Some equalities on predictors under seemingly unrelated regression models

N. Güler, M. Yiğit, M. Eriş Büyükkaya

Abstract. A system of linear regression models consists of a set of regression models that appear to be unrelated but each of them is linked to their correlated error terms across the models. Seemingly unrelated regressions (SUR) models' system is a special case of the system of linear regression models. In this paper, we establish the results by considering a system of linear regression models to present a general approach to SUR models. We consider the best linear unbiased predictors (BLUPs) and the ordinary least squares predictors (OLSPs) and give analytical expressions and properties of these predictors. We derive necessary and sufficient conditions for equalities and also for additive decompositions of the BLUPs and the OLSPs by using various rank formulas of block matrices. The corresponding results are also expressed for SUR models.

M.S.C. 2010: 62H12, 62J05.
Key words: BLUP; decomposition of predictor; OLSP; rank; SUR model; system of linear regression model.

1 Introduction

A system of linear regression models contains a group of regression models which allow correlated errors between the models. Correlations between error terms among models can occur when the same data or an amount of the same independent variables are used for the models. This case can also occur if data is collected with different periods before and after experiments. Inference results obtained from the system of linear regression models and its single models are not necessarily the same, however, there are connections between inference results obtained from the system and its individual models. Therefore, it is natural to consider the problems of establishing certain links among predictors/estimators under the system of linear regression models and its single models. One of the approaches to determining the connections among predictors/estimators is to establish the equalities between them. Another approach is to establish additive decomposition equalities for predictors/estimators. Such equalities provide us to determine the role of partial unknown vectors under linear models since single models are also associated with the system.

Applied Sciences, Vol. 24, 2022, pp. 142-154.

In this study, we consider a system of linear regression models that consists of m regression models in which the error terms are correlated across the models. One of the main purposes of the study is to establish the equalities of predictors under a system of linear regression models and its single models. The other purpose is to establish the additive decompositions of the predictors in the system and its single models. We consider the best linear unbiased predictors (BLUPs) and the ordinary least-squares predictors (OLSPs) as predictors of unknown vectors under the models. We express fundamental algebraic and statistical properties of these predictors under considered models since they need to be clearly expressed. The results on equality of the best linear unbiased estimators (BLUEs), and also additive equalities and block decompositions of them under a system of linear regression equations were derived by [20]. We extend the approach of [20] to decomposition problem of predictions of all unknown vectors under a set of linear regression models. We also consider seemingly unrelated regression (SUR) models and give the corresponding results for these models and their system. The derivation process of the results consists of heavy matrix operations including the Moore-Penrose generalized inverses of matrices. Therefore, we use some rank formulas of block matrices to simplify various matrix expressions.

The problems on connections of the BLUEs and the ordinary least-squares estimators (OLSEs) of unknown parameters under SUR models were investigated in [9], recently. SUR models were originally proposed by [25] in which a method was given for joint estimation of unknown parameters by combining the equations in SUR models. For other studies on SUR models, we may refer to [2]-[7], [10]-[12], and [17, 18, 24]. For a system of linear regression models, we may refer to the studies [14, 16, 19].

2 Preliminaries

Throughout this paper, the following terminology and notation are used. Let $\mathbb{R}^{m\times n}$ be the set of all real matrices of dimension $m \times n$. $\mathbf{I}_m$ denotes the $m \times m$ identity matrix. The notations $\mathbf{A}'$, $\mathbf{A}^+$, $\boldsymbol{r}(\mathbf{A})$, and $\mathcal{C}(\mathbf{A})$, denote the transpose, the Moore–Penrose generalized inverse, the rank, and the column space of $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. $\mathbf{P_A} = \mathbf{AA}^+$ and $\mathbf{E_A} = \mathbf{A}^\perp = \mathbf{I}_m - \mathbf{AA}^+$ stand for the orthogonal projectors. $\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$ denotes the Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

Consider the set of m regression models formulated by

$$\mathcal{M}_i : \ \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i, \ \ E(\boldsymbol{\varepsilon}_i) = \mathbf{0} \ \text{ and } \ D(\boldsymbol{\varepsilon}_i) = cov(\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_i) = \sigma^2\boldsymbol{\Sigma}_{ii}, \tag{2.1}$$

where $\mathbf{y}_i \in \mathbb{R}^{n_i\times 1}$ is an observable random vector, $\mathbf{X}_i \in \mathbb{R}^{n_i\times p_i}$ is a known matrix of arbitrary rank, $\boldsymbol{\beta}_i \in \mathbb{R}^{p_i\times 1}$ is an unknown parameter vector, $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{n_i\times 1}$ is an error vector, σ^2 is a positive unknown parameter, and $\boldsymbol{\Sigma}_{ii} \in \mathbb{R}^{n_i\times n_i}$ is a known or unknown nonnegative definite (nnd) matrix of arbitrary rank, $i = 1, \dots, m$. We can combine the m models in (2.1) by considering

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_m \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_m \end{bmatrix}, \ \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_m \end{bmatrix}, \ \boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_m \end{bmatrix},$$

and then the following model can be written

$$\mathcal{M}: \ \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \ \ E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ \ D(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{\Sigma}, \tag{2.2}$$

the combined form of the models $\mathcal{M}_i$, where $\mathbf{y} \in \mathbb{R}^{n\times 1}$, $\mathbf{X} \in \mathbb{R}^{n\times p}$, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$, $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ with $n_1 + \cdots + n_m = n$ and $p_1 + \cdots + p_m = p$, and $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}_{ik}) \in \mathbb{R}^{n\times n}$ is a nnd matrix of arbitrary rank, $i, k = 1, \ldots, m$. In general, combining the group of models is a common procedure to gain more efficiency in prediction/estimation. The combined model $\mathcal{M}$ in (2.2) allows us both to consider the models $\mathcal{M}_i$ separately by pre-multiplying $\mathcal{M}$ by transformation matrix $\big[\mathbf{0}, \ldots, \mathbf{I}_{n_i}, \ldots, \mathbf{0}\big]$. Each of the m models in (2.1) can be considered as a single regression model of the system in (2.2). In what follows, it is assumed that $\mathcal{M}$ is consistent, i.e., $\mathbf{y} \in \mathcal{C}\big[\mathbf{X},\ \boldsymbol{\Sigma}\big]$ holds with probability 1; see [15]. If $\mathcal{M}$ is consistent, then $\mathcal{M}_i$ is consistent; see [23].

$\mathcal{M}_i$ is called as a SUR model when $n_1 = \cdots = n_m = n$ and $D(\boldsymbol{\varepsilon}_i) = \sigma_{ii}\mathbf{I}_n$ in (2.1). In this case, the dispersion matrix in (2.2) is expressed as $D(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} \otimes \mathbf{I}_n$, where $\boldsymbol{\Sigma} = (\sigma_{ik}) \in \mathbb{R}^{m\times m}$ is a nnd matrix, $i, k = 1, \ldots, m$. We use the notation $\mathcal{S}_i$ for SUR models. We also use the notation $\mathcal{S}$ for SUR models' system, i.e., $\mathcal{S}$ corresponds to the combined model of $\mathcal{S}_i$, in the study.

Let $\mathbf{K} \in \mathbb{R}^{k\times p}$ and $\mathbf{H} \in \mathbb{R}^{k\times n}$ be given matrices. Since the matrix $\mathbf{X}$ in (2.2) is block diagonal, the linear transformations $\mathbf{K}\boldsymbol{\beta}$ and $\mathbf{H}\boldsymbol{\varepsilon}$ under (2.2) can be written as $\mathbf{K}\boldsymbol{\beta} = \mathbf{K}_1\boldsymbol{\beta}_1 + \cdots + \mathbf{K}_m\boldsymbol{\beta}_m$ and $\mathbf{H}\boldsymbol{\varepsilon} = \mathbf{H}_1\boldsymbol{\varepsilon}_1 + \cdots + \mathbf{H}_m\boldsymbol{\varepsilon}_m$, respectively, where $\mathbf{K} = \big[\mathbf{K}_1, \ldots, \mathbf{K}_m\big]$ with $\mathbf{K}_i \in \mathbb{R}^{k\times p_i}$ and $\mathbf{H} = \big[\mathbf{H}_1, \ldots, \mathbf{H}_m\big]$ with $\mathbf{H}_i \in \mathbb{R}^{k\times n_i}$, $n_1 + \cdots + n_m = n$ and $p_1 + \cdots + p_m = p$, $i = 1, \ldots, m$; see, also [20]. To establish some general results on simultaneous predictions/estimations of all unknown vectors under models $\mathcal{M}$ and $\mathcal{M}_i$, we can consider the following vector

$$\boldsymbol{\phi}_i = \mathbf{K}_i\boldsymbol{\beta}_i + \mathbf{H}_i\boldsymbol{\varepsilon}_i \text{ or, equivalently, } \boldsymbol{\phi}_i = \widehat{\mathbf{K}}_i\boldsymbol{\beta} + \widehat{\mathbf{H}}_i\boldsymbol{\varepsilon}, \tag{2.3}$$

where $\widehat{\mathbf{K}}_i = \big[\mathbf{0}, \ldots, \mathbf{K}_i, \ldots, \mathbf{0}\big]$ and $\widehat{\mathbf{H}}_i = \big[\mathbf{0}, \ldots, \mathbf{H}_i, \ldots, \mathbf{0}\big]$, $i = 1, \ldots, m$. Then we can take the vector

$$\boldsymbol{\phi} = \mathbf{K}\boldsymbol{\beta} + \mathbf{H}\boldsymbol{\varepsilon} = \mathbf{K}_1\boldsymbol{\beta}_1 + \mathbf{H}_1\boldsymbol{\varepsilon}_1 + \cdots + \mathbf{K}_m\boldsymbol{\beta}_m + \mathbf{H}_m\boldsymbol{\varepsilon}_m = \boldsymbol{\phi}_1 + \cdots + \boldsymbol{\phi}_m. \tag{2.4}$$

According to the assumptions in the models $\mathcal{M}_i$ and $\mathcal{M}$,

$$D(\boldsymbol{\phi}) = \sigma^2\mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' \text{ and } D(\boldsymbol{\phi}_i) = \sigma^2\mathbf{H}_i\boldsymbol{\Sigma}_{ii}\mathbf{H}_i' = \sigma^2\widehat{\mathbf{H}}_i\boldsymbol{\Sigma}\widehat{\mathbf{H}}_i', \tag{2.5}$$

$$\begin{gathered} cov(\boldsymbol{\phi}, \mathbf{y}) = \sigma^2\mathbf{H}\boldsymbol{\Sigma}, \ \ cov(\boldsymbol{\phi}_i, \mathbf{y}) = \sigma^2\widehat{\mathbf{H}}_i\boldsymbol{\Sigma}, \\ cov(\boldsymbol{\phi}_i, \mathbf{y}_i) = \sigma^2\mathbf{H}_i\boldsymbol{\Sigma}_{ii} = \sigma^2\widehat{\mathbf{H}}_i\boldsymbol{\Sigma}\mathbf{T}_i', \end{gathered} \tag{2.6}$$

where $\mathbf{T}_i = \big[\mathbf{0}, \ldots, \mathbf{I}_{n_i}, \ldots, \mathbf{0}\big]$, $i = 1, \ldots, m$. (2.5) and (2.6) turn into $D(\boldsymbol{\phi}) = \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{H}'$, $D(\boldsymbol{\phi}_i) = \sigma_{ii}\mathbf{H}_i\mathbf{H}_i' = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\widehat{\mathbf{H}}_i'$, $cov(\boldsymbol{\phi}, \mathbf{y}) = \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)$, $cov(\boldsymbol{\phi}_i, \mathbf{y}) = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)$, and $cov(\boldsymbol{\phi}_i, \mathbf{y}_i) = \sigma_{ii}\mathbf{H}_i = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{T}_i'$ when we consider the models $\mathcal{S}_i$ and $\mathcal{S}$.

We collect rank formulas of block matrices in the following two lemmas; see [13] and [20], respectively.

Lemma 2.1. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, *and* $\mathbf{D} \in \mathbb{R}^{l\times k}$. *Then,* *(a)* $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, \ \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{E_A B}) = \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{E_B A})$. *In particular,* $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, \ \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) \Leftrightarrow \mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A})$ *and* $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, \ \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{B}) \Leftrightarrow \mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{B}) = \{\mathbf{0}\}$.

(b) $\boldsymbol{r}\begin{bmatrix}\mathbf{A} \\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{C E_{A'}}) = \boldsymbol{r}(\mathbf{C}) + \boldsymbol{r}(\mathbf{A E_{C'}})$. *In particular,* $\boldsymbol{r}\begin{bmatrix}\mathbf{A} \\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) \Leftrightarrow \mathcal{C}(\mathbf{C}') \subseteq \mathcal{C}(\mathbf{A}')$.

(c) $\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{D} - \mathbf{C A^+ B})$ *if* $\mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A})$ *and* $\mathcal{C}(\mathbf{C}') \subseteq \mathcal{C}(\mathbf{A}')$.

Lemma 2.2. *Let* $\mathbf{A} \in \mathbb{R}^{m\times m}$ *be a nnd and let* $\mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{m\times k}$, *and* $\mathbf{D} \in \mathbb{R}^{t\times k}$. $\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{A}, \ \mathbf{B}\end{bmatrix} + \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{D})$ *if* $\mathcal{C}(\mathbf{C}) \subseteq \mathcal{C}(\mathbf{A})$.

3 BLUPs and OLSPs under the models

The vector $\boldsymbol{\phi}$ in (2.4) is said to be predictable under $\mathcal{M}$, if there exists a linear statistic $\mathbf{Ly}$ with $\mathbf{L} \in \mathbb{R}^{k\times n}$ such that $E(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0}$ holds, i.e., $\mathcal{C}(\mathbf{K}') \subseteq \mathcal{C}(\mathbf{X}')$. This also means that $\mathbf{K}\boldsymbol{\beta}$ is an estimable parametric function under $\mathcal{M}$; see, e.g., [1]. It is obvious that $\mathbf{X}\boldsymbol{\beta}$ is always estimable and $\mathbf{H}\boldsymbol{\varepsilon}$ (also $\boldsymbol{\varepsilon}$) is always predictable under $\mathcal{M}$. Further, if there exists a matrix $\mathbf{L}$ such that

$$D(\mathbf{Ly} - \boldsymbol{\phi}) = \min \text{ s.t. } E(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0} \tag{3.1}$$

holds in the Löwner partial ordering, then the linear statistic $\mathbf{Ly}$ is defined to be the BLUP of $\boldsymbol{\phi}$, a term introduced by [8], and is denoted by $\mathbf{Ly} = BLUP_{\mathcal{M}}(\boldsymbol{\phi}) = BLUP_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta} + \mathbf{H}\boldsymbol{\varepsilon})$. If $\mathbf{H} = \mathbf{0}$, then the linear statistic $\mathbf{Ly}$ in (3.1) is the BLUE of $\mathbf{K}\boldsymbol{\beta}$, denoted by $BLUE_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta})$, under $\mathcal{M}$. To establish the fundamental results on BLUPs, we use Theorem 3.2 in [22]. Using our notations and considerations in this theorem, we obtain the following results. Let $\boldsymbol{\phi}$ be predictable under $\mathcal{M}$. Then

$$D(\mathbf{Ly} - \boldsymbol{\phi}) = \min \text{ s.t. } E(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0} \Longleftrightarrow \mathbf{L}\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{K}, \ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}. \tag{3.2}$$

The general solution of $\mathbf{L}$ and the corresponding $BLUP_{\mathcal{M}}(\boldsymbol{\phi})$ can be written as

$$BLUP_{\mathcal{M}}(\boldsymbol{\phi}) = \mathbf{Ly} = \left([\mathbf{K}, \ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}, \tag{3.3}$$

where $\mathbf{U} \in \mathbb{R}^{k\times n}$ is an arbitrary matrix. $\mathbf{L}$ is unique $\Longleftrightarrow$ $\boldsymbol{r}\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\end{bmatrix} = n$ and $BLUP_{\mathcal{M}}(\boldsymbol{\phi})$ is unique $\Longleftrightarrow$ $\mathcal{M}$ consistent. Furthermore,

$$D[\boldsymbol{\phi} - BLUP_{\mathcal{M}}(\boldsymbol{\phi})] = \sigma^2([\mathbf{K}, \ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\mathbf{W}^{+} - \mathbf{H})\boldsymbol{\Sigma}([\mathbf{K}, \ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\mathbf{W}^{+} - \mathbf{H})', \tag{3.4}$$

where $\mathbf{W} = \begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}$. We obtain the BLUEs of $\mathbf{K}\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$, and the BLUP of $\boldsymbol{\varepsilon}$ under $\mathcal{M}$ for different choices of the matrices $\mathbf{K}$ and $\mathbf{H}$ in (3.3).

The predictability conditions of $\boldsymbol{\phi}_i$ in (2.3) under $\mathcal{M}$ and $\mathcal{M}_i$ are written as $\mathcal{C}(\widehat{\mathbf{K}}_i') \subseteq \mathcal{C}(\mathbf{X}')$ and $\mathcal{C}(\mathbf{K}_i') \subseteq \mathcal{C}(\mathbf{X}_i')$, respectively. These two conditions are equivalent according to our considerations. Let $\boldsymbol{\phi}_i$ be predictable under $\mathcal{M}_i$ (also predictable under $\mathcal{M}$). Then we obtain the following results on BLUPs by using (3.2)-(3.4).

$$BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i) = \mathbf{L}_i\mathbf{y} = \left([\widehat{\mathbf{K}}_i, \ \widehat{\mathbf{H}}_i\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}, \ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}, \tag{3.5}$$

$$D[\boldsymbol{\phi}_i - BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i)] = \sigma^2([\widehat{\mathbf{K}}_i,\ \widehat{\mathbf{H}}_i\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\mathbf{W}^{+} - \widehat{\mathbf{H}}_i)\boldsymbol{\Sigma}([\widehat{\mathbf{K}}_i,\ \widehat{\mathbf{H}}_i\boldsymbol{\Sigma}\mathbf{X}^{\perp}]\mathbf{W}^{+} - \widehat{\mathbf{H}}_i)', \tag{3.6}$$

$$BLUP_{\mathcal{M}_i}(\boldsymbol{\phi}_i) = \mathbf{G}_i\mathbf{y}_i = \left([\mathbf{K}_i,\ \mathbf{H}_i\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]\left[\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\right]^{+} + \mathbf{V}_i\left[\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\right]^{\perp}\right)\mathbf{y}_i, \tag{3.7}$$

$$D[\boldsymbol{\phi}_i - BLUP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)] = \sigma^2([\mathbf{K}_i,\ \mathbf{H}_i\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]\mathbf{W}_i^{+} - \mathbf{H}_i)\boldsymbol{\Sigma}_{ii}([\mathbf{K}_i,\ \mathbf{H}_i\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]\mathbf{W}_i^{+} - \mathbf{H}_i)', \tag{3.8}$$

where $\mathbf{U}_i \in \mathbb{R}^{k\times n}$ and $\mathbf{V}_i \in \mathbb{R}^{k\times n_i}$ are arbitrary matrices, and $\mathbf{W} = \left[\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\right]$, $\mathbf{W}_i = \left[\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\right]$. We obtain the BLUEs of $\mathbf{K}_i\boldsymbol{\beta}_i$ and $\mathbf{X}_i\boldsymbol{\beta}_i$, and the BLUP of $\boldsymbol{\varepsilon}_i$ under $\mathcal{M}_i$ and $\mathcal{M}$ for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}_i$ in (3.5) and (3.7).

The OLSP of $\boldsymbol{\phi}$ is defined to be

$$OLSP_{\mathcal{M}}(\boldsymbol{\phi}) = OLSP_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta} + \mathbf{H}\boldsymbol{\varepsilon}) = OLSE_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta}) + OLSP_{\mathcal{M}}(\mathbf{H}\boldsymbol{\varepsilon}),$$

where $OLSE_{\mathcal{M}}(\boldsymbol{\beta}) = \operatorname{argmin}_{\boldsymbol{\beta}\in\mathbb{R}^{p\times 1}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$ and $OLSP_{\mathcal{M}}(\boldsymbol{\varepsilon}) = \mathbf{y} - OLSE_{\mathcal{M}}(\mathbf{X}\boldsymbol{\beta})$. To establish the fundamental results on OLSPs, we use our notations and considerations in Theorem 5.1 in [22]. Let $\boldsymbol{\phi}$ be predictable under $\mathcal{M}$. Then

$$OLSE_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathbf{K}\mathbf{X}^{+}\mathbf{y} \text{ and } OLSP_{\mathcal{M}}(\mathbf{H}\boldsymbol{\varepsilon}) = \mathbf{H}\mathbf{X}^{\perp}\mathbf{y}, \tag{3.9}$$

$$OLSP_{\mathcal{M}}(\boldsymbol{\phi}) = (\mathbf{K}\mathbf{X}^{+} + \mathbf{H}\mathbf{X}^{\perp})\mathbf{y}, \tag{3.10}$$

$$D[\boldsymbol{\phi} - OLSP_{\mathcal{M}}(\boldsymbol{\phi})] = \sigma^2(\mathbf{K}\mathbf{X}^{+} - \mathbf{H}\mathbf{P}_{\mathbf{X}})\boldsymbol{\Sigma}(\mathbf{K}\mathbf{X}^{+} - \mathbf{H}\mathbf{P}_{\mathbf{X}})'. \tag{3.11}$$

Let $\boldsymbol{\phi}_i$ be predictable under $\mathcal{M}_i$. According to (3.10) and (3.11), we can give the following results.

$$OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i) = (\widehat{\mathbf{K}}_i\mathbf{X}^{+} + \widehat{\mathbf{H}}_i\mathbf{X}^{\perp})\mathbf{y}, \tag{3.12}$$

$$D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i)] = \sigma^2(\widehat{\mathbf{K}}_i\mathbf{X}^{+} - \widehat{\mathbf{H}}_i\mathbf{P}_{\mathbf{X}})\boldsymbol{\Sigma}(\widehat{\mathbf{K}}_i\mathbf{X}^{+} - \widehat{\mathbf{H}}_i\mathbf{P}_{\mathbf{X}})', \tag{3.13}$$

$$OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i) = (\mathbf{K}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})\mathbf{y}_i, \tag{3.14}$$

$$D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)] = \sigma^2(\mathbf{K}_i\mathbf{X}_i^{+} - \mathbf{H}_i\mathbf{P}_{\mathbf{X}_i})\boldsymbol{\Sigma}_{ii}(\mathbf{K}_i\mathbf{X}_i^{+} - \mathbf{H}_i\mathbf{P}_{\mathbf{X}_i})'. \tag{3.15}$$

Consider the SUR models $\mathcal{S}_i$ and their combined model $\mathcal{S}$. Let $\boldsymbol{\phi}$ be predictable under $\mathcal{S}$ and $\boldsymbol{\phi}_i$ be predictable under $\mathcal{S}_i$. Replacing $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}_{ii}$, $\mathbf{W}$, and $\mathbf{W}_i$ by $(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)$, $\sigma_{ii}\mathbf{I}_n$, $\left[\mathbf{X},\ (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right]$ and $\left[\mathbf{X}_i,\ \sigma_{ii}\mathbf{X}_i^{\perp}\right]$, respectively, in (3.3)-(3.8) and (3.10)-(3.15), we obtain the BLUPs and the OLSPs of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_i$ and their dispersion matrices under SUR models. We note that the BLUP and the OLSP of $\boldsymbol{\phi}_i$ under $\mathcal{S}_i$ coincide since $D(\mathbf{y}_i) = \sigma_{ii}\mathbf{I}_n$.

4 Equalities of BLUPs and OLSPs

In this section, we give results on equalities between BLUPs and OLSPs under considered models.

Theorem 4.1. *Let $\mathcal{M}_i$ and $\mathcal{M}$ be as given in (2.1) and (2.2), respectively, and $\boldsymbol{\phi}_i$ be predictable under $\mathcal{M}_i$ (also predictable under $\mathcal{M}$), $i = 1, \dots, m$. Let $BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i)$, $OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i)$, and $OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$ be as given in (3.5), (3.12), and (3.14), respectively, and denote $\widehat{\boldsymbol{\Sigma}}_i = \big[\boldsymbol{\Sigma}_{i1}, \dots, \boldsymbol{\Sigma}_{im}\big]$. Then the following statements are equivalent.*

(a) $BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$, *i.e.,* $\boldsymbol{\phi}_i - BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i) = \boldsymbol{\phi}_i - OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i) = \boldsymbol{\phi}_i - OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$,

(b) $D[\boldsymbol{\phi}_i - BLUP_{\mathcal{M}}(\boldsymbol{\phi}_i)] = D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i)] = D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)]$,

(c) $(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}\boldsymbol{\Sigma}\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}\boldsymbol{\Sigma}]'\right) \subseteq \mathcal{C}(\mathbf{X}')$,

(d) $(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\widehat{\boldsymbol{\Sigma}}_i]'\right) \subseteq \mathcal{C}(\mathbf{X}')$, *or equivalently,* $(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\boldsymbol{\Sigma}_{ij}]'\right) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1, \dots, m$,

(e) $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i)$, *i.e.,* $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp} \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i)$ *for all* $j = 1, \dots, m$,

(f) $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X})$, *i.e.,* $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_{ij} \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_j' \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X}_j)$ *for all* $j = 1, \dots, m$,

(g) $\mathcal{C}\left([\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i,\ \mathbf{0}]'\right) \subseteq \mathcal{C}\left(\left[\mathbf{X}_i'\mathbf{X}_i,\ \mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp}\right]'\right)$, *i.e.,* $\mathcal{C}\left([\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i,\ \mathbf{0}]'\right) \subseteq \mathcal{C}\left(\left[\mathbf{X}_i'\mathbf{X}_i,\ \mathbf{X}_i'\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp}\right]'\right)$ *for all* $j = 1, \dots, m$.

Proof. The expressions $\widehat{\mathbf{K}}_i\mathbf{X}^{+}\mathbf{y}$ and $\widehat{\mathbf{H}}_i\mathbf{X}^{\perp}\mathbf{y}$ are equivalently written as $\mathbf{K}_i\mathbf{X}_i^{+}\mathbf{y}_i$ and $\mathbf{H}_i\mathbf{X}_i^{\perp}\mathbf{y}_i$, respectively, by using the properties of the Moore–Penrose generalized inverse of block diagonal matrices, where $\widehat{\mathbf{K}}_i = \big[\mathbf{0}, \dots, \mathbf{K}_i, \dots, \mathbf{0}\big]$ and $\widehat{\mathbf{H}}_i = \big[\mathbf{0}, \dots, \mathbf{H}_i, \dots, \mathbf{0}\big]$. This shows $OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$ in (a) and also $D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i)] = D[\boldsymbol{\phi}_i - OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)]$ in (b). The equivalences between (a) and (c), and between (b) and (c) are written from [22, Theorems 6.1 and 6.2], respectively, by using our notations. The first equality in (c) is equivalently written as $(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp} = \mathbf{0}$ for all $j = 1, \dots, m$, i.e., (d) is obtained. The equivalences of expressions in (c) and (d) are obvious. The first equality in (d) can be written as $(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)(\mathbf{X}_i'\mathbf{X}_i)^{+}\mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} = \mathbf{0}$. Applying Lemma 2.1 (c) to this equality,

$$\boldsymbol{r}\left((\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp}\right) = \boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}_i'\mathbf{X}_i) \tag{4.1}$$

is obtained. The equalities in (e) are seen from (4.1). From Lemma 2.1 (b), it is seen that (e) and (f) are equivalent. The equivalence of (e) and (g) follows from Lemma 2.1 (a). □

Many consequences can be derived from Theorem 4.1 for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}_i$. Further, Theorem 4.1 can be applied to SUR models. We give the following corollaries that correspond to these cases.

Corollary 4.2. *Let $\mathcal{M}_i$ and $\mathcal{M}$ be as given in (2.1) and (2.2), respectively, $i = 1,\ldots,m$. Then the following statements are equivalent.*

(a) $BLUE_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta}_i) = OLSE_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta}_i) = OLSE_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\beta}_i)$,

(b) $BLUP_{\mathcal{M}}(\boldsymbol{\varepsilon}_i) = OLSP_{\mathcal{M}}(\boldsymbol{\varepsilon}_i) = OLSP_{\mathcal{M}_i}(\boldsymbol{\varepsilon}_i)$,

(c) $D[BLUE_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta}_i)] = D[OLSE_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta}_i)] = D[OLSE_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\beta}_i)]$,

(d) $D[\boldsymbol{\varepsilon}_i - BLUP_{\mathcal{M}}(\boldsymbol{\varepsilon}_i)] = D[\boldsymbol{\varepsilon}_i - OLSP_{\mathcal{M}}(\boldsymbol{\varepsilon}_i)] = D[\boldsymbol{\varepsilon}_i - OLSP_{\mathcal{M}_i}(\boldsymbol{\varepsilon}_i)]$,

(e) $\widehat{\mathbf{X}}_i\mathbf{X}^{+}\boldsymbol{\Sigma}\mathbf{X}^{\perp} = \mathbf{0}$, *where* $\widehat{\mathbf{X}}_i = \left[\mathbf{0},\ldots,\mathbf{X}_i,\ldots,\mathbf{0}\right]$,

(f) $\mathbf{P}_{\mathbf{X}_i}\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left(\widehat{\boldsymbol{\Sigma}}_i'\mathbf{P}_{\mathbf{X}_i}\right) \subseteq \mathcal{C}(\mathbf{X}')$, *or equivalently,* $\mathbf{P}_{\mathbf{X}_i}\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left(\boldsymbol{\Sigma}_{ij}'\mathbf{P}_{\mathbf{X}_i}\right) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1,\ldots,m$,

(g) $\mathbf{P}_{\mathbf{X}_i}\widehat{\boldsymbol{\Sigma}}_i = \mathbf{P}_{\mathbf{X}_i}\widehat{\boldsymbol{\Sigma}}_i\mathbf{P}_{\mathbf{X}}$, *i.e.,* $\mathbf{P}_{\mathbf{X}_i}\boldsymbol{\Sigma}_{ij} = \mathbf{P}_{\mathbf{X}_i}\boldsymbol{\Sigma}_{ij}\mathbf{P}_{\mathbf{X}_j}$ *for all* $j = 1,\ldots,m$,

(h) $\mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left(\widehat{\boldsymbol{\Sigma}}_i'\mathbf{X}_i\right) \subseteq \mathcal{C}(\mathbf{X}')$, *or equivalently,* $\mathbf{X}_i'\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left(\boldsymbol{\Sigma}_{ij}'\mathbf{X}_i\right) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1,\ldots,m$,

(i) $\mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i = \mathbf{X}_i'\widehat{\boldsymbol{\Sigma}}_i\mathbf{P}_{\mathbf{X}}$, *i.e.,* $\mathbf{X}_i'\boldsymbol{\Sigma}_{ij} = \mathbf{X}_i'\boldsymbol{\Sigma}_{ij}\mathbf{P}_{\mathbf{X}_j}$ *for all* $j = 1,\ldots,m$.

Corollary 4.3. *Consider the models $\mathcal{S}_i$ and $\mathcal{S}$ and let $\boldsymbol{\sigma}_i = \left[\sigma_{i1},\ldots,\sigma_{im}\right]$, $i = 1,\ldots,m$. Then the following results hold.*

(a) Let $\boldsymbol{\phi}_i$ be predictable under $\mathcal{S}_i$ (also predictable under $\mathcal{S}$). Then the following statements are equivalent.

  (i) $BLUP_{\mathcal{S}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{S}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{S}_i}(\boldsymbol{\phi}_i)$, *i.e.,* $\boldsymbol{\phi}_i - BLUP_{\mathcal{S}}(\boldsymbol{\phi}_i) = \boldsymbol{\phi}_i - OLSP_{\mathcal{S}}(\boldsymbol{\phi}_i) = \boldsymbol{\phi}_i - OLSP_{\mathcal{S}_i}(\boldsymbol{\phi}_i)$,

  (ii) $D[\boldsymbol{\phi}_i - BLUP_{\mathcal{S}}(\boldsymbol{\phi}_i)] = D[\boldsymbol{\phi}_i - OLSP_{\mathcal{S}}(\boldsymbol{\phi}_i)] = D[\boldsymbol{\phi}_i - OLSP_{\mathcal{S}_i}(\boldsymbol{\phi}_i)]$,

  (iii) $(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)]'\right) \subseteq \mathcal{C}(\mathbf{X}')$,

  (iv) $(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}(\boldsymbol{\sigma}_i\otimes\mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}(\boldsymbol{\sigma}_i\otimes\mathbf{I}_n)]'\right) \subseteq \mathcal{C}(\mathbf{X}')$, *or equivalently,* $\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\mathbf{X}_j^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}]'\right) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1,\ldots,m$,

  (v) $\boldsymbol{r}\begin{bmatrix}\mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'(\boldsymbol{\sigma}_i\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0}\end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i)$, *i.e.,* $\boldsymbol{r}\begin{bmatrix}\mathbf{X}_i'\mathbf{X}_i & \sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp}\\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0}\end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i)$ *for all* $j = 1,\ldots,m$,

(vi) $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \mathbf{X}_i'(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n) \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X})$, *i.e.*, $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_i'\mathbf{X}_i & \sigma_{ij}\mathbf{X}_i' \\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_j' \end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X}_j)$ *for all* $j = 1, \ldots, m$,

(vii) $\mathcal{C}\left([\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i,\ \mathbf{0}]'\right) \subseteq \mathcal{C}\left(\left[\mathbf{X}_i'\mathbf{X}_i,\ \mathbf{X}_i'(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\right]'\right)$, *i.e.*, $\mathcal{C}\left([\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i,\ \mathbf{0}]'\right) \subseteq \mathcal{C}\left(\left[\mathbf{X}_i'\mathbf{X}_i,\ \sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp}\right]'\right)$ *for all* $j = 1, \ldots, m$.

(b) *The following statements are equivalent.*

(i) $BLUE_{\mathcal{S}}(\mathbf{X}_i\boldsymbol{\beta}_i) = OLSE_{\mathcal{S}}(\mathbf{X}_i\boldsymbol{\beta}_i) = OLSE_{\mathcal{S}_i}(\mathbf{X}_i\boldsymbol{\beta}_i)$,

(ii) $BLUP_{\mathcal{S}}(\boldsymbol{\varepsilon}_i) = OLSP_{\mathcal{S}}(\boldsymbol{\varepsilon}_i) = OLSP_{\mathcal{S}_i}(\boldsymbol{\varepsilon}_i)$,

(iii) $D[BLUE_{\mathcal{S}}(\mathbf{X}_i\boldsymbol{\beta}_i)] = D[OLSE_{\mathcal{S}}(\mathbf{X}_i\boldsymbol{\beta}_i)] = D[OLSE_{\mathcal{S}_i}(\mathbf{X}_i\boldsymbol{\beta}_i)]$,

(iv) $D[\boldsymbol{\varepsilon}_i - BLUP_{\mathcal{S}}(\boldsymbol{\varepsilon}_i)] = D[\boldsymbol{\varepsilon}_i - OLSP_{\mathcal{S}}(\boldsymbol{\varepsilon}_i)] = D[\boldsymbol{\varepsilon}_i - OLSP_{\mathcal{S}_i}(\boldsymbol{\varepsilon}_i)]$,

(v) $\widehat{\mathbf{X}}_i\mathbf{X}^{+}(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *where* $\widehat{\mathbf{X}}_i = \left[\mathbf{0}, \ldots, \mathbf{X}_i, \ldots, \mathbf{0}\right]$,

(vi) $\mathbf{P}_{\mathbf{X}_i}(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.*, $\sigma_{ij}\mathbf{P}_{\mathbf{X}_i}\mathbf{X}_j^{\perp} = \mathbf{0}$ *for all* $j = 1, \ldots, m$,

(vii) $\mathbf{P}_{\mathbf{X}_i}(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n) = \mathbf{P}_{\mathbf{X}_i}(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)\mathbf{P}_{\mathbf{X}}$, *or equivalently*, $\mathbf{P}_{\mathbf{X}_i} = \mathbf{P}_{\mathbf{X}_i}\mathbf{P}_{\mathbf{X}_j}$ *for all* $j = 1, \ldots, m$,

(viii) $\mathbf{X}_i'(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.*, $\sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp} = \mathbf{0}$ *for all* $j = 1, \ldots, m$,

(ix) $\mathbf{X}_i'(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n) = \mathbf{X}_i'(\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)\mathbf{P}_{\mathbf{X}}$, *i.e.*, $\mathbf{X}_i = \mathbf{P}_{\mathbf{X}_j}\mathbf{X}_i$ *for all* $j = 1, \ldots, m$,

(x) $\mathcal{C}\left((\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)'\mathbf{P}_{\mathbf{X}_i}\right) \subseteq \mathcal{C}(\mathbf{X}')$, *or equivalently*, $\mathcal{C}(\mathbf{P}_{\mathbf{X}_i}) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1, \ldots, m$,

(xi) $\mathcal{C}\left((\boldsymbol{\sigma}_i \otimes \mathbf{I}_n)'\mathbf{X}_i\right) \subseteq \mathcal{C}(\mathbf{X}')$, *i.e.*, $\mathcal{C}(\mathbf{X}_i) \subseteq \mathcal{C}(\mathbf{X}_j')$ *for all* $j = 1, \ldots, m$.

5 Additive decomposition of BLUPs and OLSPs

In this section, we give results on additive decomposition equalities of BLUPs and OLSPs under considered models.

Theorem 5.1. *Let* $\mathcal{M}_i$ *and* $\mathcal{M}$ *be as given in (2.1) and (2.2),* $i = 1, \ldots, m$. *Assume that* $\boldsymbol{\phi}_i$ *is predictable under* $\mathcal{M}_i$ *and* $\boldsymbol{\phi}$ *is predictable under* $\mathcal{M}$. *Denote* $\mathbf{D}_{\boldsymbol{\Sigma}} = \mathrm{diag}(\boldsymbol{\Sigma}_{11}, \ldots, \boldsymbol{\Sigma}_{mm})$. *Then the following statements are equivalent.*

(a) $BLUP_{\mathcal{M}}(\boldsymbol{\phi}) = BLUP_{\mathcal{M}_1}(\boldsymbol{\phi}_1) + \cdots + BLUP_{\mathcal{M}_m}(\boldsymbol{\phi}_m)$,

(b) $\boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \\ \mathbf{0} & \mathbf{K} - \mathbf{H}\mathbf{X} & \mathbf{0} \end{bmatrix} = 2\boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\left[\mathbf{X},\ \mathbf{D}_{\boldsymbol{\Sigma}}\right]$,

(c) $\boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \\ \mathbf{0} & \mathbf{K} - \mathbf{H}\mathbf{X} & \mathbf{0} \end{bmatrix} = \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \end{bmatrix}$, *i.e.*, $\mathcal{C}\left(\left[\mathbf{0},\ \mathbf{K} - \mathbf{H}\mathbf{X},\ \mathbf{0}\right]'\right) \subseteq \mathcal{C}\left(\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \end{bmatrix}'\right)$.

Proof. The equality in (a) holds if and only if the coefficient matrices in (3.7) satisfy the equation in (3.2), i.e., $(\mathbf{G}_1\mathbf{T}_1 + \cdots + \mathbf{G}_m\mathbf{T}_m)\left[\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\right] = \left[\mathbf{K},\ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\right]$, or equivalently,

$$\begin{aligned}&\left[\mathbf{K}_1,\ \mathbf{H}_1\boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp}\right]\mathbf{W}_1^{+}\left[\widehat{\mathbf{X}}_1,\ \widehat{\boldsymbol{\Sigma}}_1\mathbf{X}^{\perp}\right] + \cdots \\ &+ \left[\mathbf{K}_m,\ \mathbf{H}_m\boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp}\right]\mathbf{W}_m^{+}\left[\widehat{\mathbf{X}}_m,\ \widehat{\boldsymbol{\Sigma}}_m\mathbf{X}^{\perp}\right] = \left[\mathbf{K},\ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\right]\end{aligned} \tag{5.1}$$

holds, and (5.1) is equivalently written as

$$\begin{aligned}&\Big[\left[\mathbf{K}_1,\ \mathbf{H}_1\boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp}\right],\ldots,\left[\mathbf{K}_m,\ \mathbf{H}_m\boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp}\right]\Big] \\ &\times \begin{bmatrix}\mathbf{W}_1 & \cdots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{W}_m\end{bmatrix}^{+}\begin{bmatrix}\widehat{\mathbf{X}}_1 & \widehat{\boldsymbol{\Sigma}}_1\mathbf{X}^{\perp}\\ \vdots & \vdots\\ \widehat{\mathbf{X}}_m & \widehat{\boldsymbol{\Sigma}}_m\mathbf{X}^{\perp}\end{bmatrix} - \left[\mathbf{K},\ \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\right] = \mathbf{0},\end{aligned} \tag{5.2}$$

where $\widehat{\mathbf{X}}_i = \left[\mathbf{0},\ldots,\mathbf{X}_i,\ldots,\mathbf{0}\right]$ and $\widehat{\boldsymbol{\Sigma}}_i = \left[\boldsymbol{\Sigma}_{i1},\ldots,\boldsymbol{\Sigma}_{im}\right]$. Applying Lemma 2.1 (c) to left hand side of (5.2), setting $\mathbf{W}_i$ and using Lemmas 2.1 and 2.2, the rank of (5.2) is equivalently written as

$$\begin{aligned}
&\boldsymbol{r}\begin{bmatrix}\mathbf{X}_1 & \boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp} & \cdots & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1 & \widehat{\boldsymbol{\Sigma}}_1\mathbf{X}^{\perp}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_m & \boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp} & \widehat{\mathbf{X}}_m & \widehat{\boldsymbol{\Sigma}}_m\mathbf{X}^{\perp}\\ \mathbf{K}_1 & \mathbf{H}_1\boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp} & \cdots & \mathbf{K}_m & \mathbf{H}_m\boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp} & \mathbf{K} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}\\
&\quad - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_1 & \boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_m & \boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp}\end{bmatrix}\\
&= \boldsymbol{r}\begin{bmatrix}\widehat{\mathbf{X}}_1 & \boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp} & \cdots & \mathbf{0} & \widehat{\boldsymbol{\Sigma}}_1\mathbf{X}^{\perp}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \widehat{\mathbf{X}}_m & \mathbf{0} & \cdots & \boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp} & \widehat{\boldsymbol{\Sigma}}_m\mathbf{X}^{\perp}\\ \mathbf{K} & \mathbf{H}_1\boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp} & \cdots & \mathbf{H}_m\boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix}\\
&\quad - \boldsymbol{r}\left[\mathbf{X}_1,\ \boldsymbol{\Sigma}_{11}\mathbf{X}_1^{\perp}\right] - \cdots - \boldsymbol{r}\left[\mathbf{X}_m,\ \boldsymbol{\Sigma}_{mm}\mathbf{X}_m^{\perp}\right]\\
&= \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp} & \boldsymbol{\Sigma}\mathbf{X}^{\perp}\\ \mathbf{K} & \mathbf{H}\mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\end{bmatrix} - \boldsymbol{r}\left[\mathbf{X}_1,\ \boldsymbol{\Sigma}_{11}\right] - \cdots - \boldsymbol{r}\left[\mathbf{X}_m,\ \boldsymbol{\Sigma}_{mm}\right]\\
&= \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp} & \boldsymbol{\Sigma}\mathbf{X}^{\perp}\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\end{bmatrix} - \boldsymbol{r}\left[\mathbf{X},\ \mathbf{D}_{\boldsymbol{\Sigma}}\right]\\
&= \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}} & \boldsymbol{\Sigma}\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix} - 2\boldsymbol{r}(\mathbf{X}) - \boldsymbol{r}\left[\mathbf{X},\ \mathbf{D}_{\boldsymbol{\Sigma}}\right]\\
&= \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\\ \mathbf{0} & \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0}\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{X} & \mathbf{D}_{\boldsymbol{\Sigma}}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix}.
\end{aligned}$$

Setting both sides of the last two expressions to zero leads to (b) and (c), respectively. The equivalence of statements in (c) is seen from Lemma 2.1 (b). □

Many consequences can be derived from the previous theorem for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}_i$, $i = 1, \ldots, m$. For example, by setting $\mathbf{H}_i = \mathbf{0}$ in Theorem 5.1, additive decomposition equalities of BLUE of $\mathbf{K}\boldsymbol{\beta}$ is obtained and they were given in [20, Theorem 4.5]. By setting $\mathbf{K}_i = \mathbf{X}_i$ and $\mathbf{H}_i = \mathbf{0}$ in Theorem 5.1, the results on decomposition of BLUEs of $\mathbf{X}\boldsymbol{\beta}$ are obtained and they were given in [20, Corollary 4.6]. In addition to these results, we give the following result on decomposition of BLUP of $\boldsymbol{\varepsilon}$ when setting $\mathbf{K}_i = \mathbf{0}$ and $\mathbf{H}_i = \mathbf{I}_{n_i}$ in Theorem 5.1 with combining the results given in [20, Corollary 4.6].

Corollary 5.2. *Let $\mathcal{M}_i$ and $\mathcal{M}$ be as given in (2.1) and (2.2), $i = 1, \ldots, m$. Then the following statements are equivalent.*

(a) $$BLUE_{\mathcal{M}}(\mathbf{X}\boldsymbol{\beta}) = \begin{bmatrix} BLUE_{\mathcal{M}}(\mathbf{X}_1\boldsymbol{\beta}_1) \\ \vdots \\ BLUE_{\mathcal{M}}(\mathbf{X}_m\boldsymbol{\beta}_m) \end{bmatrix} = \begin{bmatrix} BLUE_{\mathcal{M}_1}(\mathbf{X}_1\boldsymbol{\beta}_1) \\ \vdots \\ BLUE_{\mathcal{M}_m}(\mathbf{X}_m\boldsymbol{\beta}_m) \end{bmatrix},$$

(b) $$BLUP_{\mathcal{M}}(\boldsymbol{\varepsilon}) = \begin{bmatrix} BLUP_{\mathcal{M}}(\boldsymbol{\varepsilon}_1) \\ \vdots \\ BLUP_{\mathcal{M}}(\boldsymbol{\varepsilon}_m) \end{bmatrix} = \begin{bmatrix} BLUP_{\mathcal{M}_1}(\boldsymbol{\varepsilon}_1) \\ \vdots \\ BLUP_{\mathcal{M}_m}(\boldsymbol{\varepsilon}_m) \end{bmatrix},$$

(c) $$\boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{0} \\ \mathbf{D}_{\boldsymbol{\Sigma}} & \mathbf{0} & \mathbf{X} \end{bmatrix} = \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{D}_{\boldsymbol{\Sigma}} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} \end{bmatrix}, \text{ i.e., } \mathcal{C}\begin{bmatrix} \mathbf{X} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \cap \mathcal{C}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{D}_{\boldsymbol{\Sigma}} \\ \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix} = \{\mathbf{0}\},$$

(d) $\boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\left[\boldsymbol{\Sigma}\mathbf{X}^{\perp},\ \mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp}\right] = \boldsymbol{r}\left[\boldsymbol{\Sigma}\mathbf{X}^{\perp},\ \mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp},\ \mathbf{X}\right]$, *i.e.,* $\mathcal{C}(\mathbf{X}) \cap \mathcal{C}\left[\boldsymbol{\Sigma}\mathbf{X}^{\perp},\ \mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp}\right] = \{\mathbf{0}\}$,

(e) $$\boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} & \mathbf{0} \\ \mathbf{D}_{\boldsymbol{\Sigma}} & \mathbf{0} & \mathbf{X} \end{bmatrix} = \boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\left[\mathbf{D}_{\boldsymbol{\Sigma}},\ \mathbf{X}\right],$$

(f) $\mathcal{C}(\boldsymbol{\Sigma}\mathbf{X}^{\perp}) \subseteq \mathcal{C}(\mathbf{D}_{\boldsymbol{\Sigma}}\mathbf{X}^{\perp})$, *i.e.,* $\mathcal{C}(\boldsymbol{\Sigma}_{ij}\mathbf{X}_j^{\perp}) \subseteq \mathcal{C}(\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp})$, $j = 1, \ldots, m$, $i \neq j$.

Theorem 5.1 can be applied to SUR models, given in the following corollary.

Corollary 5.3. *Consider the models $\mathcal{S}_i$ and $\mathcal{S}$, $i = 1, \ldots, m$. Let $\mathbf{D}_{\boldsymbol{\sigma}} = \operatorname{diag}\left[\sigma_{11}, \ldots, \sigma_{mm}\right]$. Then the following results hold.*

(a) Let $\boldsymbol{\phi}_i$ be predictable under $\mathcal{S}_i$ and $\boldsymbol{\phi}$ be predictable under $\mathcal{S}$. Then the following statements are equivalent.

(i) $BLUP_{\mathcal{S}}(\boldsymbol{\phi}) = BLUP_{\mathcal{S}_1}(\boldsymbol{\phi}_1) + \cdots + BLUP_{\mathcal{S}_m}(\boldsymbol{\phi}_m)$,

(ii) $$\boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \\ \mathbf{0} & \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} \end{bmatrix} = 2\boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\left[\mathbf{X},\ \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\right] = 2\boldsymbol{r}(\mathbf{X}) + nm,$$

(iii) $\boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\\ \mathbf{0} & \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix}$, *i.e.,* $\mathcal{C}\left(\left[\mathbf{0},\ \mathbf{K}-\mathbf{H}\mathbf{X},\ \mathbf{0}\right]'\right) \subseteq \mathcal{C}\left(\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix}'\right)$.

(b) The following statements are equivalent.

(i) $BLUE_{\mathcal{S}}(\mathbf{X}\boldsymbol{\beta}) = \begin{bmatrix}BLUE_{\mathcal{S}}(\mathbf{X}_1\boldsymbol{\beta}_1)\\ \vdots\\ BLUE_{\mathcal{S}}(\mathbf{X}_m\boldsymbol{\beta}_m)\end{bmatrix} = \begin{bmatrix}BLUE_{\mathcal{S}_1}(\mathbf{X}_1\boldsymbol{\beta}_1)\\ \vdots\\ BLUE_{\mathcal{S}_m}(\mathbf{X}_m\boldsymbol{\beta}_m)\end{bmatrix}$,

(ii) $BLUP_{\mathcal{S}}(\boldsymbol{\varepsilon}) = \begin{bmatrix}BLUP_{\mathcal{S}}(\boldsymbol{\varepsilon}_1)\\ \vdots\\ BLUP_{\mathcal{S}}(\boldsymbol{\varepsilon}_m)\end{bmatrix} = \begin{bmatrix}BLUP_{\mathcal{S}_1}(\boldsymbol{\varepsilon}_1)\\ \vdots\\ BLUP_{\mathcal{S}_m}(\boldsymbol{\varepsilon}_m)\end{bmatrix}$,

(iii) $\boldsymbol{r}(\mathbf{X})+\boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{X}'\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\end{bmatrix}$, *i.e.,* $\mathcal{C}\begin{bmatrix}\mathbf{X}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} \cap \mathcal{C}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{X}'\end{bmatrix} = \{\mathbf{0}\}$,

(iv) $\boldsymbol{r}\left[(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp},\ (\mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n)\mathbf{X}^{\perp},\ \mathbf{X}\right] = \boldsymbol{r}\left[(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp},\ (\mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] + \boldsymbol{r}(\mathbf{X})$, *i.e.,* $\mathcal{C}(\mathbf{X}) \cap \mathcal{C}\left[(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp},\ (\mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] = \{\mathbf{0}\}$,

(v) $\boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n\\ \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{X}'\end{bmatrix} = \boldsymbol{r}(\mathbf{X}) + \boldsymbol{r}\left[\mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n,\ \mathbf{X}\right] = \boldsymbol{r}(\mathbf{X}) + nm$,

(vi) $\mathcal{C}\left((\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right) \subseteq \mathcal{C}\left((\mathbf{D}_{\boldsymbol{\sigma}}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right)$.

Assume that $\boldsymbol{\phi}$ is predictable under $\mathcal{M}$ and $\boldsymbol{\phi}_i$ is predictable under $\mathcal{M}_i$, $i = 1,\ldots,m$. We note that the following additive decomposition of OLSPs always holds:

$$\begin{aligned} OLSP_{\mathcal{M}}(\boldsymbol{\phi}) &= OLSP_{\mathcal{M}}(\boldsymbol{\phi}_1) + \cdots + OLSP_{\mathcal{M}}(\boldsymbol{\phi}_m)\\ &= OLSP_{\mathcal{M}_1}(\boldsymbol{\phi}_1) + \cdots + OLSP_{\mathcal{M}_m}(\boldsymbol{\phi}_m), \end{aligned}$$

where the equality $OLSP_{\mathcal{M}}(\boldsymbol{\phi}_i) = OLSP_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$ is seen from Theorem 4.1. Moreover, the following decomposition always holds:

$$BLUP_{\mathcal{M}}(\boldsymbol{\phi}) = BLUP_{\mathcal{M}}(\boldsymbol{\phi}_1) + \cdots + BLUP_{\mathcal{M}}(\boldsymbol{\phi}_m).$$

This equality is obtained using similar way in Theorem 5.1; see also [21, Corollary 3.4].

6 Conclusion

In this study, we consider a system of linear regression models. This system consists of regression models in which the error terms are correlated across the models.

Necessary and sufficient conditions for equalities of BLUPs and OLSPs and additive decomposition equalities of these predictors under the system of linear regression models and its single models need to be characterized and the formulations of the equivalency conditions are worth being considered to obtain statistical inference of these models since there are connections between the single models and their combined model. We establish the results on equality of the BLUPs and the OLSPs, and derive the equalities of additive decompositions of the BLUPs and the OLSPs under the system of linear regression models and its single models. Such equalities will present useful and general aspects for connections between BLUPs/BLUEs and OLSPs/OLSEs under SUR models since they are a special case of a system of linear regression models and these equalities will also provide new contributions to SUR models from the theoretical point of view.

References

[1] I. S. Alalouf, G. P. H. Styan, *Characterizations of estimability in the general linear model*, Ann. Stat., 7 (1979), 194-200.

[2] J. K. Baksalary, R. Kala, *On the prediction problem in the seemingly unrelated regression equations model*, Statistics, 10 (1979), 203-208.

[3] J. K. Baksalary, G. Trenkler, *The efficiency of OLS in a seemingly unrelated regressions model*, Econom. Theory, 5 (1989), 463-465.

[4] B. H. Baltagi, *The efficiency of OLS in a seemingly unrelated regressions model*, Econom. Theory, 4 (1988), 536-537.

[5] R. Bartels, D. G. Fiebig, *A simple characterization of seemingly unrelated regressions models in which OLS is BLUE*, Am. Stat., 45 (1991), 137-140.

[6] T. D. Dwivedi, V. K. Srivastava, *Optimality of least squares in the seemingly unrelated regression equation model*, J. Econom., 7 (1978), 391-395.

[7] P. Foschi, E. J. Kontoghiorghes, *Seemingly unrelated regression model with unequal size observations: computational aspects*, Comput. Stat. Data Anal., 41 (2002), 211-229 .

[8] A. S. Goldberger, *Best linear unbiased prediction in the generalized linear regression model*, J. Am. Stat. Assoc., 57 (1962), 369-375.

[9] L. Gong, *Establishing equalities of OLSEs and BLUEs under seemingly unrelated regression models*, J. Stat. Theory Pract., 13, 5 (2019), doi: 10.1007/s42519-018-0015-6.

[10] N. Güler, M. E. Büyükkaya, M. Yiğit, *Comparison of covariance matrices of predictors in seemingly unrelated regression models*, Indian J. Pure Appl. Math., (2021), doi: 10.1007/s13226-021-00174-w.

[11] J. Hou, Y. Zhao, *Some remarks on a pair of seemingly unrelated regression models*, Open Math., 17 (2019), 979-989.

[12] H. Jiang, J. Qian, Y. Sun, *Best linear unbiased predictors and estimators under a pair of constrained seemingly unrelated regression models*, Stat. Probab. Lett., 158 (2020), 108669.

[13] G. Marsaglia, G. P. H. Styan, *Equalities and inequalities for ranks of matrices*, Linear Multilinear Algebra, 2 (1974), 269-292.

[14] R. W. Parks, *Efficient estimation of a system of regression equations when disturbances are both serially and contemporaneously correlated*, J. Am. Stat. Assoc., 62 (1967), 500-509.

[15] C. R. Rao, *Representations of best linear unbiased estimators in the Gauss–Markoff model with a singular dispersion matrix*, J. Multivar. Anal., 3 (1973), 276-292.

[16] C. R. Rao, *Simultaneous estimation of parameters in different linear models and applications to biometric problems*, Biometrics, 31 (1975), 545-554.

[17] V. K. Srivastava, D. E. A. Giles, *Seemingly Unrelated Regression Equations Model* Marcel Dekker, New York, 1987.

[18] Y. Sun, R. Ke, Y. Tian, *Some overall properties of seemingly unrelated regression models*, Adv. Stat. Anal., 98, 2 (2014), 103-120.

[19] L. G. Telser, *Iterative estimation of a set of linear regression equations*, J. Am. Stat. Assoc., 59 (1964), 845-862.

[20] Y. Tian, *Estimations of parametric functions under a system of linear regression equations with correlated errors*, Acta Math. Sin. Engl. Ser., 26, 10 (2010), 1927-1942.

[21] Y. Tian, *A matrix handling of predictions under a general linear random-effects model with new observations*, Electron. J. Linear Algebra, 29 (2015), 30-45.

[22] Y. Tian, *Matrix rank and inertia formulas in the analysis of general linear models*, Open Math., 15, 1 (2017), 126-150.

[23] Y. Tian, *Transformation approaches of linear random-effects models*, Stat. Methods Appl., 26, 4 (2017), 583-608.

[24] Y. Tian, P. Xie, *Simultaneous optimal predictions under two seemingly unrelated linear random-effects models*, J. Ind. Manag. Optim., (2020), doi: 10.3934/jimo.2020168.

[25] A. Zellner, *An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias*, J. Am. Stat. Assoc., 57 (1962), 348-368.

Authors' addresses:

Nesrin Güler
Department of Econometrics,
Sakarya University, TR-54187 Sakarya, Turkey.
E-mail: nesring@sakarya.edu.tr

Melike Yiğit
Department of Mathematics,
Sakarya University, TR-54187 Sakarya, Turkey.
E-mail: melikeyigitt@gmail.com

Melek Eriş Büyükkaya
Department of Statistics and Computer Sciences,
Karadeniz Technical University, TR-61080 Trabzon, Turkey.
E-mail: melekeris@ktu.edu.tr