Cubic filters of semigroups

T. Gaketem and A. Iampan

Abstract. Cubic sets which are the generalized form of fuzzy sets and intuitionistic fuzzy sets, and introduced by Jun et al. [Y. B. Jun, C. S. Kim, K. O. Yang, *Cubic sets*, Ann. Fuzzy Math. Inform. 6, 1 (2012), 83-98.]. From the above concept, it inspired us to introduce the concepts of cubic filters, cubic left (right) filters, and cubic bi-filters of semigroups and study some properties of their concepts. Finally, we study some properties of cubic transformations and inverse cubic transformations induced by a homomorphism of semigroups.

M.S.C. 2010: 08A72, 03E72.
Key words: semigroup; cubic filter; cubic left (right) filter; cubic bi-filter; cubic transformation; inverse cubic transformation; homomorphism.

1 Introduction

Existing approaches are insufficient to address uncertainty in many real-world applications; thus, theories such as the hypothesis of fuzzy sets [14] and interval-valued fuzzy sets [10] were presented.

In 1965, Zadeh [14] introduced the concept of fuzzy sets. Since fuzzy set has been applied to many branches in mathematics. The fuzzifications of algebraic structures were initiated by Rosenfeld in 1971 [12] and he introduced the notion of fuzzy subgroups. The fuzzy algebraic structures play an important role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces. In 1981, Kuroki [6] is the pioneer of fuzzy ideal theory of semigroups. Fuzzy filters have been studied by Ma et al. in 2009 [8]. In 1975, the concept of interval-valued fuzzy sets was introduced by Zadeh [15], as a generalization of the concept of fuzzy sets. In 2006, Narayanan and Manikanran [10] initiated the concept of interval-valued fuzzy ideals in semigroups. In 2012, Jun et al. [3] gave the concept of cubic sets which are the very useful generalization of fuzzy sets, where one is allowed to extend the output through a subinterval of $[0,1]$ and a number from $[0,1]$. They investigated several properties and introduced cubic subsemigroups and cubic left(right) ideals of semigroups. In 2016, Hadi and Khan [1] discussed cubic generalized bi-ideals of semigroups. In 2020, Deetae and Khamrot

Applied Sciences, Vol. 24, 2022, pp. 131-141.

[5] studied Q-cubic bi-quasi ideals of semigroups. In the same year, Songsaeng and Iampan [13] used acknowledges of cubic sets for studying neutrosophic cubic set in UP-algebras.

In this paper, we introduce the concepts of cubic filters, cubic left (right) filters, and cubic bi-filters of semigroups and study some properties of their concepts. Finally, we study some properties of cubic transformations and inverse cubic transformations induced by a homomorphism of semigroups.

2 Preliminaries

In this topic, some basic definitions are given as the followings.

Let S be a semigroup. A non-empty subset F of S is called a *subsemigroup* of S if $xy \in F$ for all $x, y \in F$, and a *left (right) ideal* of S if for any $x \in S, a \in F, xa \in F$ ($ax \in F$). An ideal of S is a non-empty subset of S which is both a left ideal and a right ideal of S. A subsemigroup F of S is called a *bi-ideal* of S if for any $x \in S, a, b \in F, axb \in F$, a *filter* of S if for any $x, y \in S, xy \in F$ implies $x, y \in F$, a *left (right) filter* of S if for any $x, y \in S, xy \in F$ implies $y \in F (x \in F)$, and a *bi-filter* of S if for any $x, y, z \in S, xyz \in F$ implies $x, z \in F$.

Now, we will review the definitions of fuzzy subsemigroups and many types of fuzzy subsemigroups.

Definition 2.1. [14] Let T be a non-empty set. A function $\varpi : T \to [0,1]$ is called a *fuzzy set* in T.

Definition 2.2. [9] Let S be a semigroup. A fuzzy set ϖ in S is called

(1) a *fuzzy subsemigroup* of S if $\varpi(uv) \geq \min\{\varpi(u), \varpi(v)\}$ for all $u, v \in S$,

(2) a *fuzzy left (right) ideal* of S if $\varpi(uv) \geq \varpi(v)(\varpi(uv) \geq \varpi(u))$ for all $u, v \in S$. A *fuzzy ideal* of S is a fuzzy set in S which is both a fuzzy left ideal and a fuzzy right ideal of S,

(3) a *fuzzy bi-ideal* of S if ϖ is fuzzy subsemigroup of S and $\varpi(uvw) \geq \min\{\varpi(u), \varpi(w)\}$ for all $u, v, w \in S$.

Definition 2.3. [11] Let S be a semigroup. A fuzzy subsemigroup ϖ of S is called

(1) a *fuzzy filter* of S if $\varpi(uv) = \min\{\varpi(u), \varpi(v)\}$ for all $u, v \in S$,

(2) a *fuzzy left (right) filter* of S if $\varpi(uv) \leq \varpi(v)(\varpi(uv) \leq \varpi(u))$ for all $u, v \in S$,

(3) a *fuzzy bi-filter* of S if $\varpi(uvw) \leq \max\{\varpi(u), \varpi(w)\}$ for all $u, v, w \in S$.

For fuzzy sets ϖ and ρ in T, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$(\varpi \vee \rho)(x) = \max\{\varpi(x), \rho(x)\} = \varpi(x) \vee \rho(x)$$

and

$$(\varpi \wedge \rho)(x) = \min\{\varpi(x), \rho(x)\} = \varpi(x) \wedge \rho(x),$$

respectively, for all $x \in T$.

Now, we will review the concept of interval numbers.

Definition 2.4. [2] An *interval number* on $[0,1]$, say $\overline{a}$ is a closed subinterval of $[0,1]$, that is, $\overline{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denoted the family of all closed subinterval of $[0,1]$, i.e.,

$$D[0,1] = \{\overline{a} = [a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

The interval number $[a,a]$ is identified with the number $a \in [0,1]$.

Note that $[a,a] = \{a\}$ for all $a \in [0,1]$. For $a = 0$ or 1, we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

Definition 2.5. [2] Let $\overline{a}_i = [a_i^-, a_i^+] \in D[0,1]$ for all $i \in I$, where I is an index set. We define

$$\operatorname*{rinf}_{i\in I} \overline{a}_i = \left[\inf_{i\in I} a_i^-, \inf_{i\in I} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in I} \overline{a}_i = \left[\sup_{i\in I} a_i^-, \sup_{i\in I} a_i^+\right].$$

Define the binary relations "$\succeq$", "$\preceq$", "$=$" and the binary operations "rmin", "rmax" on $D[0,1]$ as the following. For two interval numbers $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ in $D[0,1]$,

(1) $\overline{a} \succeq \overline{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\overline{a} \preceq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,

(4) $\text{rmin}\{\overline{a}, \overline{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$,

(5) $\text{rmax}\{\overline{a}, \overline{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Definition 2.6. [2] An *interval-valued fuzzy set* in a non-empty set T is a function $\overline{\mu} : T \to D[0,1]$.

Definition 2.7. [2] For a family $\{\overline{\mu}_i \mid i \in I\}$ of interval-valued fuzzy sets in T, we define two interval-valued fuzzy sets $\sqcup_{i\in I}\overline{\mu}_i$ and $\sqcap_{i\in I}\overline{\mu}_i$ as follows:

$$(\bigsqcup_{i\in I} \overline{\mu}_i)(x) = \text{rsup}\, \overline{\mu}_i(x)$$

and

$$(\bigsqcap_{i\in I} \overline{\mu}_i)(x) = \text{rinf}\, \overline{\mu}_i(x),$$

respectively, for all $x \in T$. If $I = \{1,2\}$, then $(\overline{\mu}_1 \sqcup \overline{\mu}_2)(x) = \text{rmax}\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ and $(\overline{\mu}_1 \sqcap \overline{\mu}_2)(x) = \text{rmin}\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x \in T$.

Now, we will review the definitions of cubic subsemigroups and many types of cubic subsemigroups.

Definition 2.8. [2] Let T be a non-empty set. A *cubic set* $\mathcal{A}$ in T is a structure of the form

$$\mathcal{A} = \{\langle x, \overline{\mu}(x), \varpi(x)\rangle \mid x \in T\}$$

and denoted by $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$, where $\overline{\mu} = [\mu^-, \mu^+]$ is an interval-valued fuzzy set in T and ϖ is a fuzzy set in T. In this case, we will use

$$\mathcal{A}(x) = \langle \overline{\mu}(x), \varpi(x) \rangle = \langle [\mu^-(x), \mu^+(x)], \varpi(x) \rangle$$

for all $x \in T$. Note that the concept of cubic sets is a generalization of intuitionistic fuzzy sets.

Definition 2.9. [2] Let F be a non-empty subset of a semigroup S. Then characteristic cubic set $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is defined as follows:

$$\overline{\mu}_{\chi_F}(u) = \begin{cases} \overline{1} & \text{if} \quad u \in F, \\ \overline{0} & \text{if} \quad u \notin F \end{cases}$$

and

$$\varpi_{\chi_F}(u) = \begin{cases} 0 & \text{if} \quad u \in F, \\ 1 & \text{if} \quad u \notin F. \end{cases}$$

Definition 2.10. [2] The whole cubic set $\mathcal{S}$ in a semigroup S is defined to be a structure

$$\mathcal{S} = \{ \langle u, \overline{1}_S(u), 0_S(u) \rangle \mid u \in S \},$$

where $\overline{1}_S(u) = \overline{1}$ and $0_S(u) = 0$ for all $\in S$. It will briefly denoted by $\mathcal{S} = \langle \overline{1}_S, 0_S \rangle$.

Definition 2.11. [2] Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be two cubic sets in a semigroup S. Then the intersection of $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A} \sqcap \mathcal{B}$, is the cubic set

$$\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle,$$

and the union of $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A} \sqcup \mathcal{B}$, is the cubic set

$$\mathcal{A} \sqcup \mathcal{B} = \langle \overline{\mu} \sqcup \overline{\lambda}, \varpi \wedge \rho \rangle.$$

Definition 2.12. [2] A cubic set $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ in a semigroup S is called a *cubic subsemigroup* of S if it satisfies

(1) $\overline{\mu}(uv) \succeq \text{rmin}\{\overline{\mu}(u), \overline{\mu}(v)\}$,

(2) $\varpi(uv) \leq \max\{\varpi(u), \varpi(v)\}$ for all $u, v \in S$.

Theorem 2.1. *[2] Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be cubic subsemigroups of a semigroup S. Then $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic subsemigroup of S.*

Theorem 2.2. *[2] Let F be a non-empty subset of a semigroup S. Then F is a subsemigroup of S if and only if the characteristic cubic set $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic subsemigroup of S.*

Definition 2.13. [2] A cubic set $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ in a semigroup S is called a *cubic left (right) ideal* of S if it satisfies

(1) $\overline{\mu}(uv) \succeq \overline{\mu}(v)$ $(\overline{\mu}(uv) \succeq \overline{\mu}(u))$,

(2) $\varpi(uv) \leq \varpi(v)$, $(\varpi(uv) \leq \varpi(u))$ for all $u, v \in S$.

A cubic set $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ in S is called a *cubic ideal* of S if it is both a cubic left ideal and a cubic right ideal of S.

Theorem 2.3. *[2] Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be cubic left (right) ideals of a semigroup S. Then $A \sqcap B = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic left (right) ideal of S.*

Theorem 2.4. *[2] Let F be a non-empty subset of a semigroup S. Then F is a left (right) ideal of S if and only if the characteristic cubic set $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic left (right) ideal of S.*

3 Cubic filters of semigroups

Definition 3.1. A cubic subsemigroup $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ of a semigroup S is called a *cubic filter* of S if it satisfies

(1) $\overline{\mu}(uv) = \text{rmin}\{\overline{\mu}(u), \overline{\mu}(v)\}$,

(2) $\varpi(uv) = \max\{\varpi(u), \varpi(v)\}$ for all $u, v \in S$.

Example 1. Consider a semigroup $(S, \cdot)$ defined by the following table:

$S : \cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

Define a cubic set $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ in S as follows:

$S :$	$\overline{\mu}(x)$	$\varpi(x)$
a	$[0.2, 0.4]$	0.4
b	$[0.3, 0.5]$	0.2
c	$[0.4, 0.6]$	0.1

Then, by routine calculation one can easily verify that $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ of S is a cubic filter of S.

Theorem 3.1. *Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be cubic filters of a semigroup S. Then $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic filter of S.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cubic filters of S, we have $\mathcal{A}$ and $\mathcal{B}$ are cubic subsemigroups of S. Thus by Theorem 2.1, we have $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic subsemigroup of S. Let $u, v \in S$. Then

$$\begin{aligned}
(\overline{\mu} \sqcap \overline{\lambda})(uv) &= \text{rmin}\{\overline{\mu}(uv), \overline{\lambda}(uv)\} \\
&= \text{rmin}\{\text{rmin}\{\overline{\mu}(u), \overline{\mu}(v)\}, \text{rmin}\{\overline{\lambda}(u), \overline{\lambda}(v)\}\} \\
&= \text{rmin}\{\text{rmin}\{\overline{\mu}(u), \overline{\lambda}(u)\}, \text{rmin}\{\overline{\mu}(v), \overline{\lambda}(v)\}\} \\
&= \text{rmin}\{(\overline{\mu} \sqcap \overline{\lambda})(u), (\overline{\mu} \sqcap \overline{\lambda})(v)\}
\end{aligned}$$

and

$$\begin{aligned}(\varpi \vee \rho)(uv) &= \max\{\varpi(uv), \rho(uv)\}\\ &= \max\{\max\{\varpi(u), \varpi(v)\}, \max\{\rho(u), \rho(v)\}\}\\ &= \max\{\max\{\varpi(u), \rho(u)\}, \max\{\varpi(v), \rho(v)\}\}\\ &= \max\{(\varpi \vee \rho)(u), (\varpi \vee \rho)(v)\}.\end{aligned}$$

Hence, $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic filter of S. □

Corollary 3.2. *The intersection of any family of cubic filters of a semigroup S is a cubic filter of S.*

Proof. It is straightforward. □

Theorem 3.3. *Let S be a semigroup and F a non-empty subset of S. Then F is a filter of S if and only if the characteristic cubic set $\chi_F = \langle \mu_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic filter of S.*

Proof. Suppose that F is a filter of S. Then by Theorem 2.2, we have $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic subsemigroup of S. Let $u, v \in S$.

If $uv \in F$, then $\overline{\mu}_{\chi_F}(uv) = \overline{1}$ and $\varpi_{\chi_F}(uv) = 0$. Since F is a filter of S, we have $u, v \in F$. Thus $\overline{\mu}_{\chi_F}(u) = \overline{\mu}_{\chi_F}(v) = \overline{1}$ and $\varpi_{\chi_F}(u) = \varpi_{\chi_F}(v) = 0$. Hence, $\overline{\mu}_{\chi_F}(uv) = \overline{1} = \text{rmin}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(v)\}$ and $\varpi_{\chi_F}(uv) = 0 = \max\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(v)\}$.

If $uv \notin F$, then $\overline{\mu}_{\chi_F}(uv) = \overline{0}$ and $\varpi_{\chi_F}(uv) = 1$. Since F is a filter of S and $uv \notin F$, we have $u \notin F$ or $v \notin F$. Thus $\overline{\mu}_{\chi_F}(u) = \overline{0}$ or $\overline{\mu}_{\chi_F}(v) = \overline{0}$, and $\varpi_{\chi_F}(u) = 1$ or $\varpi_{\chi_F}(v) = 1$. Hence, $\overline{\mu}_{\chi_F}(uv) = \overline{0} = \text{rmin}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(v)\}$ and $\varpi_{\chi_F}(uv) = 1 = \max\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(v)\}$.

This implies that $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic filter of S.

Conversely, suppose that $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic filter of S. Then by Theorem 2.2, we have F is a subsemigroup of S. Let $u, v \in S$ be such that $uv \in F$. Then $\overline{\mu}_{\chi_F}(uv) = \overline{1}$ and $\varpi_{\chi_F}(uv) = 0$. Since $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic filter of S, we have $\overline{\mu}_{\chi_F}(uv) = \text{rmin}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(v)\}$ and $\varpi_{\chi_F}(uv) = \max\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(v)\}$. Thus $\text{rmin}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(v)\} = \overline{1}$ and $\max\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(v)\} = 0$. It implies that $\overline{\mu}_{\chi_F}(u) = \overline{\mu}_{\chi_F}(v) = \overline{1}$ and $\varpi_{\chi_F}(u) = \varpi_{\chi_F}(v) = 0$. Hence, $u, v \in F$. Therefore, F is a filter of S. □

Definition 3.2. A cubic subsemigroup $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ of a semigroup S is called a *cubic left (right) filter* of S if it satisfies

(1) $\overline{\mu}(uv) \preceq \overline{\mu}(v)$ ($\overline{\mu}(uv) \preceq \overline{\mu}(u)$),

(2) $\varpi(uv) \geq \varpi(v)$ ($\varpi(uv) \geq \varpi(u)$) for all $u, v \in S$.

Theorem 3.4. *Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be cubic left (right) filters of a semigroup S. Then $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic left (right) filter of S.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cubic left filters of S, we have $\mathcal{A}$ and $\mathcal{B}$ are cubic subsemigroups of S. Thus by Theorem 2.1, we have $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic subsemigroup of S. Let $u, v \in S$. Then

$$(\overline{\mu} \sqcap \overline{\lambda})(uv) = \text{rmin}\{\overline{\mu}(uv), \overline{\lambda}(uv)\} \preceq \text{rmin}\{\overline{\mu}(v), \overline{\lambda}(v)\} = (\overline{\mu} \sqcap \overline{\lambda})(v)$$

and

$$(\varpi \vee \rho)(uv) = \max\{\varpi(uv), \rho(uv)\} \geq \max\{\varpi(v), \rho(v)\} = (\varpi \vee \rho)(v).$$

Hence, $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic left filter of S. □

Corollary 3.5. *The intersection of any family of cubic left (right) filters of a semigroup S is a cubic left (right) filter of S.*

Proof. It is straightforward. □

Theorem 3.6. *Let S be a semigroup and F a non-empty subset of S. Then F is a left (right) filter of S if and only if the characteristic cubic set $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic left (right) filter of S.*

Proof. Suppose that F is a left filter of S. Then by Theorem 2.2, we have $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic subsemigroup of S. Let $u, v \in S$.

If $uv \in F$, then $\overline{\mu}_{\chi_F}(uv) = \overline{1}$ and $\varpi_{\chi_F}(uv) = 0$. Since F is a left filter of S, we have $v \in F$. Thus $\overline{\mu}_{\chi_F}(v) = \overline{1}$ and $\varpi_{\chi_F}(v) = 0$. Hence, $\overline{\mu}_{\chi_F}(uv) = \overline{1} \preceq \overline{1} = \overline{\mu}_{\chi_F}(v)$ and $\varpi_{\chi_F}(uv) = 0 \geq 0 = \varpi_{\chi_F}(v)$.

If $uv \notin F$, then $\overline{\mu}_{\chi_F}(uv) = \overline{0}$ and $\varpi_{\chi_F}(uv) = 1$. Hence, $\overline{\mu}_{\chi_F}(uv) = \overline{0} \preceq \overline{\mu}_{\chi_F}(v)$ and $\varpi_{\chi_F}(uv) = 1 \geq \varpi_{\chi_F}(v)$.

This implies that $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic left filter of S.

Conversely, suppose that $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic left filter of S. Then by Theorem 2.2, we have F is a subsemigroup of S. Let $u, v \in S$ be such that $uv \in F$. Then $\overline{\mu}_{\chi_F}(uv) = \overline{1}$ and $\varpi_{\chi_F}(uv) = 0$. Since $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic left filter of S, we have $\overline{\mu}_{\chi_F}(uv) \preceq \overline{\mu}_{\chi_F}(v)$ and $\varpi_{\chi_F}(uv) \geq \varpi_{\chi_F}(v)$. Thus $\overline{1} = \overline{\mu}_{\chi_F}(uv) \preceq \overline{\mu}_{\chi_F}(v)$ and $0 = \varpi_{\chi_F}(uv) \geq \varpi_{\chi_F}(v)$, that is, $\overline{1} = \overline{\mu}_{\chi_F}(v)$ and $0 = \varpi_{\chi_F}(v)$. Hence, $v \in F$. Therefore, F is a left filter of S. □

Definition 3.3. A cubic subsemigroup $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ of a semigroup S is called a *cubic bi-filter* of S if it satisfies

(1) $\overline{\mu}(uvw) \preceq \text{rmax}\{\overline{\mu}(u), \overline{\mu}(w)\}$,

(2) $\varpi(uvw) \geq \min\{\varpi(u), \varpi(w)\}$ for all $u, v, w \in S$.

Theorem 3.7. *Let $\mathcal{A} = \langle \overline{\mu}, \varpi \rangle$ and $\mathcal{B} = \langle \overline{\lambda}, \rho \rangle$ be cubic bi-filters of a semigroup S. Then $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic bi-filter of S.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cubic bi-filters of S, we have $\mathcal{A}$ and $\mathcal{B}$ are cubic subsemigroups of S. Thus by Theorem 2.1, we have $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic subsemigroup of S. Let $u, v, w \in S$. Then

$$\begin{aligned}
(\overline{\mu} \sqcap \overline{\lambda})(uvw) &= \text{rmax}\{\overline{\mu}(uvw), \overline{\lambda}(uvw)\} \\
&\preceq \text{rmax}\{\text{rmax}\{\overline{\mu}(u), \overline{\mu}(w)\}, \text{rmax}\{\overline{\lambda}(u), \overline{\lambda}(w)\}\} \\
&= \text{rmax}\{\text{rmax}\{\overline{\mu}(u), \overline{\lambda}(u)\}, \text{rmax}\{\overline{\mu}(w), \overline{\lambda}(w)\}\} \\
&= \text{rmax}\{(\overline{\mu} \sqcap \overline{\lambda})(u), (\overline{\mu} \sqcap \overline{\lambda})(w)\}
\end{aligned}$$

and

$$\begin{array}{rcl}(\varpi \vee \rho)(uvw) & = & \min\{\varpi(uvw), \rho(uvw)\} \\ & \geq & \min\{\min\{\varpi(u), \varpi(w)\}, \min\{\rho(u), \rho(w)\}\} \\ & = & \min\{\min\{\varpi(u), \rho(u)\}, \min\{\varpi(w), \rho(w)\}\} \\ & = & \min\{(\varpi \vee \rho)(u), (\varpi \vee \rho)(w)\}.\end{array}$$

Hence, $\mathcal{A} \sqcap \mathcal{B} = \langle \overline{\mu} \sqcap \overline{\lambda}, \varpi \vee \rho \rangle$ is a cubic bi-filter of S. □

Corollary 3.8. *The intersection of any family of cubic bi-filters of a semigroup S is a cubic bi-filter of S.*

Proof. It is straightforward. □

Theorem 3.9. *Let S be a semigroup and F a non-empty subset of S. If F is a bi-filter of S, then the characteristic cubic set $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic bi-filter of S.*

Proof. Suppose that F is a bi-filter of S. Then by Theorem 2.2, we have $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic subsemigroup of S. Let $u, v, w \in S$.

If $uvw \in F$, then $\overline{\mu}_{\chi_F}(uvw) = \overline{1}$ and $\varpi_{\chi_F}(uvw) = 0$. Since F is a bi-filter of S, we have $u, w \in F$. Thus $\overline{\mu}_{\chi_F}(u) = \overline{\mu}_{\chi_F}(w) = \overline{1}$ and $\varpi_{\chi_F}(u) = \varpi_{\chi_F}(w) = 0$. Hence, $\overline{\mu}_{\chi_F}(uvw) = \overline{1} \preceq \overline{1} = \text{rmax}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(w)\}$ and $\varpi_{\chi_F}(uvw) = 0 \geq 0 = \min\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(w)\}$.

If $uvw \notin F$, then $\overline{\mu}_{\chi_F}(uvw) = \overline{0}$ and $\varpi_{\chi_F}(uvw) = 1$. Hence, $\overline{\mu}_{\chi_F}(uvw) = \overline{0} \preceq \text{rmax}\{\overline{\mu}_{\chi_F}(u), \overline{\mu}_{\chi_F}(w)\}$ and $\varpi_{\chi_F}(uvw) = 1 \geq \min\{\varpi_{\chi_F}(u), \varpi_{\chi_F}(w)\}$.

This implies that $\chi_F = \langle \overline{\mu}_{\chi_F}, \varpi_{\chi_F} \rangle$ is a cubic bi-filter of S. □

4 Cubic transformations and inverse cubic transformations

In this section, we study some properties of cubic transformations and inverse cubic transformations induced by a homomorphism of semigroups.

Definition 4.1. [2] Let $\mathcal{C}(X)$ be the family of cubic sets in a non-empty set X. Let X and Y be given classical sets. A mapping $h : X \to Y$ induces two mapping $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$, $\mathcal{A} \mapsto \mathcal{C}_h(\mathcal{A})$ and $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$, $\mathcal{B} \mapsto \mathcal{C}_h^{-1}(\mathcal{B})$. The *image* $\mathcal{C}_h(\mathcal{A}) = \langle \mathcal{C}_h(\overline{\mu}), \mathcal{C}_h(f) \rangle$ is defined by

$$\mathcal{C}_h(\overline{\mu})(y) = \begin{cases} \text{rsup}_{y=h(x)} \overline{\mu}(x), & \text{if} \quad h^{-1}(y) \neq \varnothing, \\ \overline{0}, & \text{otherwise} \end{cases}$$

$$\mathcal{C}_h(f)(y) = \begin{cases} \inf_{y=h(x)} f(x), & \text{if} \quad h^{-1}(y) \neq \varnothing, \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in Y$. The *inverse image* $\mathcal{C}_h^{-1}(\mathcal{B}) = \langle \mathcal{C}_h^{-1}(\overline{\lambda}), \mathcal{C}_h^{-1}(g) \rangle$ is defined by $\mathcal{C}_h^{-1}(\overline{\lambda})(x) = \overline{\lambda}(h(x))$ and $\mathcal{C}_h^{-1}(g)(x) = g(h(x))$ for all $x \in X$. Then the mapping $\mathcal{C}_h$ (resp., $\mathcal{C}_h^{-1}$) is called a *cubic transformation* (resp., inverse cubic transformation) induced by h. A cubic set $\mathcal{A} = \langle \overline{\mu}, f \rangle$ in X has the *cubic property* if for any subset T of X, there exists $x_0 \in T$ such that $\overline{\mu}(x_0) = \text{rsup}_{x \in T} \overline{\mu}(x)$ and $f(x_0) = \inf_{x \in T} f(x)$.

Theorem 4.1. *For an onto homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$ be the cubic transformation induced by h. If $\mathcal{A} = \langle \overline{\mu}, f\rangle \in \mathcal{C}(X)$ is a cubic filter of X which has the cubic property, then $\mathcal{C}_h(\mathcal{A})$ is a cubic filter of Y.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}, f\rangle \in \mathcal{C}(X)$ is a cubic filter of X and let $h(x), h(y) \in Y$, where $x, y \in X$. By the cubic property, let $x_0 \in h^{-1}(h(x))$, $y_0 \in h^{-1}(h(y))$ be such that $\overline{\mu}(x_0) = \text{rsup}_{a\in h^{-1}(h(x))} \overline{\mu}(a)$, $f(x_0) = \inf_{a\in h^{-1}(h(x))} f(a)$ and $\overline{\mu}(y_0) = \text{rsup}_{b\in h^{-1}(h(y))} \overline{\mu}(a)$, $f(y_0) = \inf_{b\in h^{-1}(h(y))} f(b)$, respectively. Then

$$\begin{aligned} \mathcal{C}_h(\overline{\mu})(h(x)h(y)) &= \text{rsup}_{z\in h^{-1}(h(x)h(y))} \overline{\mu}(z) \\ &= \overline{\mu}(x_0y_0) \\ &\succeq \text{rmin}\{\overline{\mu}(x_0), \overline{\mu}(y_0)\} \\ &= \text{rmin}\{\text{rsup}_{a\in h^{-1}(h(x))} \overline{\mu}(a), \text{rsup}_{b\in h^{-1}(h(y))} \overline{\mu}(b)\} \\ &= \text{rmin}\{\mathcal{C}_h(\overline{\mu}(a))(h(x)), \mathcal{C}_h(\overline{\mu}(a))(h(y))\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_h(f)(h(x)h(y)) &= \inf_{z\in h^{-1}(h(x)h(y))} f(z) \\ &= f(x_0y_0) \\ &\leq \max\{f(x_0), f(y_0)\} \\ &= \max\{\inf_{a\in h^{-1}(h(x))} f(a), \inf_{b\in h^{-1}(h(y))} f(b)\} \\ &= \max\{\mathcal{C}_h(h(x)), \mathcal{C}_h(h(y))\}. \end{aligned}$$

Hence, $\mathcal{C}_h(\mathcal{A})$ is a cubic subsemigroup of Y.

Similarly,

$$\begin{aligned} \mathcal{C}_h(\overline{\mu})(h(x)h(y)) &= \text{rsup}_{z\in h^{-1}(h(x)h(y))} \overline{\mu}(z) \\ &= \overline{\mu}(x_0y_0) \\ &= \text{rmin}\{\overline{\mu}(x_0), \overline{\mu}(y_0)\} \\ &= \text{rmin}\{\text{rsup}_{a\in h^{-1}(h(x))} \overline{\mu}(a), \text{rsup}_{b\in h^{-1}(h(y))} \overline{\mu}(b)\} \\ &= \text{rmin}\{\mathcal{C}_h(\overline{\mu}(a))(h(x)), \mathcal{C}_h(\overline{\mu}(a))(h(y))\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_h(f)(h(x)h(y)) &= \inf_{z\in h^{-1}(h(x)h(y))} f(z) \\ &= f(x_0y_0) \\ &= \max\{f(x_0), f(y_0)\} \\ &= \max\{\inf_{a\in h^{-1}(h(x))} f(a), \inf_{b\in h^{-1}(h(y))} f(b)\} \\ &= \max\{\mathcal{C}_h(h(x)), \mathcal{C}_h(h(y))\}. \end{aligned}$$

Hence, $\mathcal{C}_h(\mathcal{A})$ is a cubic filter of Y. □

We can similarly prove the following.

Theorem 4.2. *For an onto homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$ be the cubic transformation induced by h. If $\mathcal{A} = \langle \overline{\mu}, f\rangle \in \mathcal{C}(X)$ is a cubic left (right) filter of X which has the cubic property, then $\mathcal{C}_h(\mathcal{A})$ is a cubic left (right) filter of Y.*

Theorem 4.3. *For an onto homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$ be the cubic transformation induced by h. If $\mathcal{A} = \langle \overline{\mu}, f\rangle \in \mathcal{C}(X)$ is a cubic bi-filter of X which has the cubic property, then $\mathcal{C}_h(\mathcal{A})$ is a cubic bi-filter of Y.*

Theorem 4.4. *For a homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the inverse cubic transformation induced by h. If $\mathcal{B} = \langle \overline{\lambda}, g\rangle \in \mathcal{C}(Y)$ is a cubic filter of Y, then $\mathcal{C}_h^{-1}(\mathcal{B})$ is a cubic filter of X.*

Proof. Suppose that $\mathcal{B} = \langle \overline{\lambda}, g\rangle \in \mathcal{C}(Y)$ is a cubic filter of Y and let $x, y \in X$. Then

$$\begin{array}{rcl} \mathcal{C}_h^{-1}(\overline{\lambda})(xy) & = & \overline{\lambda}(h(xy)) \\ & = & \overline{\lambda}(h(x)h(y)) \\ & \succeq & \text{rmin}\{\overline{\lambda}(h(x)), \overline{\lambda}(h(y))\} \\ & = & \text{rmin}\{\mathcal{C}_h^{-1}(\overline{\lambda})(x), \mathcal{C}_h^{-1}(\overline{\lambda})(y)\} \end{array}$$

and

$$\begin{array}{rcl} \mathcal{C}_h^{-1}(g)(xy) & = & g(h(xy) \\ & = & g(h(x)h(y)) \\ & \leq & \max\{g(h(x)), g(h(y))\} \\ & = & \max\{\mathcal{C}_h^{-1}(g)(x), \mathcal{C}_h^{-1}(g)(y)\}. \end{array}$$

Hence, $\mathcal{C}_h^{-1}(\mathcal{B})$ is a cubic subsemigroup of X.

Similarly,

$$\begin{array}{rcl} \mathcal{C}_h^{-1}(\overline{\lambda})(xy) & = & \overline{\lambda}(h(xy)) \\ & = & \overline{\lambda}(h(x)h(y)) \\ & = & \text{rmin}\{\overline{\lambda}(h(x)), \overline{\lambda}(h(y))\} \\ & = & \text{rmin}\{\mathcal{C}_h^{-1}(\overline{\lambda})(x), \mathcal{C}_h^{-1}(\overline{\lambda})(y)\} \end{array}$$

and

$$\begin{array}{rcl} \mathcal{C}_h^{-1}(g)(xy) & = & g(h(xy)) \\ & = & g(h(x)h(y)) \\ & = & \max\{g(h(x)), g(h(y))\} \\ & = & \max\{\mathcal{C}_h^{-1}(g)(x), \mathcal{C}_h^{-1}(g)(y)\}. \end{array}$$

Therefore, $\mathcal{C}_h^{-1}(\mathcal{B})$ is a cubic filter of X. □

In similar we can prove the following.

Theorem 4.5. *For a homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the inverse cubic transformation induced by h. If $\mathcal{B} = \langle \overline{\lambda}, g\rangle \in \mathcal{C}(Y)$ is a cubic left (right) filter of Y, then $\mathcal{C}_h^{-1}(\mathcal{B})$ is a cubic left (right) filter of X.*

Theorem 4.6. *For a homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the inverse cubic transformation induced by h. If $\mathcal{B} = \langle \overline{\lambda}, g\rangle \in \mathcal{C}(Y)$ is a cubic bi-filter of Y, then $\mathcal{C}_h^{-1}(\mathcal{B})$ is a cubic bi-filter of X.*

5 Conclusion

Cubic sets which are the generalized form of fuzzy sets and intuitionistic fuzzy sets, and introduced by Jun et al. [3]. This paper has studied the concepts of cubic filters, cubic left (right) filters, and cubic bi-filters of semigroups. Moreover, cubic transformations and inverse cubic transformations induced by a homomorphism of semigroups are studied.

In future work, we will study the cubic filters in hypersemigroups defined by Kehayopulu [4].

Acknowledgment

This work is partially supported by School of Science, University of Phayao. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

References

[1] A. Hadi, A. Khan, *Cubic generalized bi-ideal in semigroups*, Discuss. Math., Gen. Algebra Appl. 36 (2016), 131-146.

[2] Y. B. Jun, A. Khan, *Cubic ideal in semigroups*, Honam Math. J. 35 (2013), 607-623.

[3] Y. B. Jun, C. S. Kim, K. O. Yang, *Cubic sets*, Ann. Fuzzy Math. Inform. 6, 1 (2012), 83-98.

[4] N. Kehayopulu, *On hypersemigroups*, Pure Math. Appl. (PU.M.A.) 25, 2 (2015), 151-156.

[5] N. Deetae, P. Khamrot, *Q-cubic bi-quasi ideals of semigroups*, Glob. J. Pure Appl. Math. 16, 4 (2020), 553-566.

[6] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli 5, (1979), 128-132.

[7] D. Mandal, *Fuzzy bi-ideal and fuzzy quasi-ideal in ordered semigroup*, Gulf J. Math. 2, 4 (2014), 60-67.

[8] X. Ma, J. Zhan, Y. B. Jun, *On* $(\epsilon, \epsilon, \vee_q)$*-fuzzy filters of* R_0*-algebras*, Math. Log. Q. 55, (2009), 493-508.

[9] J. N. Mordeson, D. S. Malik, N. Kuroki, *Fuzzy Semigroup*, Springer Science and Business Media, 2003.

[10] Al. Naraynan, T. Manikantan, *Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semigroups*, J. Appl. Math. Comput. 20, 1-2 (2006), 455-464.

[11] M. Shabir, A. Khan, *Intuitionistic fuzzy fileters of ordered semigroups*, J. Appl. Math. Inform. 26, 5-6 (2008), 1071-1784.

[12] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. 35, 3 (1971), 512-517.

[13] M. Songsaeng, A. Iampan, *Neutrosophic cubic set theory applied to UP-algebras*, Thai J. Math. 18, 3 (2020), 1433-1446.

[14] L. A. Zadeh, *Fuzzy sets*, Inf. Cont. **8**, 3 (1965), 338–353.

[15] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Inf. Sci. 8 (1975), 199-249.

Authors' address:

Thiti Gaketem and Aiyared Iampan (corresponding author)
Department of Mathematics, School of Science, University of Phayao,
Mae Ka, Mueang, Phayao 56000, Thailand.
E-mail: thiti.ga@up.ac.th , aiyared.ia@up.ac.th