Results for multivalued mappings for Kannan type contractions in ordered ρ_q- metric spaces

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Abstract. The ambition of this work is to obtain fixed point results for a pair of multivalued mappings on the intersection of a sequence and an open ball in left (right) ρ_q-sequentially complete ordered ρ_q- metric spaces. A helpful example has been worked to highlight the importance of the established results. These results broaden the conceptual results of Altun et al. (J. Funct. Spaces, Article ID 6759320, 2016).

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1 Introduction

The outstanding result called Banach Contraction principle was assembled by the French mathematician Stefan Banach, which is a very significant result of metric fixed point theory. A huge amount of literature affects with the extension of this remarkable theorem. A point y is said to be a fixed point of single-valued mapping H, if it's image under H remains the same y. If the image of a point x under some multivalued mapping F contains the same point x, then x will be the fixed point of F. The first very important theorem for fixed point of multivalued mappings was developed by Nadler. Nadler's theorem for multivalued contraction has been modified in different directions, see [5, 14, 15, 16, 17, 18].

A ρ_q- metric is a function which satisfies only one and a half condition. This class of functions is wider than the class of functions like quasi-partial metric [8, 10], dislocated metric, partial metric, quasi metric and metric functions. For more details on ρ_q- metric spaces, see [4, 6, 13, 23, 24].

The fixed point results related to partial order was first developed by Ran and Reurings [12] and Nieto et al. [11]. For more outcomes with order relation see [2, 3, 6, 7, 9]. A new methodology and constraints to obtain common fixed were introduced by Altun et al. [1]. Another type of generalization the fixed point results was given by Arshad et al. [3]. They weakened the contractive condition and imposed it only on closed ball. For more outcomes, see [16, 17, 19, 20, 21, 22].

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In this paper, we broaden the aftereffect of Altun et al. [1] in five unique ways by utilizing

(i) set-valued functions rather than single-valued functions;
(ii) open balls rather than whole spaces or closed balls;
(iii) Kannan type contractions rather than Banach type contractions;
(iv) ρ_q-metric spaces rather than complete metric spaces.
(v)) a generalized restriction of order.

Our outcome bind together, broadens and sum up a few tantamount outcomes in the current writing. We give the accompanying definitions and results which will be helpful to comprehend the paper.

Definition 1.1. [15] Let $\Im \neq \{\}$ and $\rho_q : \Im \times \Im \to [0, \infty)$ be a function. If the following satisfy for all $l, n, z \in \Im$:

(i) $\rho_q(l, n) \leq \rho_q(l, z) + \rho_q(z, n)$;
(ii) if$\rho_q(l, n) = \rho_q(n, l) = 0$, then $l = n$.

Then ρ_q is called a ρ_q- metric. The pair $(\Im, \rho_q)$ is called a ρ_q- metric space. For $l \in \Im$ and $\varepsilon > 0$, $\overline{B_{\rho_q}(l, \varepsilon)} = \{y \in \Im : \rho_q(l, y) \leq \varepsilon$ and $\rho_q(y, l) \leq \varepsilon\}$ and $B_{\rho_q}(l, \varepsilon) = \{y \in \Im : \rho_q(l, y) < \varepsilon$ and$\rho_q(y, l) < \varepsilon\}$ are closed ball and open ball in $(\Im, \rho_q)$ respectively.

Example 1.2. [15] Let $\Im = [0, \infty)$ and$\rho_q(l, y) = \max\{l, y\} + l$ for any $l, y \in \Im$. Then, $(\Im, \rho_q)$ is a ρ_q- metric space, because ρ_q satisfies (i) and (ii) of Definition 1.1.

Definition 1.3. [15] Let $(\Im, \rho_q)$ be a ρ_q- metric space and $\{l_n\}$ be a sequence in $(\Im, \rho_q)$. Then
(i) $\{l_n\}$ is said to be left (respectively right) ρ_q-Cauchy if $\forall\, \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n > m \geq n_0$ (respectively $\forall\, m > n \geq n_0$), $\rho_q(l_m, l_n) < \varepsilon$.
(ii) $\{l_n\}$ is said to be ρ_q-converges to l if $\lim\limits_{n\to\infty} \rho_q(l_n, l) = \lim\limits_{n\to\infty} \rho_q(l, l_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $\rho_q(l, l_n) < \varepsilon$ and $\rho_q(l_n, l) < \varepsilon$. In this case l is called a ρ_q-limit of $\{l_n\}$.
(iii) If every right (respectively left) ρ_q-Cauchy sequence in $\Im$, ρ_q-converges to a point $l \in \Im$ and $\rho_q(l, l) = 0$, then $(\Im, \rho_q)$ is said to be right (respectively left) ρ_q-sequentially complete.

Definition 1.4. [15] $(\Im, \preceq, \rho_q)$ is know as an ordered ρ_q- metric space, if $\preceq$ is a partial order relation on $\Im$ and $(\Im, \rho_q)$ is ρ_q- metric space.

Definition 1.5. [15] Let $(\Im, \rho_q)$ be a ρ_q- metric space. Let $A \neq \{\}$ and let $l \in \Im$. An element $y_0 \in A$ is called a best approximation in A if

$$\begin{aligned} \rho_q(l, A) &= \rho_q(l, y_0),\ \text{where } \rho_q(l, A) = \inf_{y \in A} \rho_q(l, y) \\ \text{and } \rho_q(A, l) &= \rho_q(y_0, l),\ \text{where } \rho_q(A, l) = \inf_{y \in A} \rho_q(y, l). \end{aligned}$$

If each $l \in \Im$ has a best approximation in A, then A is know as proximinal set. $P(\Im)$ is equal to the set of all proximinal subsets of $\Im$.

Definition 1.6. [15] The function $H_{\rho_q} : P(\Im) \times P(\Im) \to [0, \infty)$, defined by

$$H_{\rho_q}(K, M) = \max\left\{\sup_{b \in M} \rho_q(K, b), \sup_{a \in K} \rho_q(a, M)\right\}$$

is known as ρ_q- Hausdorff metric on $P(\Im)$. The pair $(P(\Im), H_{\rho_q})$ is called ρ_q- Hausdorff metric space.

Lemma 1.1. *[15] Let $(\Im, \rho_q)$ be a ρ_q- metric space. Let $(P(\Im), H_{\rho_q})$ be the ρ_q-Hausdorff metric space on $P(\Im)$. Then, for all $K, M \in P(\Im)$ and for each $a \in K$, there exists $b_a \in M$, such that $H_{\rho_q}(K, M) \geq \rho_q(a, b_a)$ and $H_{\rho_q}(M, K) \geq \rho_q(b_a, a)$.*

Lemma 1.2. *[15] Let Y be a closed set in a right (left) ρ_q-sequentially complete ρ_q-metric space $(\Im, \rho_q)$. Then Y is also right (left) ρ_q-sequentially complete under ρ_q-metric.*

2 Main Result

Let $(\Im, \rho_q)$ be a ρ_q- metric space, $l_0 \in \Im$ and $T : \Im \to P(\Im)$ be a multivalued mapping on $\Im$. As Tl_0 is a proximinal set, then there exists $l_1 \in Tx_0$ such that $\rho_q(l_0, Tl_0) = \rho_q(l_0, l_1)$ and $\rho_q(Tl_0, l_0) = \rho_q(l_1, l_0)$. Now, for $l_1 \in \Im$, there exist $l_2 \in Tl_1$ be such that $\rho_q(l_1, Tl_1) = \rho_q(l_1, l_2)$ and $\rho_q(Tl_1, l_1) = \rho_q(l_2, l_1)$. By induction, we obtain a sequence l_n, which satisfies $l_{n+1} \in Tl_n$, $\rho_q(l_n, Tl_n) = \rho_q(l_n, l_{n+1})$ and $\rho_q(Tl_n, l_n) = \rho_q(l_{n+1}, l_n)$. We denote this iterative sequence $\{\Im T(l_n)\}$ and say that $\{\Im T(l_n)\}$ is a sequence in $\Im$ generated by l_0.

Theorem 2.1. *Let $(\Im, \preceq, \rho_q)$ be an ordered left (right) ρ_q-sequentially complete ρ_q-metric space, $r > 0$, $l_0 \in B_{\rho_q}(l_0, r)$, and $S, T : \Im \to P(\Im)$ be two multivalued mappings on $B_{\rho_q}(x_0, r)$. Suppose that, we have the following:*

(i)

$$H_{\rho q}(Tl, Ty) \leq t(\rho_q(l, Tl) + \rho_q(y, Ty)),$$

for all $l, y \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}$ with $(l, y) \in B(S)$, where

$$B(S) = \{(l, y) : l \succeq Sl,\ y \preceq Sy,\ \text{or } l \preceq Sl,\ y \succeq Sy\} \text{ and } t \in [0, \frac{1}{2}).$$

(ii) If $l \in B_{\rho_q}(l_0, r)$, $\rho_q(l, Tl) = \rho_q(l, y)$ and $\rho_q(Tl, l) = \rho_q(y, l)$ then

$$(a) \text{ If } l \preceq Sl \Longrightarrow y \succeq Sy \ (b) \text{ If } l \succeq Sl \Longrightarrow\ y \preceq Sy.$$

(iii) The set $G(S) = \left\{l : l \preceq Sl \text{ and } l \in B_{\rho_q}(l_0, r)\right\}$ is closed and contains l_0.

(iv)

$$\max\{\rho_q(l_1, l_0), \rho_q(l_0, l_1)\} < (1 - \mu)r,$$

where $\mu = \frac{t}{1-t}$. Then the subsequence $\{l_{2n}\}$ of $\{\Im T(l_n)\}$ is a sequence in $G(S)$ and $\{l_{2n}\} \to l^* \in G(S)$ and $\rho_q(l^*, l^*) = 0$. Also, if the inequality (i) holds for $l, y \in \{l^*\}$. Then S and T have a common fixed point l^* in $B_{\rho_q}(l_0, r)$.

Proof. As l_0 be an element of $G(S)$, from condition (iii)

$$l_0 \preceq Sl_0.$$

Consider the sequence $\{\Im T(l_n)\}$, then there exists $l_1 \in Tl_0$ such that

$$\rho_q\left(l_0, Tl_0\right) = \rho_q\left(l_0, l_1\right) \text{ and } \rho_q\left(Tl_0, l_0\right) = \rho_q\left(l_1, l_0\right).$$

From condition (ii)

$$l_1 \succeq Sl_1.$$

Now, by (iv)

$$\max\{\rho_q(l_1,l_0),\rho_q(l_0,l_1)\} < (1-\mu)r < r.$$

It follows that, $\rho_q(l_1,l_0) < r$ and $\rho_q(l_0,l_1) < r$. So, we have $l_1 \in B_{\rho_q}(l_0,r)$. Also,

$$\rho_q(l_1,Tl_1) = \rho_q(l_1,l_2) \text{ and } \rho_q(Tl_1,l_1) = \rho_q(l_2,l_1).$$

As $l_1 \succeq Sl_1$, so from condition (ii), we have $l_2 \preceq Sl_2$, by triangular inequality, we have

$$\rho_q(l_0,l_2) \le \rho_q(l_0,l_1) + \rho_q(l_1,l_2)$$

$$\rho_q(l_0,l_2) \le \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\} + \max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\}. \tag{2.1}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_1,l_2) \le H_{\rho q}(Tl_0,Tl_1).$$

As $l_0,l_1 \in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\}$, $l_1 \succeq Sl_1$ and $l_0 \preceq Sl_0$, then by (i), we have

$$\rho_q(l_1,l_2) \le t(\rho_q(l_0,Tl_0) + \rho_q(l_1,Tl_1))$$

$$\rho_q(l_1,l_2) \le t\begin{pmatrix} \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}+ \\ \max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\}\end{pmatrix}. \tag{2.2}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_2,l_1) \le H_{\rho q}(Tl_1,Tl_0).$$

As $l_0,l_1 \in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\}$, $l_1 \succeq Sl_1$ and $l_0 \preceq Sl_0$, then by (i), we have

$$\rho_q(l_2,l_1) \le t\begin{pmatrix} \max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\}+ \\ \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}\end{pmatrix}. \tag{2.3}$$

From inequalities (2.2) and (2.3), we have

$$\max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\} \le t(\max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\} + \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\})$$

$$\max\{\rho_q(l_1,l_2),\rho_q(l_2,l_1)\} \le \mu(\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}). \tag{2.4}$$

Using (2.4) in (2.1), we have

$$\begin{aligned}\rho_q(l_0,l_2) &\le \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\} + \mu(\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}) \\ &\le (1+\mu)\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\} \\ &< (1+\mu)(1-\mu)r < r.\end{aligned}$$

Now,

$$\begin{aligned}\rho_q(l_2,l_0) &\leq \rho_q(l_2,l_1)+\rho_q(l_1,l_0)\\ &\leq \max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}+\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}\\ &\leq \mu(\max\{\rho_q(l_1,l_0),\rho_q(l_0,l_1)\})+\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\} \qquad \text{using (2.4)}\\ &< (1+\mu)(1-\mu)r<r.\end{aligned}$$

It follows that, $\rho_q(l_2,l_0)<r$ and $\rho_q(l_0,l_2)<r$, so, we have $l_2\in B_{\rho_q}(l_0,r)$. Also,

$$\rho_q(l_2,Tl_2)=\rho_q(l_2,l_3) \text{ and } \rho_q(Tl_2,l_2)=\rho_q(l_3,l_2).$$

As $l_2\preceq Sl_2$, so from condition (ii), we have $l_3\succeq Sl_3$. By triangle inequality, we have

$$\begin{aligned}\rho_q(l_0,l_3) &\leq \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}+\max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}\\ &\quad +\max\{\rho_q(l_2,l_3),\rho_q(l_3,l_2)\}.\end{aligned} \tag{2.5}$$

As $l_1,l_2\in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\}$, $l_1\succeq Sl_1$ and $l_2\preceq Sl_2$, then by (i), we have

$$\rho_q(l_2,l_3)\leq t\begin{pmatrix}\max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}+\\ \max\{\rho_q(l_2,l_3),\rho_q(l_3,l_2)\}\end{pmatrix}. \tag{2.6}$$

As $l_1,l_2\in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\}$, $l_1\succeq Sl_1$ and $l_2\preceq Sl_2$, then by (i), we have

$$\rho_q(l_3,l_2)\leq t\begin{pmatrix}\max\{\rho_q(l_2,l_3),\rho_q(l_3,l_2)\}+\\ \max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}\end{pmatrix}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\max\{\rho_q(l_2,l_3),\rho_q(l_3,l_2)\}\leq\mu(\max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}). \tag{2.8}$$

Using (2.4) and (2.8) in (2.5) , we have

$$\begin{aligned}\rho_q(l_0,l_3) &\leq (1-\mu)r+\mu(1+\mu)(1-\mu)r\\ &= (1-\mu)(1+\mu+\mu^2)r=(1-\mu^3)r<r.\end{aligned}$$

Now by triangular ineqality, we have

$$\begin{aligned}\rho_q(l_3,l_0) &\leq \max\{\rho_q(l_2,l_3),\rho_q(l_3,l_2)\}+\max\{\rho_q(l_2,l_1),\rho_q(l_1,l_2)\}\\ &\quad +\max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}.\end{aligned} \tag{2.9}$$

Using (2.4) and (2.8) in (2.9), we have

$$\rho_q(l_3,l_0)\leq(1-\mu^3)r<r.$$

It follows that, $\rho_q(l_3,l_0)<r$ and $\rho_q(l_0,l_3)<r$. So, we have $l_3\in B_{\rho_q}(l_0,r)$. Also,

$$\rho_q(l_3,Tl_3)=\rho_q(l_3,l_4) \text{ and} \rho_q(Tl_3,l_3)=\rho_q(l_4,l_3).$$

As $l_3 \succeq Sl_3$, so from condition (ii), we have $l_4 \preceq Sl_4$. Let $l_4, \cdots, l_j \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}$, $l_5 \succeq Sl_5$, $l_6 \preceq Sl_6$, $l_7 \succeq Sl_7$ up to $l_j \preceq Sl_j$ and $l_{j-1} \succeq Sl_{j-1}$ for some $j \in \mathbb{N}$, where $j = 2i$, $i = 2, 3 ... \frac{j}{2}$. Now, by Lemma 1.1, we have

$$\rho_q(l_{2i}, l_{2i+1}) \leq H_{\rho q}(Tl_{2i-1,}Tl_{2i}).$$

As $l_{2i-1}, l_{2i} \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}, l_{2i-1} \succeq Sl_{2i-1}$, $l_{2i} \preceq Sl_{2i}$, then by (i), we have

$$\rho_q(l_{2i}, l_{2i+1}) \leq t(\rho_q\,(l_{2i-1}, Tl_{2i-1}) + \rho_q\,(l_{2i}, Tl_{2i}))$$

$$\rho_q(l_{2i}, l_{2i+1}) \leq t \begin{pmatrix} \max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1})\} + \\ \max\{\rho_q(l_{2i}, l_{2i+1}), \rho_q(x_{2i+1}, x_{2i}\} \end{pmatrix}. \tag{2.10}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_{2i+1}, l_{2i}) \leq H_{\rho q}(Tl_{2i,}Tl_{2i-1}).$$

As $l_{2i-1}, l_{2i} \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}$, $l_{2i-1} \succeq Sl_{2i-1}$, $l_{2i} \preceq Sl_{2i}$, then by (i), we have

$$\rho_q(l_{2i+1}, l_{2i}) \leq t \begin{pmatrix} \max\{\rho_q(l_{2i}, l_{2i+1}), \rho_q(l_{2i+1}, l_{2i})\} + \\ \max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1})\} \end{pmatrix}. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\max\{\rho_q(l_{2i}, l_{2i+1}), \rho_q(l_{2i+1}, l_{2i})\} \leq \mu(\max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1}\}). \tag{2.12}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_{2i-1}, l_{2i}) \leq H_{\rho q}(Tl_{2i-2,}Tl_{2i-1}).$$

As $l_{2i-2}, l_{2i-1} \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}$, $l_{2i-1} \succeq Sl_{2i-1}$, $l_{2i-2} \preceq Sl_{2i-2}$, then by (i), we have

$$\rho_q(l_{2i-1}, l_{2i}) \leq t \left(\max \left\{ \begin{array}{c} \rho_q(l_{2i-2}, l_{2i-1}), \rho_q(l_{2i-1}, l_{2i-2})\} + \\ \max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1}) \end{array} \right\} \right). \tag{2.13}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_{2i}, l_{2i-1}) \leq H_{\rho q}(Tl_{2i-1,}Tl_{2i-2}).$$

As $l_{2i-2}, l_{2i-1} \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}, l_{2i-1} \succeq Sl_{2i-1}$, $l_{2i-2} \preceq Sl_{2i-2}$ then by (i), we have

$$\rho_q(l_{2i}, l_{2i-1}) \leq t \begin{pmatrix} \max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1})\} + \\ \max\{\rho_q(l_{2i-2}, l_{2i-1}), \rho_q(l_{2i-1}, l_{2i-2})\} \end{pmatrix}. \tag{2.14}$$

By combining (2.13) in (2.14), we have

$$\max\{\rho_q(l_{2i-1}, l_{2i}), \rho_q(l_{2i}, l_{2i-1})\} \leq \mu \left(\max \left\{ \begin{array}{c} \rho_q(l_{2i-2}, l_{2i-1}), \\ \rho_q(l_{2i-1}, l_{2i-2}) \end{array} \right\} \right). \tag{2.15}$$

Using (2.15) in (2.12), we have

$$\max\{\rho_q(l_{2i},l_{2i+1}),\rho_q(l_{2i+1},l_{2i})\}\leq\mu^2\left(\max\left\{\begin{array}{c}\rho_q(l_{2i-2},l_{2i-1}),\\ \rho_q(l_{2i-1},l_{2i-2})\end{array}\right\}\right). \tag{2.16}$$

Now, by Lemma 1.1 we have

$$\rho_q(l_{2i-2},l_{2i-1})\leq H_{\rho q}(Tl_{2i-3,}Tl_{2i-2}).$$

As $l_{2i-2},l_{2i-1}\in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\}$, $l_{2i-1}\succeq Sl_{2i-1}$, $l_{2i-2}\preceq Sl_{2i-2}$ then by (i), we have

$$\rho_q(l_{2i-2},l_{2i-1})\leq t\left(\begin{array}{c}\max\{\rho_q(l_{2i-3},l_{2i-2}),\rho_q(l_{2i-2},l_{2i-3})\}+\\ \max\{\rho_q(l_{2i-2},l_{2i-1}),\rho_q(l_{2i-1},l_{2i-2})\}\end{array}\right). \tag{2.17}$$

Now, by Lemma 1.1, we have

$$\rho_q(l_{2i-1},l_{2i-2})\leq H_{\rho q}(Tl_{2i-2,}Tl_{2i-3}).$$

As $l_{2i-2},l_{2i-1}\in B_{\rho_q}(l_0,r)\cap\{\Im T(l_n)\},l_{2i-1}\succeq Sl_{2i-1}$, $l_{2i-2}\preceq Sl_{2i-2}$ then by (i) we have

$$\rho_q(l_{2i-1},l_{2i-2})\leq t\left(\begin{array}{c}\max\{\rho_q(l_{2i-2},l_{2i-1}),\rho_q(l_{2i-1},l_{2i-2})\}+\\ \max\{\rho_q(l_{2i-3},l_{2i-2}),\rho_q(l_{2i-2},l_{2i-3})\}\end{array}\right). \tag{2.18}$$

From (2.17) and (2.18), we have

$$\mu^2(\max\{\rho_q(l_{2i-2},l_{2i-1}),\rho_q(l_{2i-1},l_{2i-2})\})\leq\mu^3(\max\{\rho_q(l_{2i-3},l_{2i-2}),\rho_q(l_{2i-2},l_{2i-3})\}). \tag{2.19}$$

Using (2.19) in (2.16), we have

$$\max\{\rho_q(l_{2i},l_{2i+1}),\rho_q(l_{2i+1},l_{2i})\}\leq\mu^3(\max\{\rho_q(l_{2i-3},l_{2i-2}),\rho_q(l_{2i-2},l_{2i-3})\}).$$

Continuing in this way, we get

$$\max\{\rho_q(l_{2i},l_{2i+1}),\rho_q(l_{2i+1},l_{2i})\}\leq\max\left\{\begin{array}{c}\mu^{2i}(\rho_q(l_0,l_1)),\\ \mu^{2i}(\rho_q(l_1,l_0))\end{array}\right\}. \tag{2.20}$$

Similarly, we have

$$\max\{\rho_q(l_{2i-1},l_{2i}),\rho_q(l_{2i},l_{2i-1})\}\leq\max\left\{\mu^{2i-1}(\rho_q(l_0,l_1)),\mu^{2i-1}(\rho_q(l_1,l_0))\right\}. \tag{2.21}$$

Combining the inequalities (2.20) and (2.21), we have

$$\max\{\rho_q(l_j,l_{j+1}),\rho_q(l_{j+1},l_j)\}\leq\max\left\{\mu^j(\rho_q(l_0,l_1)),\mu^j(\rho_q(l_1,l_0))\right\}. \tag{2.22}$$

By using inequalities (2.22), (iv) and triangle inequality, we have

$$\begin{aligned}\rho_q(l_0,l_{j+1}) &\leq \rho_q(l_0,l_1)+\dots\dots\dots.+\rho_q(l_j,l_{j+1})\\ &\leq \max\{\rho_q(l_0,l_1),\rho_q(l_1,l_0)\}+\dots\dots\dots.+\max\left\{\mu^j(\rho_q(l_1,l_0)),\mu^j(\rho_q(l_0,l_1))\right\}\end{aligned}$$

$$\rho_q(l_0,l_{j+1})\leq\sum_{i=0}^{j}\max\{\mu^i(\rho_q(l_1,l_0),\mu^i(\rho_q(l_0,l_1))\}<r. \tag{2.23}$$

Similarly, by using inequalities (2.22), (iv) and triangle inequality, we have

$$\rho_q(l_{j+1}, l_0) \leq \sum_{i=0}^{j} \max\{\mu^i(\rho_q(l_1, l_0), \mu^i(\rho_q(l_0, l_1))\} < r. \tag{2.24}$$

By Inequality (2.23) and (2.24), we have $l_{j+1} \in B_{\rho q}(l_0, r)$. Also,

$$\rho_q\left(l_{j+1}, Tl_{j+1}\right) = \rho_q\left(l_{j+1}, l_{j+2}\right) \text{ and} \rho_q\left(Tl_{j+1}, l_{j+1}\right) = \rho_q\left(l_{j+2}, l_{j+1}\right).$$

As $l_{j+1} \succeq Sl_{j+1}$, so from condition (ii), we have $l_{j+2} \preceq Sl_{j+2}$. Similarly, we get

$$\rho_q(l_{j+1}, l_{j+2}) \leq \max\left\{\mu^{j+1}(\rho_q(l_1, l_0)), \mu^{j+1}(\rho_q(l_0, l_1))\right\}. \tag{2.25}$$

And

$$\rho_q(l_{j+2}, l_{j+1}) \leq \max\left\{\mu^{j+1}(\rho_q(l_1, l_0)), \mu^{j+1}(\rho_q(l_0, l_1))\right\}. \tag{2.26}$$

Also,

$$\rho_q(l_0, l_{j+2}) < r \text{ and } \rho_q(l_{j+2}, l_0) < r.$$

It follows that, $l_{j+2} \in B_{\rho_q}(l_0, r)$. Also,

$$\rho_q\left(l_{j+2}, Tl_{j+2}\right) = \rho_q\left(l_{j+2}, l_{j+3}\right) \text{ and } \rho_q\left(Tl_{j+2}, l_{j+2}\right) = \rho_q\left(l_{j+3}, l_{j+2}\right).$$

As $l_{j+2} \preceq Sl_{j+2}$, so from condition (ii), we have $l_{j+3} \succeq Sl_{j+3}$. Hence by mathematical induction $l_n \in B_{\rho q}(l_0, r)$, $l_{2n} \preceq Sl_{2n}$ and $l_{2n+1} \succeq Sl_{2n+1}$ for all $n \in \mathbb{N}$. Also $l_{2n} \in G(S)$. Now inequalities (2.22), (2.25), and (2.26) can be merged as

$$\max\{\rho_q(l_n, l_{n+1}), \rho_q(l_{n+1}, l_n)\} \leq \max\left\{\mu^n(\rho_q(l_1, l_0)), \mu^n(\rho_q(l_0, l_1))\right\}. \tag{2.27}$$

For all $n \in N$. Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that $\sum_{k \geq k_1(\varepsilon)} \max\{\mu^k(\rho_q(l_1, l_0)), \mu^k(\rho_q(l_0, l_1))\} < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$, then

$$\begin{aligned} \rho_q(l_n, l_m) &\leq \sum_{k=n}^{m-1} \rho_q(l_k, l_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\{\mu^k(\rho_q(l_1, l_0)), \mu^k(\rho_q(l_0, l_1))\}, \text{ by inequality (2.27)} \end{aligned}$$

$$\rho_q(l_n, l_m) \leq \sum_{k \geq k_1(\varepsilon)} \max\{\mu^n(\rho_q(l_1, l_0)), \mu^n(\rho_q(l_0, l_1))\} < \varepsilon. \tag{2.28}$$

Thus we proved that $\{\Im T(l_n)\}$ is a left ρ_q-Cauchy sequence in $(B_{\rho_q}(l_0, r), \rho_q)$. Similarly, by using (2.27), we have

$$\rho_q(l_m, l_n) \leq \sum_{k=n}^{m-1} \rho_q(l_{k+1}, l_k) < \varepsilon. \tag{2.29}$$

From inequalities (2.28) and (2.29), we have

$$\max\{\rho_q(l_n,l_m),\rho_q(l_m,l_n)\}\le\sum_{k\ge k_1(\varepsilon)}\max\{\mu^n(\rho_q(l_1,l_0)),\mu^n(\rho_q(l_0,l_1))\}<\varepsilon.$$

Hence, $\{\Im T(l_n)\}$ is a right ρ_q-Cauchy sequence in $(B_{\rho_q}(l_0,r),\rho_q)$. As every closed set in left(right) ρ_q-sequentially complete ρ_q- metric space is left(right) ρ_q-sequentially complete and $G(S)$ is closed set, so, by Lemma 1.2 $G(S)$ is left(right) ρ_q-sequentially complete. As $\{l_{2n}\}$ is a left(right) ρ_q-Cauchy sequence in $G(S)$, so there exists $l^*\in G(S)$ such that $\{l_{2n}\}\to l^*$, that is

$$\lim_{n\to\infty}\rho_q(l_{2n},l^*)=\lim_{n\to\infty}\rho_q(l^*,l_{2n})=0. \tag{2.30}$$

Also $l^*\preceq Sl^*$. Now,

$$\rho_q(l^*,l^*)\le\rho_q(l^*,l_{2n})+\rho_q(l_{2n},l^*).$$

This implies $\rho_q(l^*,l^*)=0$ as $n\to\infty$. Now,

$$\begin{aligned}\rho_q(l^*,Tl^*)&\le\rho_q(l^*,l_{2n+2})+\rho_q(l_{2n+2},Tl^*)\\&\le\rho_q(l^*,l_{2n+2})+H_{\rho_q}(Tl_{2n+1},Tl^*).\end{aligned}$$

By assumption, inequality (i) holds for l^*. Also $l_{2n+1}\succeq Sl_{2n+1}$ and $l^*\preceq Sl^*$, so

$$\begin{aligned}\rho_q(l^*,Tl^*)&\le\rho_q(l^*,l_{2n+2})+t(\rho_q(l_{2n+1},Tl_{2n+1})+\rho_q(l^*,Tl^*))\\&\le\rho_q(l^*,l_{2n+2})+t(\rho_q(l_{2n+1},l_{2n+2})+\rho_q(l^*,Tl^*)).\end{aligned}$$

Letting $n\longrightarrow\infty$ and by using inequalities (2.27) and (2.30), we obtain

$$\rho_q(l^*,Tl^*)\le t(\max\{\rho_q(l^*,Tl^*),\rho_q(Tl^*,l^*)\}). \tag{2.31}$$

Now,

$$\begin{aligned}\rho_q(Tl^*,l^*)&\le\rho_q(Tl^*,l_{2n+2})+\rho_q(l_{2n+2},l^*)\\&\le H_{\rho_q}(Tl^*,Tl_{2n+1})+\rho_q(l_{2n+2},l^*).\end{aligned}$$

By assumption, inequality (i) holds for l^*. Also $l_{2n+1}\succeq Sl_{2n+1}$ and $l^*\preceq Sl^*$, so

$$\rho_q(Tl^*,l^*)\le t(\rho_q(l^*,Tl^*)+\rho_q(l_{2n+1},Tl_{2n+1}))+\rho_q(l_{2n+2},l^*)$$

Letting $n\longrightarrow\infty$ and by using inequalities (2.27) and (2.30), we obtain

$$\rho_q(l^*,Tl^*)\le t(\max\{\rho_q(l^*,Tl^*),\rho_q(Tl^*,l^*)\}). \tag{2.32}$$

By inequalities (2.31) and (2.32), we have

$$\max\{\rho_q(l^*,Tl^*),\rho_q(Tl^*,l^*)\}\le t(\max\{\rho_q(l^*,Tl^*),\rho_q(Tl^*,l^*)\}).$$

This implies that

$$\rho_q(l^*,Tl^*)=0\text{ and }\rho_q(Tl^*,l^*)=0. \tag{2.33}$$

From inequality (2.33) , we have $l^* \in Tl^*$. As $l^* \preceq Sl^*$ and $\rho_q(l^*, Tl^*) = \rho_q(Tl^*, l^*) = 0 = \rho_q(l^*, l^*)$, then from (ii)

$$l^* \succeq Sl^*.$$

Now, we have $l^* \preceq Sl^* \preceq l^*$. This implies $l^* \preceq y \preceq l^*$, for all $y \in Sl^*$. Therefore $l^* = y$, for all $y \in Sl^*$ or $Sl^* = \{l^*\}$. Hence, l^* is a common fixed point for S and T. □

Corollary 2.2. *Let $(\Im, \rho_q)$ be an ordered left (right) ρ_q-sequentially complete ρ_q-metric space, $r > 0$, $l_0 \in B_{\rho_q}(l_0, r)$, $S, T : \Im \to \Im$ be a self mapping on $B_{\rho_q}(l_0, r)$ and the picard sequence $\{l_n\}$ in $\Im$ generated by l_0 . Assume that for some $t \in [0, \frac{1}{2})$, such that*

(i) *The set $G(S) = \{l : l \preceq Sl \text{ and } l \in B_{\rho_q}(l_0, r)\}$ is closed and contains l_0.*

(ii) *For every $l \in B_{\rho_q}(l_0, r)$, we have $l \preceq Sl \Longrightarrow Tl \succeq STl$,and $l \succeq Sl \Longrightarrow Tl \preceq STl$.*

(iii) There exists a function $\mu \in \Psi$, $l_0 \in \Im$ and $r > 0$ such that for every $(l, y) \in \Im \times \Im$, we have

$$\rho_q(Tl, Ty) \le t(\rho_q(l, Tl) + \rho_q(y, Ty)).$$

For all $l, y \in B_{\rho_q}(l_0, r) \cap \{l_n\}$ with $(l, y) \in B(S)$,where

$$B(S) = \{(l, y) : l \succeq Sl,\ y \preceq Sy, \text{or } l \preceq Sl,\ y \succeq Sy\}.$$

(iv)

$$\max\{\rho_q(l_1, l_0), \rho_q(l_0, l_1)\} < (1 - \mu)r.$$

Where $\mu = \frac{t}{1-t}$. Then the subsequence $\{l_{2n}\}$ of $\{l_n\}$ is a sequence in $G(S)$ and $\{l_{2n}\} \to l^* \in G(S)$ and $\rho_q(l^*, l^*) = 0$. Also, if the inequality (i) holds for l^*. Then S and T have a common fixed point l^* in $B_{\rho_q}(l_0, r)$.

Example 2.1. Let $\Im = [0, \infty)$ and let

$$\rho_q(l, y) = 2l + 5y,\ (l, y) \in \Im \times \Im.$$

Then $(\Im, \preceq, \rho_q)$ be an ordered left(right) K sequentially complete ρ_q- metric space. Let $\mathcal{R}$ be the binary relation on $\Im$ defined by

$$\begin{aligned} B(S) \quad = \quad & \mathcal{R} = \{(l, l) : l \in \Im\} \cup \left\{(l, \frac{l}{5}) : l \in \{1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\}\right\} \\ & \cup \left\{(\frac{l}{5}, l) : l \in \{\frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\}\right\}. \end{aligned}$$

Consider the partial order on $\Im$ defined by

$$(l, y) \in \Im \times \Im,\ l \preceq y \text{ if and only if } (l, y) \in R.$$

Define the pair of mapping $T, S : \Im \to P(\Im)$ by

$$Tl = \left[\frac{l}{5}, \frac{l}{4}\right],\ Sl = \left\{ \begin{array}{l} \{\frac{l}{5}\} : l \in [0, 1] \\ \{l + 5\} : l > 1 \end{array} \right. .$$

Let

$$\begin{aligned} A &= \{l : l \preceq Sl\} = \left\{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\right\}, \\ B &= \{y : y \succeq Sy\} = \left\{0, \frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\right\}. \end{aligned}$$

Let $l_0 = 1$ and $r = 28$, then

$$B_{\rho q}(l_0, r) = \{y : \rho_q(1, y) < 28 \wedge \rho_q(y, 1) < 28\} = [0, \frac{23}{2}).$$

Then

$$\begin{aligned} G(S) &= \{l : l \preceq Sl \text{ and } l \in B_{\rho q}(l_0, r)\} \\ &= \left\{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\right\}. \end{aligned}$$

Now, as $\frac{1}{5^{n-1}} \in B_{\rho q}(l_0, r)$,

$$\rho_q(\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}}) = \rho_q(\frac{1}{5^{n-1}}, \frac{1}{5 \times 5^{n-1}})$$

and

$$\rho_q(T\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}) = \rho_q(\frac{1}{5 \times 5^{n-1}}, \frac{1}{5^{n-1}}).$$

Also, $(\frac{1}{5^{n-1}}, \frac{1}{5 \times 5^{n-1}}) \in \mathcal{R}$ when n is odd, so $\frac{1}{5^{n-1}} \preceq S\frac{1}{5^{n-1}}$. As $(\frac{1}{25 \times 5^{n-1}}, \frac{1}{5 \times 5^{n-1}}) \in \mathcal{R}$, when n is odd, so $\frac{1}{5 \times 5^{n-1}} \succeq S\frac{1}{5 \times 5^{n-1}}$. Hence, condition (ii)(a) is satisfied. Now, as $\frac{1}{5 \times 5^{n-1}} \in B_{\rho q}(l_0, r)$,

$$\rho_q(\frac{1}{5 \times 5^{n-1}}, T\frac{1}{5 \times 5^{n-1}}) = \rho_q(\frac{1}{5 \times 5^{n-1}}, \frac{1}{25 \times 5^{n-1}})$$

and

$$\rho_q(T\frac{1}{5 \times 5^{n-1}}, \frac{1}{5 \times 5^{n-1}}) = \rho_q(\frac{1}{25 \times 5^{n-1}}, \frac{1}{5 \times 5^{n-1}}).$$

Also, $\frac{1}{5 \times 5^{n-1}} \succeq S\frac{1}{5 \times 5^{n-1}}$, when n is odd. Also $\frac{1}{25 \times 5^{n-1}} \preceq S\frac{1}{25 \times 5^{n-1}}$, when n is odd. Hence, condition (ii)(b) is satisfied. Also, condition (ii)(a) and (ii)(b) are trivially satisfied for $0 \in B_{\rho q}(l_0, r)$. Now

$$B_{\rho q}(l_0, r) \cap \{\Im T(l_n)\} = \left\{1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}, \frac{1}{15625}, \frac{1}{78125}, ...\right\}.$$

Now for $l, y \in B_{\rho_q}(l_0, r) \cap \{\Im T(l_n)\}$ with $l \succeq Sl$ and $y \preceq Sy$, then $l \in B$ and $y \in A$. In general for some $n, m \in \mathbb{N}$

$$l = \frac{1}{5 \times 5^{m-1}}, \ y = \frac{1}{5^{n-1}}, \ t = \frac{3}{7}.$$

Case i: Let $n \geq m$, we have

$$\begin{aligned}
H(Ty, Tl) &= H\left(\left[\frac{1}{5 \times 5^{n-1}}, \frac{1}{4 \times 5^{n-1}}\right], \left[\frac{1}{25 \times 5^{m-1}}, \frac{1}{20 \times 5^{m-1}}\right]\right) \\
&= \max\left\{\frac{1}{2 \times 5^{n-1}} + \frac{1}{5 \times 5^{m-1}}, \frac{2}{5 \times 5^{n-1}} + \frac{1}{4 \times 5^{m-1}}\right\} \\
&= \max\left\{\frac{5 + 2 \times 5^{n-m}}{10 \times 5^{n-1}}, \frac{8 + 5 \times 5^{n-m}}{20 \times 5^{n-1}}\right\} \\
&= \frac{8 + 5 \times 5^{n-m}}{20 \times 5^{n-1}}.
\end{aligned}$$

Now,

$$\begin{aligned}
t\left\{\rho_q(l, Tl) + \rho_q(y, Ty)\right\} &= \frac{3}{7}\left\{\rho_q(\frac{1}{5 \times 5^{m-1}}, T\frac{1}{5 \times 5^{m-1}}) + \rho_q(\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}})\right\} \\
&= \frac{3}{7}\left\{\frac{2}{5 \times 5^{m-1}} + \frac{1}{5 \times 5^{m-1}} + \frac{2}{5^{n-1}} + \frac{1}{5^{n-1}}\right\} \\
&= \frac{3}{7}\left\{\frac{3 \times 5^{n-m} + 15}{5 \times 5^{n-1}}\right\}.
\end{aligned}$$

As

$$\begin{aligned}
56 + 35 \times 5^{n-m} &< 36 \times 5^{n-m} + 180 \\
\frac{56 + 35 \times 5^{n-m}}{20 \times 5^{n-1}} &< \frac{9 \times 5^{n-m} + 45}{5 \times 5^{n-1}} \\
\frac{8 + 5 \times 5^{n-m}}{20 \times 5^{n-1}} &< \frac{3}{7}\left(\frac{3 \times 5^{n-m} + 15}{5 \times 5^{n-1}}\right) \\
H_q\left(Ty, Tl\right) &< t\left\{\rho_q(l, Tl) + \rho_q(y, Ty)\right\}.
\end{aligned}$$

Now,

$$\begin{aligned}
H(Tl, Ty) &= \max\left\{\frac{5^{n-m} + 10}{10 \times 5^{n-1}}, \frac{8 \times 5^{n-m} + 125}{100 \times 5^{n-1}}\right\} \\
&= \frac{5^{n-m} + 10}{10 \times 5^{n-1}}.
\end{aligned}$$

Now, we have

$$t\left\{\rho_q(l, Tl) + \rho_q(y, Ty)\right\} = \frac{3}{7}\left\{\frac{3 \times 5^{n-m} + 15}{5 \times 5^{n-1}}\right\}.$$

As

$$\begin{aligned}
70 + 7 \times 5^{n-m} &< 18 \times 5^{n-m} + 90 \\
\frac{70 + 7 \times 5^{n-m}}{10 \times 5^{n-1}} &< \frac{9 \times 5^{n-m} + 45}{5 \times 5^{n-1}} \\
\frac{10 + 5^{n-m}}{10 \times 5^{n-1}} &< \frac{3}{7}\left(\frac{3 \times 5^{n-m} + 15}{5 \times 5^{n-1}}\right) \\
H_q\left(Tl, Ty\right) &< t\left\{\rho_q(l, Tl) + \rho_q(y, Ty)\right\}
\end{aligned}$$

Case ii: Let $n < m$, we have

$$\begin{aligned} H(Ty, Tl) &= \max\left\{\frac{5\times 5^{m-n}+2}{10\times 5^{m-1}}, \frac{8\times 5^{m-n}+5}{20\times 5^{m-1}}\right\} \\ &= \frac{5\times 5^{m-n}+2}{10\times 5^{m-1}}. \end{aligned}$$

Now, we have

$$t\left\{\rho_q(l, Tl)+\rho_q(y, Ty)\right\} = \frac{3}{7}\left\{\frac{3+15\times 5^{m-n}}{5\times 5^{m-1}}\right\}.$$

As

$$\begin{aligned} 35\times 5^{m-n}+14 &< 18+90\times 5^{m-n} \\ \frac{5\times 5^{m-n}+2}{10\times 5^{m-1}} &< \frac{3}{7}\left\{\frac{3+15\times 5^{m-n}}{5\times 5^{m-1}}\right\} \\ H_q\left(Ty, Tl\right) &< t\left\{\rho_q(l, Tl)+\rho_q(y, Ty)\right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} H_q\left(Tl, Ty\right) &= \max\left\{\frac{1+10\times 5^{m-n}}{10\times 5^{m-1}}, \frac{8+125\times 5^{m-n}}{100\times 5^{m-1}}\right\} \\ &= \frac{8+125\times 5^{m-n}}{100\times 5^{m-1}}. \end{aligned}$$

Now, we have

$$t\left\{\rho_q(l, Tl)+\rho_q(y, Ty)\right\} = \frac{3}{7}\left\{\frac{3+15\times 5^{m-n}}{5\times 5^{m-1}}\right\}.$$

As

$$\begin{aligned} 56+875\times 5^{m-n} &< 180+900\times 5^{m-n} \\ \frac{56+875\times 5^{m-n}}{100\times 5^{m-1}} &< \frac{9+45\times 5^{m-n}}{5\times 5^{m-1}} \\ \frac{8+125\times 5^{m-n}}{100\times 5^{m-1}} &< \frac{3}{7}\left\{\frac{3+15\times 5^{m-n}}{5\times 5^{m-1}}\right\} \\ \max\left\{H_q\left(Tl, Ty\right), H_q\left(Ty, Tl\right)\right\} &< t\left\{\rho_q(l, Tl)+\rho_q(y, Ty)\right\}. \end{aligned}$$

Case iii: Let

$$l = 0,\ y = \frac{1}{5^{n-1}}.$$

We have

$$\begin{aligned} H(Tl, Ty) &= \max\left\{[0,0], \left[\frac{1}{5\times 5^{n-1}}, \frac{1}{4\times 5^{n-1}}\right]\right\} \\ &= \max\left\{\frac{1}{5^{n-1}}, \frac{1}{20\times 5^{n-1}}\right\} = \frac{1}{5^{n-1}}. \end{aligned}$$

Now,

$$t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}=\frac{3}{7}\left\{\frac{3}{5^{n-1}}\right\}.$$

Clearly

$$H_q\left(Tl,Ty\right)<t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}.$$

Similarly, we have

$$H(Ty,Tl)=\max\left\{\frac{1}{2\times 5^{n-1}},\frac{2}{5\times 5^{n-1}}\right\}=\frac{1}{2\times 5^{n-1}}.$$

Now, we have

$$t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}=\frac{3}{7}\left\{\frac{3}{5^{n-1}}\right\}.$$

Clearly

$$H_q\left(Tl,Ty\right)<t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}.$$

Case iv: Let

$$l=\frac{1}{5\times 5^{m-1}},\ y=0.$$

We have

$$H(Tl,Ty)=\max\left\{\frac{1}{10\times^{m-1}},\frac{2}{25\times 5^{m-1}}\right\}=\frac{1}{10\times^{m-1}}.$$

Now,

$$t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}=\frac{3}{7}\left\{\frac{3}{5\times 5^{m-1}}\right\}.$$

Now,

$$\max\left\{H_q\left(Tl,Ty\right),H_q\left(Ty,Tl\right)\right\}=\frac{1}{3\times 9^{n-1}}.$$

Clearly

$$H_q\left(Tl,Ty\right)<t\left\{\rho_q(l,Tl)+\rho_q(y,Ty)\right\}.$$

Case v: The contraction trivially holds for $l=0$ and $y=0$.

Now, $l_0=1$ and $l_1=\frac{1}{5}$, we have

$$\max\{\rho_q(l_1,l_0),\rho_q(l_0,l_1)\}=\max\{\frac{27}{5},3\}=\frac{27}{5}$$

and

$$(1-\mu)r=(1-\frac{3}{4})28=7.$$

Now, we have

$$\max\{\rho_q(l_1,l_0),\rho_q(l_0,l_1)\}<(1-\mu)r.$$

Thus all the conditions of Theorem 2.1 are satisfied. Hence S and T have a common fixed point.

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