The approximate solution of the third order boundary value problem with an internal boundary condition using a hybrid finite difference method

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Abstract. In the present article, we considered and proposed a numerical technique for the solution of differential equations of order three. The boundary conditions are prescribed on the two points on the boundary and an interior point in the domain of the reference differential equation. Our technique based on method of finite difference approximations and a system of linear equations obtained after discretization of the continuous problem. The development and the convergence of the proposed method discussed in detail. Additionally, in the numerical experiment considered both linear and nonlinear model problems. The computational efficiency and quadratic order of the convergence of the proposed HFD method tested by experiment. The proposed HFD technique is accurate and precise, it may be seen from tabulated numerical outcomes.

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1 Introduction

The mathematical modelling of flow in fluid dynamics, the deflection in shape in the theory of beams, formation of layers in the theory of boundary layer are some specific subject in natural science, engineering and technology yields differential equation of order three.

For the development and discussion of HFD method, we have considered following differential equation of order three in canonical form,

$$u'''(x) = f(x, u), \quad x \in [a, b] - \{c\}. \tag{1.1}$$

subject to the internal / boundary conditions

$$u'(a) = \beta \quad u(c) = \alpha, \quad \text{and} \quad u'(b) = \gamma \tag{1.2}$$

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where constants α, β and γ are real.

We have assumed that $f(x, u)$ is a continuous and monotonic function in $[a, b] \times \mathrm{R}$. Further, any other assumption to ensure the existence and uniqueness of the solution of the specific problem (1.1) will not be considered and presume the existence and uniqueness of the solution of the problem (1.1). However, in detail the development of theoretical concepts such as existence and uniqueness of the solution of the problem (1.1) can be found in the literature [4, 7].

In general, mathematical modeling of physical problems leads to study of a nonlinear boundary value problems. The acceptable analytical solution of these problems is important, but in general it is difficult /impossible to achieve. To have an acceptable solution, we usually rely on approximate solution another available mathematical tool, i.e. numerical method for the approximate numerical solution of these BVPs. The emphasis in this article will be on the finite difference approximation approach for the numerical solution of the reference problem (1.1) and the detail about an internal boundary condition can be found in [15].

Though there are many researchers reported their work on the approximate solution of third order differential equations and corresponding BVPs. For instance, we found the method of superposition and chasing [11], the spline approximation method [6, 9, 13], the finite difference method [2, 11, 12, 16], the differential transformation method [1] and references therein reported in the literature of numerical analysis but not featured with interior boundary conditions. Recently, a work featuring interior boundary conditions reported in the literature of [10] based on the concept of boundary shape function. It is difficult to satisfy exactly the boundary conditions in three point BVPs (1.1). So, there are not enrich the work reported in the literature of the approximate numerical solution of problems (1.1). Therefore, the purpose of this article is to design and develop a hybrid finite difference method to exactly satisfy all the boundary conditions for the accurate approximate numerical solution of problems (1.1).

The work in this article presented as: In the section 2, there we provided a hybrid finite difference method for the approximate numerical solution of the problem (1.1). We have discussed in detail the derivation of the method in the Section 3 and the convergence analysis of the proposed method under appropriate conditions in the Section 4. The experimental discussion and outcome of the proposed method to test the efficiency and accuracy presented in the Section 5. A concluding discussion on the performance of the proposed method presented in Section 6.

2 The hybrid finite difference method

We consider $N \geq 5$, an odd numbers of nodal points in the domain (a,b] i.e. $a = x_0 < x_1 < .. < x_{N-1} < x_N = b$. These nodal points are shown in figure 1. Let us define uniform step length h between two successive nodes $x_i = a + ih, \quad i = 0, 1, 2, ..., N$ is $h = \frac{b-a}{N}$. Let $c = a + \frac{N+1}{2N}$, hence we have $u(c) = \alpha$.
We wish to determine the numerical approximation of the theoretical solution $u(x)$ of

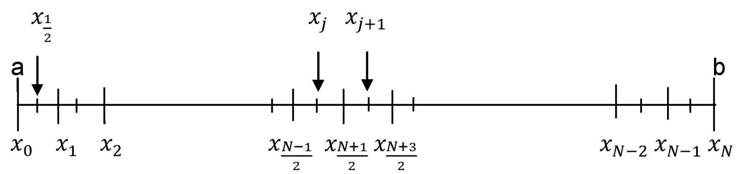

Figure 1: Nodal points

the problem (1.1). Let us denote u_i and f_i respectively the numerical approximation of $u(x)$ and $f(x,u)$ at these $x = x_i$, $i = 0, 1, 2, ..., N$ nodal points. Let us convert continuous boundary value problem (1.1) into a discrete problem at defined node $x = x_i$ as,

$$u_i''' = f_i, \quad a \le x_i \le b, \tag{2.1}$$

and continuous boundary conditions as the discrete boundary conditions

$$u_j = \alpha, \quad u_0' = \beta \quad \text{and} \quad u_N' = \gamma \tag{2.2}$$

where $x_j = c$ and $0 < j < N$. Let us define nodes $x_{i\pm\frac{k}{2}} = x_i \pm \frac{kh}{2}$, $k = 0, 1, \cdots$ and following approximations,

$$\overline{u}_{i-\frac{1}{2}} = \begin{cases} u_{i+\frac{1}{2}} - hu_{i-1}', & i = 1 \\ \frac{1}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{3}{2}}), & 1 < i < \frac{N+1}{2} \\ \frac{1}{3}(2u_i + u_{i-\frac{3}{2}}), & i = \frac{N+1}{2} \\ \frac{1}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{3}{2}}), & \frac{N+1}{2} < i < N \\ u_{i-\frac{3}{2}} + hu_i', & i = N \end{cases} \tag{2.3}$$

and

$$\overline{f}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, \overline{u}_{i-\frac{1}{2}}), \quad i = 1, 2, \cdots, N. \tag{2.4}$$

We pursue the thought in [13, 14] and propose the discretization technique for the problem (2.1) in $(a, c) \cup (c, b)$ at nodes $x_{i-\frac{1}{2}}$, $i = 1, \cdots, N$,

$$\begin{aligned} 2u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} &= -hu_{i-1}' + \frac{h^3}{48}(21\overline{f}_{i-\frac{1}{2}} + 25f_{i+\frac{1}{2}}) + T_i, \quad i = 1 \\ -u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} &= \frac{h^3}{2}(\overline{f}_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + T_i, \quad 1 < i < \frac{N+1}{2} \\ -u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} &= 8u_i - \frac{h^3}{16}(f_{i-\frac{3}{2}} - 9\overline{f}_{i-\frac{1}{2}}) + T_i, \quad i = \frac{N+1}{2} \\ u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= -\frac{h^3}{2}(f_{i-\frac{3}{2}} + \overline{f}_{i-\frac{1}{2}}) + T_i, \quad \frac{N+1}{2} < i < N \\ u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} &= hu_i' - \frac{h^3}{48}(25f_{i-\frac{3}{2}} + 21\overline{f}_{i-\frac{1}{2}}) + T_i, \quad i = N \end{aligned} \tag{2.5}$$

where $T_i, i = 1, 2, ..., N$ is truncation error.

We got our proposed hybrid method after neglecting the truncation error terms T_i, $i = 1, 2, ..., N$ in (2.5). Thus, we obtained the $N \times N$ system of linear equations. The solution of system of equations is the approximate solution of the problem (1.1). We have incorporated internal boundary condition $u_{\frac{N+1}{2}}$ in a natural way in proposed method.

We computed approximate solution $u_i, i = 1, 2, \cdots, N$ and $i \neq \frac{N+1}{2}$ of problem (1.1) by using following second order difference approximation,

$$(2.6) \qquad u_j = \begin{cases} u_{j+\frac{1}{2}} - \frac{h}{2}u'_j, & j = 0 \\ \frac{1}{2}(u_{j-\frac{1}{2}} + u_{j+\frac{1}{2}}), & 0 < j < \frac{N+1}{2} \\ \frac{1}{2}(u_{j-\frac{1}{2}} + u_{j+\frac{1}{2}}), & \frac{N+1}{2} < j < N \\ u_{j-\frac{1}{2}} + \frac{h}{2}u'_j, & j = N. \end{cases}$$

3 The derivation of the hybrid finite difference method

In the section we outline the derivation of the proposed hybrid finite difference method (2.5). Consider following,

$$\begin{aligned} \overline{u}_{i-\frac{1}{2}} &= u_{i+\frac{1}{2}} - hu'_{i-1}, \quad i = 1 \\ &= u_{i-\frac{1}{2}} + h^2u''_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} + O(h^2) \end{aligned}$$

and from (2.4),

$$(3.1) \quad \begin{aligned} \overline{f}_{i-\frac{1}{2}} &= f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}} + O(h^2)), \quad i = 1. \\ &= f_{i-\frac{1}{2}} + O(h^2). \end{aligned}$$

Consider

$$(3.2) \qquad c_0u_{i-\frac{1}{2}} + c_1u_{i+\frac{1}{2}} + c_2u_{i+\frac{3}{2}} + hc_3u'_{i-1} + h^3(c_4\overline{f}_{i-\frac{1}{2}} + c_5f_{i+\frac{1}{2}}) = 0,\ i = 1.$$

where c_0, c_1, c_2, c_3, c_4 and c_5 are constant. Using (2.1), (3.1) and write each term of (3.2) in Taylor series about point $x_{i-\frac{1}{2}}$. Compare the coefficients of terms $h^p, p = 0, 1, .., 4$ to determine the constants $c_0, \cdots, c_5$. Thus we find

$$(3.3) \qquad \begin{aligned} c_0 + c_1 + c_2 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ c_1 + 4c_2 - c_3 &= 0 \\ 4c_1 + 32c_2 + 3c_3 + 24c_4 + 24c_5 &= 0 \\ 2c_1 + 32c_2 - c_3 + 48c_5 &= 0 \end{aligned}$$

Solving the system of equations (3.3), we have

$$(3.4) \qquad (c_0, c_1, c_2, c_3, c_4, c_5) = \frac{1}{48}(96, -144, 48, 48, -21, -25)$$

Substituting the values of the constants from (3.4) in (3.2), we have

$$2u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} = -hu'_i + \frac{h^3}{48}(21\overline{f}_{i-\frac{1}{2}} + 25f_{i+\frac{1}{2}}), \quad i = 1. \tag{3.5}$$

Using above method, similarly we can derive other equations in (2.5). We compared coefficients of $h^p, p = 0, \cdots, 4$ in determining constants implies that the leading truncating term is of $O(h^5)$. Thus, we conclude that method (2.5), the discretization of the problem (1.1) at nodes $x_{i-\frac{1}{2}}, \quad i = 1, 2, ..., N$ is of $O(h^2)$.

4 Convergence analysis

We will discuss the convergence of the proposed hybrid method (2.5) for the approximate solution of the problem (1.1)-(1.2). Let us consider the linearization of a forcing function $f(x, u)$ in (1.1). Let $u(x)$ and $U(x)$ are respectively approximate and exact solution of the problem (1.1). The linearization of a forcing function $f(x, u)$ at $U(x)$ is given as,

$$f(x, u) = f(x, U) + (u - U)\frac{\partial f}{\partial U}. \tag{4.1}$$

Let us define

$$F = f(x, U), \quad f = f(x, u) \quad \text{and} \quad G = \frac{\partial f}{\partial U}. \tag{4.2}$$

We can write proposed hybrid method (2.5) for the exact solution $\mathbf{U} = [U_{\frac{1}{2}}, \cdots, U_{N-\frac{1}{2}}]^T$ and approximate solution $\mathbf{u} = [u_{\frac{1}{2}}, \cdots, u_{N-\frac{1}{2}}]^T$ as in the following matrix equations,

$$\mathbf{DU} = \mathbf{R}_e + \mathbf{T} \tag{4.3}$$

and

$$\mathbf{Du} = \mathbf{R}_a \tag{4.4}$$

where $\mathbf{R}_e$ and $\mathbf{R}_a$ are respectively value of the right hand side of (2.5) at the exact and approximate value of solution of the problem (1.1). The term $\mathbf{T}$ is named as truncation error in (2.5). The coefficient matrix $\mathbf{D}$ in (4.3) and (4.4) is defined as,

$$\mathbf{D} = \begin{pmatrix} 2 & -3 & 1 & & & & & & & 0 \\ -1 & 3 & -3 & 1 & & & & & & \\ & .. & .. & .. & .. & & & & & \\ & & -1 & 3 & -3 & 1 & & & & \\ & & & -1 & 6 & 3 & & & & \\ & & & 1 & -3 & 3 & -1 & & & \\ & & & & & .. & .. & .. & .. & \\ & & & & & & 1 & -3 & 3 & -1 \\ 0 & & & & & & & 1 & -3 & 2 \end{pmatrix}_{N\times N}.$$

and $\mathbf{T} = [T_1, T_2, \cdots, T_N]^T$ in (4.3) is defined as,

$$T_i = \frac{h^5}{1920}\begin{cases} -(31u^{(5)} + 840u''\frac{\partial f}{\partial u})_{i-\frac{1}{2}}, & i = 1 \\ -480(u''\frac{\partial f}{\partial u})_{i-\frac{1}{2}} + O(h), & 1 < i < N \quad \text{and} \quad i \neq \frac{N+1}{2} \\ (12u^{(5)} - 270u''\frac{\partial f}{\partial u})_{i-\frac{1}{2}}, & i = \frac{N+1}{2} \\ (31u^{(5)} + 840u''\frac{\partial f}{\partial u})_{i-\frac{1}{2}}, & i = N. \end{cases} \tag{4.5}$$

Let us define an error vector $\boldsymbol{\epsilon} = [\epsilon_{\frac{1}{2}}, \epsilon_{\frac{3}{2}}, \cdots, \epsilon_{N-\frac{1}{2}}]^T$, where

$$e_{i-\frac{1}{2}} = U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}, \quad i = 1, 2, .., N \quad \text{and} \quad G^* = \max_{a \le x \le b}(|G|). \tag{4.6}$$

Subtracting (4.4) from (4.3) and using (4.1), we will obtained

$$(\mathbf{D} - \mathbf{D}_R)\boldsymbol{\epsilon} = \mathbf{T} \tag{4.7}$$

where

$$\mathbf{D}_R = \frac{h^3}{24}G^*\begin{pmatrix} 0 & 23 & & & & & & & & 0 \\ 6 & 0 & 18 & & & & & & & \\ & .. & .. & .. & & & & & & \\ & & 6 & 0 & 18 & & & & & \\ & & & 3 & 0 & 0 & & & & \\ & & & & & -18 & 0 & -6 & & \\ & & & & & & .. & .. & .. & \\ & & & & & & & -18 & 0 & -6 \\ 0 & & & & & & & & -23 & 0 \end{pmatrix}_{N\times N}. \tag{4.8}$$

Let matrix $\mathbf{B} = (b_{l,m})_{N\times N}$ be the explicit inverse of matrix $\mathbf{D}$ and

$$b_{l,m} = \frac{1}{8N}\begin{cases} (mN(3N - 4m + 2) - 4l(l-1)(N-m)), & l \le m < \frac{N+1}{2} \\ N, \quad m = \frac{N+1}{2} & \\ (N + 2l)(2l - N - 2)(N - m + 1), & \frac{N+l}{2} + 1 \le m, l \le m \\ m(3N - 2l)(N - 2l + 2), & \frac{N+1}{2} > m, m \le l \\ ((m-1)(N(N-2) - 4(N-l)(N-l+1)) & \\ -N(2m - N - 2)(2m - N - 4)), & \frac{N+1}{2} + 1 \le m \le l, m \le l \end{cases} \tag{4.9}$$

Let us assume $||\mathbf{D}_R\mathbf{B}|| < 1$ then by [3],

$$||(\mathbf{D} - \mathbf{D}_R)^{-1}|| \le \frac{||\mathbf{B}||}{1 + ||\mathbf{D}_R\mathbf{B}||} \tag{4.10}$$

So from (4.7) and (4.10), we have obtained following as in [5],

$$\|\boldsymbol{\epsilon}\| \le \frac{||\mathbf{B}||}{1 + ||\mathbf{D}_R\mathbf{B}||}||\mathbf{T}|| \tag{4.11}$$

Let us assume

$$T^* = \max_{x\in[a,b]}|T(x)| \tag{4.12}$$

Computing $\|\mathbf{B}\|$ from (4.9) and using (4.12) in (4.11), we have

$$\|\boldsymbol{\epsilon}\| \leq \frac{(b-a)^3}{96h^3} \frac{1}{1+||\mathbf{D}_R\mathbf{B}||} ||\mathbf{T}^*|| \tag{4.13}$$

From (4.13), we observed $\|\mathbf{T}^*\|$ of $O(h^5)$. It follows that $\|\boldsymbol{\epsilon}\| \to 0$ as $h \to 0$. Thus we have established the convergence of the method (2.5) and the order of convergence of method (2.5) is $O(h^2)$.

5 Numerical results

To illustrate proposed hybrid method and demonstrate its computational efficiency, we have considered two model problems. In each model problem, we took a uniform step size h. In Tables 1-3 we have presented MAE, the maximum absolute error in the solution $u(x)$ of the considered model problem similar to the problems (1.1) for different values of N. We have used the following formula in computation of MAE,

$$MAE = \max_{\substack{0 \leq i \leq N \\ i \neq \frac{N+1}{2}}} |u(x_i) - u_i|.$$

We have applied Gauss Seidel method iterative method for the solution of system of equations arises from proposed hybrid method (2.5). In parametric problem 2, we tested our method for different values of the parameters. In Table 2, we took $A = 1.0$ and in Table 3 we took a range of values of parameter A.

All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 10^3.

Problem 1. The linear model problem in [8] with internal boundary condition is given by

$$u'''(x) = xu(x) + (x^3 - 2x^2 - 5x - 3)\exp(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(c) = c(1-c)\exp(c) \quad , \quad u'(0) = 1 \qquad \text{and} \quad u'(1) = -\exp(1)$$

The analytical solution of the problem is $u(x) = x(1-x)\exp(x)$. The MAE computed by method (2.5) for different values of N are presented in Table 1.

Problem 2. The nonlinear parametric model problem with internal boundary condition is given by,

$$u'''(x) = A\exp(-x - 2u(x)) + f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u'(0) = -A \quad , \quad u'(1) = -\frac{A}{A + \exp(1)} \qquad \text{and} \quad u(c) = \ln(1 + A\exp(-c))$$

where function $f(x)$ is computed so that the analytical solution of the problem is $u(x) = \ln(1 + A\exp(-x))$. The MAE computed by method (2.5) and $A = 1$ for different values of N in Table 2. The MAE computed for different values of parameter A and N are presented in Table 3.

Table 1: Maximum absolute error (Problem 1).

N	MAE	N	MAE	N	MAE
5	.48435701e -1	7	.25413571e -1	9	.15617082e -1
15	.57705889e -2	21	.29653969e -2	27	.18010154e -2
45	.65435735e -3	63	.33423394e -3	81	.20232276e -3
135	.73163639e -4	189	.37342529e -4	243	.22638173e -4
405	.82124860e -5	567	.42069748e -5	729	.25547240e -5
1215	.91996164e -6	1701	.46963010e -6	2187	.28411380e -6

Table 2: Maximum absolute error (Problem 2).

N	MAE	N	MAE	N	MAE
5	.11765188e -2	7	.60555540e -3	9	.37200210e -3
15	.13205232e -3	21	.67635896e -4	27	.42819078e -4
45	.14692666e -4	63	.75202200e -5	81	.47627666e -5
135	.16337313e -5	189	.84295391e -6	243	.52922050e -6
405	.18987478e -6	567	.97105757e -7	729	.58802384e -7
1215	.21168856e -7	1701	.10800446e -7	2187	.65335990e -8

Table 3: Maximum absolute error (Problem 2).

		A				
	N	.04	0.2	1	5	25
MAE	5	.17820097e -3	.66411644e -3	.11765188e -2	.11591921e -2	.43392172e -3
	15	.20251329e -4	.75257404e -4	.13205032e -3	.12892064e -3	.48991660e -4
	45	.22664716e -5	.84113747e -5	.14692666e -4	.14343407e -4	.54739805e -5
	135	.25244313e -6	.93649810e -6	.16337313e -5	.15944800e -5	.60930717e -6
	405	.28172684e -7	.10575079e -6	.18987478e -6	.17494358e -6	.67384226e -7
	1215	.31310913e -8	.11759400e -7	.21168856e -7	.19318040e -7	.74887862e -8
	3645	.34792842e -9	.13066814e -8	.23520904e -8	.21465070e -8	.83214825e -9

In the tables, we presented computational value of MAE for the different values of N and c. We observed from the numerical results in tables, the maximum absolute error reduces as step length h reduces and the order of accuracy remains unchanged with change in c i.e. the order of exactness of proposed method is remains at least $O(h^2)$. In the numerical outcomes of parametric model problem 2, we infer that the change in parametric value the exactness of our proposed method and order stays unaltered. Thus, we observed the proposed method (2.5) is convergent and the order of convergence is quadratic.

6 Conclusion

An algorithm for the approximate numerical solution of problems (1.1) based on method of finite differences has been developed and discussed. The method designed and developed exactly satisfy all three boundary conditions in a natural way. The discretization of (1.1) at the discrete nodal points in the domain of the problem, we obtained a system of linear equations (2.5). The system of linear equations (2.5) is the our proposed method and generates an approximate numerical solution of the model problem (1.1) at these discrete nodal points. The computational results are in perfect accordance with the theoretical discussion of the proposed method. The idea presented in this article leads to the possibility to develop hybrid finite difference methods for the approximate solution of higher order differential equation with boundary condition(s) as in problem (1.1).

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