therefore, the family $\{f^{-1}[P_i^{-1}[B]], B \in \tau_i, i \in I\}$ contains $\sigma-$open bipolar fuzzy sets in Y. We know that the union of finite intersection of members of the collection $\{P_i^{-1}[B], B \in \tau_i, i \in I\}$ is a member of τ, and f^{-1} preserves union and intersection, since f is continuous. This implies that f^{-1} maps τ-open bipolar fuzzy sets onto σ-open bipolar fuzzy sets. So f is continuous. □

Theorem 4.2. *The product bipolar fuzzy topological space (X, τ) is second countable, if the countable family $\{(X_n, \tau_n) : n \in N\}$ of bipolar fuzzy topological spaces is second countable.*

Proof. Let $\mathcal{B}_n$ be a countable base for τ_n (this exists, since τ_n is second countable). Let $\mathcal{S} = \{P_n^{-1}[B] : B \in \mathcal{B}_n, n \in N\}$ and let the family of all finite intersection of members of $\mathcal{S}$ be $\mathcal{B}$. Therefore $\mathcal{B}$ us a countable family of τ. We show that the base for τ is $\mathcal{B}$. Let $U \in \tau$, then U is the union of open bipolar fuzzy sets of the form $\bigcap\limits_{i=1}^{k} P_{n_i}^{-1}[A_i]$, where $A_i \in \tau_{n_i}$ by the definition of the product bipolar fuzzy topology. Since $\mathcal{B}_{n_i}$ is a base for τ_{n_i},

$$A_i = \bigcup_{j_i \in J_i} B_{j_i}, \quad B_{j_i} \in \mathcal{B}_{n_i},$$

Consequently, let

$$P_{n_i}^{-1}[A_i] = P_{n_i}^{-1}\left[\bigcup_{j_i \in J_i} B_{j_i}\right] = \bigcup_{j_i \in J_i} P_{n_i}^{-1}[B_{j_i}].$$

Therefore,

$$\bigcap_{i=1}^{k} P_{n_i}^{-1}[A_i] = \bigcap_{i=1}^{k} \bigcup_{j_i \in J_i} P_{n_i}^{-1}[B_{j_i}]$$

is the union of finite intersections of members of $\mathcal{S}$. So $\mathcal{B}$ is a base for τ. □

Acknowledgements. The first author is grateful to the University Grants Commission, New Delhi for providing financial support in the form Junior Research Fellowship through Grant No. F.16-6(DEC.2016)/2017(NET)(404363) to complete this study.

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