Analytic Gevrey well-posedness and regularity for class of coupled periodic KdV systems of Majda-Biello type

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Abstract. In the present paper, we consider an important problem from the point of view of application in sciences and engineering, namely, new class of coupled periodic KdV systems of Majda-Biello type. A new minimal conditions on the exponents s, β, σ, δ and the relationship between them are used to show the existence of unique solution in analytic Gevrey spaces. Moreover, the Gevrey-3σ regularity in time is given in addition to the failure of Gevrey-d.

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1 Introduction and overview

Our purpose in this article is to study the well-posedness and regularity for the coupled Kotewege-de Varie (KdV) system

$$\left\{ \begin{array}{l} u_t + u_{xxx} + ww_x = 0 \\ w_t + \beta w_{xxx} + (uw)_x = 0, \quad x \in \mathbb{T}_\gamma \ , t \in \mathbb{R}, \ 0 < \beta < 1 \\ (u, w) \mid_{t=0} = (u_0, w_0). \end{array} \right. \tag{1.1}$$

where $\mathbb{T}_\gamma = [0, 2\pi\gamma)$ for some $\gamma \geq 1$.

This model, whose discoverers are considered as a single equation to be two Dutch mathematicians Korteweg and de Vries, was proposed for long surface waves of water in a narrow and shallow channel. Problem (1.1) is inspired from [1, 25], where Majda and Biello proposed a related system, as a simplified asymptotic model for the behavior of certain atmospheric Rossby waves. Rossby waves are long atmospheric or oceanic waves which have important effects on weather patterns and ocean currents. In [25], the authors introduced the system (1.1) where ww_x is taken instead of $\frac{1}{2}(w^2)_x$ and they studied solutions in $H^s(\mathbb{T})$.

Nonlinear wave phenomena have been the subject of research by such outstanding scientists as Poisson, Stokes, Airy, Rayleigh, Boussinesq, Riemann. However, as a

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modern sciences, great development was known in the late 1960s - early 1970s, which became the years of its rapid development. This quick interest is due to the development of computer technology, which made it possible to approach the direct numerical solution of partial differential equations that describe the propagation of waves in various media.

The theory of nonlinear KdV is still a young science, although research in this direction was carried out even in the 19th century, mainly in connection with the problems of gas and hydrodynamics (See [9, 10, 13, 17]). For example, the work of T. Oh [26], who observed local well-posedness for problem of coupled KdV-type systems in the periodic/non-periodic cases. Dates back to 1895, the Korteweg-de Vries equation representing the basis of the mathematical description of dynamics of solutions was obtained by [24]. This type of equations describes the propagation of waves on water with small dispersion and small nonlinearity. It serves as a model equation for any physical system with an approximate dispersion. Equations of the KdV or Burgers type play an extremely important role in the theory of nonlinear waves in the study of weakly nonlinear long-wave processes in media with dispersion and (or) dissipation. Above in (1.1), the coupled system of the KdV equations can be considered with specific physical examples related to plasma physics, gas and hydrodynamics, and radio physics.

This article describes a new results on the analytic Gevrey spaces. There exist so far essentially results, for this kind of systems.

It is well known that it is not new to study the KdV equations in the classical Sobolev spaces H^s, there are many discussed results according to different value of the exponent s.(See [6, 7, 8, 20, 21, 22, 23])

And Since Gevrey functions on the circle belong to every Sobolev spaces, it is very important to study KdV equation on theses spaces, especially when it comes from a system of coupled equations with nonlinearity.

To motivate our work, we review some related results. In [14], H. Hannah et *al.* proposed

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0, \ x \in \mathbb{T}, \ t \in \mathbb{R} \\ u(x,0) = \phi(x), \end{cases} \tag{1.2}$$

with $k = 1, 2, 3$ and they proved existence and regularity results in Gevrey spaces, if $\phi(x)$ belongs to $G^\sigma(\mathbb{T})$. However, in [11], a periodic Cauchy problem for a KdV equation with dispersion of order $m = 2j+1, \ j > 0$ is proposed

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u + u\partial_x u = 0, \ x \in \mathbb{T}, \ t \in \mathbb{R} \\ u(x,0) = \phi(x). \end{cases} \tag{1.3}$$

An extension to previous works, [12, 19, 22], the authors showed that the local in time well-posedness holds when $\phi(x) \in G^\sigma$. Moreover, they showed that the solution is not necessarily G^σ in t and belongs to $G^{m\sigma}(\mathbb{R})$ near zero for any x in the circle.

In [18], for $k = 1, 2, 3, \dots$ an initial-value problem for the generalized Burgers equation is considered

$$\begin{cases} u_t + u_{xx} + u^k u_x = 0, \ x \in \mathbb{T}, \ t \in \mathbb{R} \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

and its well-posedness in $G^{\sigma,\delta,s}$ and the regularity properties of the solution in G^σ in x and in $G^{2\sigma}$ in t is studied in the case when $u_0(x)$ belongs to a class of analytic

Gevrey spaces.
Very recently, the same research team issued a series of articles of related results with regard to existence and uniqueness and different regularity's properties in analytic Gevrey spaces (See [2, 3, 4, 5]).
In the first part of this article, we demonstrate that the unique solution of (1.1) is well-posed in analytic an appropriate Gevrey spaces. Next, as a last section $G^{3\sigma}$ regularity in t is given and the failure of G^d regularity in t is shown.

2 Preliminary estimates and Function spaces

We often use without mention, in the whole manuscript, nation $A+$ and we mean $A+\varepsilon$, for arbitrarily small $\epsilon \ll 1$. It is now necessary to recall a definition of the needed spaces, where the analytic Gevrey spaces with $\gamma \geq 1$ are given by $\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \mathcal{G}_{\sigma,\delta,s}$. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let us define

$$\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \left\{ f \in L^2(\mathbb{T}_\gamma); \|f\|^2_{\mathcal{G}^{\sigma,\delta,s}(\mathbb{T}_\gamma)} = \sum_{k\in\mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 d\xi < \infty \right\},$$

where $\langle \cdot \rangle = (1 + |\cdot|)$.
At a time, the analytic Gevrey-Bourgain spaces $X^\beta_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X^\beta_{\sigma,\delta,s,b}$ and $X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X_{\sigma,\delta,s,b}$ are defined by

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \left\langle \tau - k^3 \right\rangle^{2b} |\ \widehat{u}(k,\tau)\ |^2\ d\tau \right)^{\frac{1}{2}}, \tag{2.1}$$

$$\|w\|_{X^\beta_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \left\langle \tau - \beta k^3 \right\rangle^{2b} |\ \widehat{w}(k,\tau)\ |^2\ d\tau \right)^{\frac{1}{2}}. \tag{2.2}$$

The proof of local well-posedness is based on the iteration in the spaces $X_{\sigma,\delta,s,1/2} \times X^\beta_{\sigma,\delta,s,1/2}$. However, these spaces barely fails to be in $C(\mathbb{R}; \mathcal{G}_{\sigma,\delta,s} \times \mathcal{G}_{\sigma,\delta,s})$, what made us think in a second way , to introduce slightly smaller spaces $Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y_{\sigma,\delta,s}$ and $Y^\beta_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y^\beta_{\sigma,\delta,s}$ defined via the norms

$$\|u\|_{Y_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k \rangle^s \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \tag{2.3}$$

and

$$\|w\|_{Y^\beta_{\sigma,\delta,s}} = \|w\|_{X^\beta_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k \rangle^s \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \tag{2.4}$$

For any interval $I \subset \mathbb{R}$, we define the localized spaces $Y^I_{\sigma,\delta,s} = Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ and $Y^{\beta,I}_{\sigma,\delta,s} = Y^\beta_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ with the norms

$$\|u\|_{Y^I_{\sigma,\delta,s}} = inf \left\{ \|U\|_{Y_{\sigma,\delta,s}}; U|_{(\mathbb{T}_\gamma\times I)} = u \right\} \tag{2.5}$$

and

$$\|w\|_{Y^{\beta,I}_{\sigma,\delta,s}} = inf\left\{\|W\|_{Y^{\sigma,\beta}_{\delta,s}}; W|_{(\mathbb{T}_\gamma\times I)} = w\right\} \tag{2.6}$$

For $s \in \mathbb{R}$, $\sigma \geq 1$ and $\delta > 0$, we have, for all $T > 0$

$$Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) \hookrightarrow C\left(\mathbb{R}, \mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right),$$

and

$$Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) \hookrightarrow C\left(\mathbb{R}, \mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right).$$

Define the spaces $Z_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Z_{\sigma,\delta,s}$ and $Z^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Z^{\beta}_{\sigma,\delta,s}$ via the norms

$$\|u\|_{Z_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \left\langle \tau - k^3\right\rangle^{-1} \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}, \tag{2.7}$$

and

$$\|w\|_{Z^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \left\langle \tau - \beta k^3\right\rangle^{-1} \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}. \tag{2.8}$$

2.1 Linear estimates

Owing to the Fourier transform with respect to x of (1.1), we obtain a differential equation and then solving it in t. We consider for two operators Θ, Ξ, the following integral system which is equivalent to (1.1)

$$\begin{cases} \Theta_{u_0}(t) = S(t)u_0 - \displaystyle\int_0^t S(t-\nu)G_1(\nu)d\nu \\ \\ \Xi_{w_0}(t) = S_\beta(t)w_0 - \displaystyle\int_0^t S_\beta(t-\nu)G_2(\nu)d\nu, \end{cases} \tag{2.9}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_\beta(t) = e^{-t\beta\partial_x^3}$. The nonlinear terms are defined by $G_1(\nu) = \partial_x(\frac{w^2}{2})(\nu)$ and $G_2(\nu) = \partial_x(uw)(\nu)$.

Lemma 2.1. *(Lemma 7.1 in [8]) Let $s \in \mathbb{R}$, $\delta >$ and $\sigma \geq 1$. For some constant $C > 0$ and any time interval I which contains $t = 0$ and has length $| I |\leq 1$, we have*

$$\|S(t)u_0\|_{Y^I_{\sigma,\delta,s}} \leq C\ \|u_0\|_{\mathcal{G}_{\sigma,\delta,s}}, \tag{2.10}$$

and

$$\|S_\beta(t)w_0\|_{Y^{\beta,I}_{\sigma,\delta,s}} \leq C\ \|w_0\|_{\mathcal{G}_{\sigma,\delta,s}}, \tag{2.11}$$

for all $w_0, u_0 \in \mathcal{G}_{\sigma,\delta,s}$.

Lemma 2.2. *(Lemma 7.2 in [8]) Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, then for some constant $C > 0$ and any time interval I which contains $t = 0$ and has length $| I |\leq 1$, we have*

$$\left\|\int_0^t S(t-\nu)G_1(\nu)\mathrm{d}\nu\right\|_{Y^I_{\sigma,\delta,s}} \leq C\|G_1(\nu)\|_{Z^I_{\sigma,\delta,s}}, \tag{2.12}$$

and

$$\left\|\int_0^t S_\beta(t-\nu)G_2(\nu)\mathrm{d}\nu\right\|_{Y^{\beta,I}_{\sigma,\delta,s}} \leq C\|G_2(\nu)\|_{Z^{\beta,I}_{\sigma,\delta,s}}. \tag{2.13}$$

2.2 Bilinear estimates

In the following Lemma, we state the desired bilinear estimate.

Lemma 2.3. *Let* $s \geq \min\{1, \frac{1}{2}+\frac{1}{2}\varrho_1+\}$, $\delta > 0$ *and* $\sigma \geq 1$, *then*

$$\| \partial_x(w_1 w_2) \|_{Z_{\sigma,\delta,s}} \leq C_0(\gamma) \| w_1 \|_{Y^\beta_{\sigma,\delta,s}} \| w_2 \|_{Y^\beta_{\sigma,\delta,s}}, \tag{2.14}$$

where

$$C_0(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\varrho_1+}, & for\ 0 \leq \varrho_1 < 1, \\ \gamma^{0+}, & for\ \varrho_1 \geq 1. \end{cases} \tag{2.15}$$

Proof. Define the operator A by

$$\widehat{Aw}^x(\xi, t) = e^{\delta|\xi|^{1/\sigma}} \widehat{w}^x(\xi, t). \tag{2.16}$$

we have

(2.17)

$$\begin{aligned} e^{\delta|k|^{1/\sigma}} \widehat{w_1 w_2} &= (2\pi)^{-2} e^{\delta|k|^{1/\sigma}} \widehat{w_1} * \widehat{w_2} \\ &\leq (2\pi)^{-2} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{\delta|k-k_1|^{1/\sigma}} \widehat{w_1}(k-k_1, \tau-\tau_1) e^{\delta|k_1|^{1/\sigma}} \widehat{w_2}(k_1,\tau_1) d\tau_1 \\ &= \widehat{Aw_1 Aw_2}, \end{aligned}$$

since $\delta \mid k \mid^{1/\sigma} \leq \delta \mid k-k_1 \mid^{1/\sigma} + \delta \mid k_1 \mid^{1/\sigma}, \quad \forall \sigma \geq 1$. Then

$$\begin{aligned} \|\partial_x(w_1 w_2)\|_{Z_{\sigma,\delta,s}} &= \|\partial_x(w_1 w_2)\|_{X_{\sigma,\delta,s,-1/2}} \\ &+ \| e^{\delta|k|^{1/\sigma}} \langle k \rangle^s \langle \tau - k^3 \rangle^{-1} \partial_x \widehat{(w_1 w_2)}(k,\tau) \|_{L^2_k(\mathbb{T}/\gamma) L^1_\tau(\mathbb{R})} \\ &\leq \|\partial_x(Aw_1 Aw_2)\|_{X_{s,-1/2}} \\ &+ \| \langle k \rangle^s \langle \tau - k^3 \rangle^{-1} \partial_x (\widehat{Aw_1 Aw_2})(k,\tau) \|_{L^2_k(\mathbb{T}/\gamma) L^1_\tau(\mathbb{R})} \\ &= \| \partial_x(Aw_1 Aw_2) \|_{Z_s}. \end{aligned}$$

Now, by using Proposition 3.7 of [26], there exists $C_0(\gamma)$ where

$$C_0(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\varrho_1+}, & for\ 0 \leq \varrho_1 < 1, \\ \gamma^{0+}, & for\ \varrho_1 \geq 1. \end{cases} \tag{2.18}$$

such that

$$\begin{aligned} \| \partial_x(Aw_1 Aw_2) \|_{Zs} &\leq C_0(\gamma) \| Aw_1 \|_{Y^\beta_s} \| Aw_2 \|_{Y^\beta_s} \\ &= C_0(\gamma) \| w_1 \|_{Y^\beta_{\sigma,\delta,s}} \| w_2 \|_{Y^\beta_{\sigma,\delta,s}}. \end{aligned}$$

□

Lemma 2.4. *Let* $s \geq \min\{1, \frac{1}{2} + \frac{1}{2}\max\{\varrho_2, \varrho_3\}+\}$, $\delta > 0$ *and* $\sigma \geq 1$, *we have*

$$\| \partial_x(uw) \|_{Z^{\beta}_{\sigma,\delta,s}} \leq C_1(\gamma) \| u \|_{Y_{\sigma,\delta,s}} \| w \|_{Y^{\beta}_{\sigma,\delta,s}}, \tag{2.19}$$

where

$$C_1(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\max\{\varrho_2,\varrho_3\}+}, \ for \ 0 \leq \max\{\varrho_2, \varrho_3\} < 1, \\ \gamma^{0+}, \ for \ \max\{\varrho_2, \varrho_3\} \geq 1. \end{cases} \tag{2.20}$$

Proof. We have

$$\begin{aligned} \|\partial_x(uw)\|_{Z^{\beta}_{\sigma,\delta,s}} &= \|\partial_x(uw)\|_{X^{\beta}_{\sigma,\delta,s,-1/2}} \\ &+\|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \left\langle \tau - \beta k^3\right\rangle^{-1} \widehat{\partial_x(uw)}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \\ &\leq \|\partial_x(AuAw)\|_{X^{\beta}_{s,-1/2}} \\ &+\| \langle k\rangle^s \left\langle \tau - \beta k^3\right\rangle^{-1} \partial_x\widehat{(AuA}w)(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \\ &=\| \partial_x(AuAw) \|_{Z^{\beta}_s} . \end{aligned}$$

Now, by using Proposition 3.8 of [26], there exists $C_1(\gamma)$ where

$$C_1(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\max\{\varrho_2,\varrho_3\}+}, \ for \ 0 \leq \max\{\varrho_2, \varrho_3\} < 1, \\ \gamma^{0+}, \ for \ \max\{\varrho_2, \varrho_3\} \geq 1. \end{cases} \tag{2.21}$$

such that

$$\begin{aligned} \| \partial_x(AuAw) \|_{Z^{\beta}_s} &\leq C_1(\gamma) \| Au \|_{Y_s} \| Aw \|_{Y^{\beta}_s} \\ &= C_1(\gamma) \| u \|_{Y_{\sigma,\delta,s}} \| w \|_{Y^{\beta}_{\sigma,\delta,s}} . \end{aligned}$$

□

Corollary 2.5 ([16])**.** *Under the hypotheses of Lemma 2.3, for any time interval I, we have*

$$\| \partial_x(w_1w_2) \|_{Z^{I}_{\sigma,\delta,s}} \leq C_0(\gamma) \mid I \mid^{0+} \| w_1 \|_{Y^{\beta,I}_{\sigma,\delta,s}} \| w_2 \|_{Y^{\beta,I}_{\sigma,\delta,s}}, \tag{2.22}$$

Corollary 2.6 ([16])**.** *Under the hypotheses of Lemma 2.4, for any time interval I, we have*

$$\| \partial_x(uw) \|_{Z^{\beta,I}_{\sigma,\delta,s}} \leq C_1(\gamma) \mid I \mid^{0+} \| u \|_{Y^{I}_{\sigma,\delta,s}} \| w \|_{Y^{\beta,I}_{\sigma,\delta,s}}, \tag{2.23}$$

3 Local well-posedness and Proof

We are now ready to estimate all terms in (2.9) by using the trilinear estimates in the above Lemmas. We define the spaces

$$\mathcal{Y}_{\sigma,\delta,s} = Y^I_{\sigma,\delta,s} \times Y^{\beta,I}_{\sigma,\delta,s}, \quad \mathcal{Z}_{\sigma,\delta,s} = Z^I_{\sigma,\delta,s} \times Z^{\beta,I}_{\sigma,\delta,s} \quad \text{and} \quad \mathcal{B}^{\sigma,\delta,s} = \mathcal{G}^{\sigma,\delta,s} \times \mathcal{G}^{\sigma,\delta,s},$$

with norms

$$\|(u,w)\|_{\mathcal{Y}_{\sigma,\delta,s}} = max\{\|u\|_{Y^I_{\sigma,\delta,s}}, \|w\|_{Y^{\beta,I}_{\sigma,\delta,s}}\},$$

and similar for $\mathcal{Z}_{\sigma,\delta,s}$ and $\mathcal{B}^{\sigma,\delta,s}$.

Theorem 3.1. *Let* $s \geq \min\{1, s_0+\}$, $s_0 = \frac{1}{2} + \frac{1}{2}\max\{\varrho_1, \varrho_2, \varrho_3\}$ $0 < \beta < 1, \gamma \geq 1, \sigma \geq 1, \delta > 0$ *and* $(u_0, w_0) \in \mathcal{B}^{\sigma,\delta,s}$. *There exists* $T > 0$, *which depends on* (u_0, w_0), *such that the Cauchy problem (1.1) has a unique solution*

$$(u,w) \in C\left([-T,T], \mathcal{G}^{\sigma,\delta,s}\right) \times C\left([-T,T], \mathcal{G}^{\sigma,\delta,s}\right). \tag{3.1}$$

We will show that $\Theta \times \Xi$ is a contraction on the ball $\mathcal{Y}(0,R)$ to $\mathcal{Y}(0,R)$.

Lemma 3.2. *Let* $s \geq \min\{1, s_0+\}$, $\sigma \geq 1$ *and* $\delta > 0$. *Then, for all* $(u_0, w_0) \in \mathcal{B}^{\sigma,\delta,s}$, *such that the map* $\Theta\times\Xi : \mathcal{Y}(0,R) \to \mathcal{Y}(0,R)$ *is a contraction, where* $\mathcal{Y}(0,R)$ *is given by*

$$\mathcal{Y}(0,R) = \{(u,w) \in \mathcal{Y}_{\sigma,\delta,s}; \|(u,w)\|_{\mathcal{Y}_{\sigma,\delta,s}} \leq R\},$$

where $R = 2C\|(u_0,w_0)\|_{\mathcal{B}^{\sigma,\delta,s}}$.

Proof. Combining Lemma 2.1, (2.2) and Corollary 2.5,2.6 we obtain

$$\begin{aligned} \| \Theta[u,w] \|_{Y^I_{\sigma,\delta,s}} &\leq C \| u_0 \|_{\mathcal{G}^{\sigma,\delta,s}} + CC_0(\gamma) \| w \|^2_{Y^{\beta,I}_{\sigma,\delta,s}} \\ &\leq C \| (u_0,w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u,w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \| \Xi[u,w] \|_{Y^{\beta,I}_{\sigma,\delta,s}} &\leq C \| w_0 \|_{\mathcal{G}^{\sigma,\delta,s}} + CC_1(\gamma) \| u \|_{Y^I_{\sigma,\delta,s}} \| w \|_{Y^{\beta,I}_{\sigma,\delta,s}} \\ &\leq C \| (u_0,w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u,w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}. \end{aligned} \tag{3.3}$$

Where $C(\gamma) = \max\{C_0(\gamma), C_1(\gamma)\}$. Therefore, from (3.2) and (3.3), we obtain

$$\| (\Theta[u,w], \Xi[u,w]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq C \| (u_0,w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u,w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}. \tag{3.4}$$

For all $(u,w) \in \mathcal{Y}(0,R)$, we have

$$\| (\Theta[u,w], \Xi[u,w]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq R,$$

where $T^\epsilon \leq \frac{1}{4CC(\gamma)R}$.

Thus, $\Theta \times \Xi : \Big(\mathcal{Y}(0,R), \mathcal{Y}(0,R)\Big)$, is a contraction, since

$$\| (\Theta[u,w] - \Theta[u^*,w^*], \Xi[u,w] - \Xi[u^*,w^*]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq \frac{1}{2} \| (u-u^*, w-w^*) \|_{\mathcal{Y}_{\sigma,\delta,s}}.$$

□

Proof. (Of Theorem 3.1) The rest of the proof follows a standard argument, see [2]. □

4 Gevrey's regularity

Replacing t with $-t$ we can write our system as follows

$$\begin{cases} \partial_t u = \partial_x^3 u + w\partial_x w, \\ \partial_t w = \beta\partial_x^3 w + \partial_x(uw), \qquad 0 < \beta < 1 \\ u(x,0) = u_0(x), \\ w(x,0) = w_0(x), \end{cases} \tag{4.1}$$

4.1 Gevrey-3σ regularity in time

Theorem 4.1. *Let $s \geq \min\{1, s_0+\}$, for $0 < \beta < 1$, $\delta > 0$, $\sigma \geq 1$ and $(u, w) \in C\left([-T, T]; \mathcal{G}^{\sigma,\delta,s}\right) \times C\left([-T, T]; \mathcal{G}^{\sigma,\delta,s}\right)$ be the solution of (4.1). Then $(u, w) \in \mathcal{G}^{3\sigma}([-T, T]) \times \mathcal{G}^{3\sigma}([-T, T])$ in the time variable t.*

In order to prove Theorem 4.1 it is enough to prove the following results.

Proposition 4.2. *Let $n, k \in \mathbb{Z}_+^*$, we have*

$$|\partial_t^n \partial_x^k u(x,t)| \leq C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x \in \mathbb{T}_\gamma, \tag{4.2}$$

and

$$|\partial_t^n \partial_x^k w(x,t)| \leq C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x \in \mathbb{T}_\gamma, \tag{4.3}$$

where $M = C^2 + \frac{C}{2^\sigma}$.

Proof. We will use the proof by induction on n. For $n = 0$, inequality (4.2) follows from the following result.

$$|\partial_x^k u(x,t)| \leq C^{k+1}(k!)^\sigma, \quad \forall x \in \mathbb{T}, \forall k \in \mathbb{Z}_+^*, \tag{4.4}$$

and

$$|\partial_x^k w(x,t)| \leq C^{k+1}(k!)^\sigma, \quad \forall x \in \mathbb{T}, \forall k \in \mathbb{Z}_+^*. \tag{4.5}$$

For the proof of these inequalities, one can see Proposition 5 in [11]. We now suppose that (4.2) holds for all derivatives in t of order $\leq n$ and $k \in \mathbb{Z}_+^*$ and we then prove that (4.2) holds for $n+1$ and $k \in \mathbb{Z}_+^*$. We have from (4.1) that

$$\begin{aligned} \partial_t^{n+1} u &= \partial_t^n \partial_x^{k+3} u + \partial_t^n \partial_x^k (w\partial_x w) \\ &= \partial_t^n \partial_x^{k+3} u + \partial_t^n \left(\sum_{q=0}^{k} \binom{k}{q} \partial_x^{k-q} w \partial_x^{q+1} w \right) \\ &= \partial_t^n \partial_x^{k+3} u + \left(\sum_{q=0}^{k} \binom{k}{q} (\partial_t^n \partial_x^{k-q} w)(\partial_x^{q+1} w) \right) \\ &+ \left(\sum_{q=0}^{k} \binom{k}{q} (\partial_x^{k-q} w)(\partial_t^n \partial_x^{q+1} w) \right) \\ &+ \left(\sum_{m=1}^{n-1} \sum_{q=0}^{k} \binom{n}{m} \binom{k}{q} (\partial_t^{n-m} \partial_x^{k-q} w)(\partial_t^m \partial_x^{q+1} w) \right), \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
\partial_t^{n+1} w &= \partial_t^n \partial_x^{k+3} w + \partial_t^n \partial_x^{k+1}(uw) \\
&= \partial_t^n \partial_x^{k+3} u + \partial_t^n \left(\sum_{q=0}^{k} \binom{k}{q} \partial_x^{k-q} u \partial_x^{q+1} w \right) \\
&= \partial_t^n \partial_x^{k+3} w + \left(\sum_{q=0}^{k} \binom{k}{q} (\partial_t^n \partial_x^{k-q} u)(\partial_x^{q+1} w) \right) \\
&+ \left(\sum_{q=0}^{k} \binom{k}{q} (\partial_x^{k-q} u)(\partial_t^n \partial_x^{q+1} w) \right) \\
&+ \left(\sum_{m=1}^{n-1} \sum_{q=0}^{k} \binom{n}{m} \binom{k}{q} (\partial_t^{n-m} \partial_x^{k-q} u)(\partial_t^m \partial_x^{q+1} w) \right).
\end{aligned}
\tag{4.7}
$$

We will proof (4.6) and (4.7) will be similar. By using the induction assumption, we obtain

$$
\begin{aligned}
\mid \partial_t^n \partial_x^{k+3} u \mid &\leq C^{n+k+3+1}((3n+k+3)!)^\sigma M^n \\
&= C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n C^2.
\end{aligned}
\tag{4.8}
$$

For the second term in (4.6), by using the induction assumption, we obtain

$$
\begin{aligned}
&\left| \sum_{q=0}^{k} \binom{k}{q} (\partial_t^n \partial_x^{k-q} w)(\partial_x^{q+1} w) \right| \\
&\leq \sum_{q=0}^{k} \frac{k!}{q!(k-q)!} C^{n+k-q+1}((3n+k-q)!)^\sigma M^n C^{q+1+1}((q+1)!)^\sigma \\
&\leq C^{n+k+3} M^n \sum_{q=0}^{k} \frac{(k!)^\sigma}{(q!(k-q)!)^\sigma}((3n+k-q)!)^\sigma ((q+1)!)^\sigma \\
&\leq \frac{1}{3} C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n \frac{C}{2^\sigma}.
\end{aligned}
\tag{4.9}
$$

For the third term of (4.6), we have

$$
\begin{aligned}
&\left|\sum_{q=0}^{k}\binom{k}{q}(\partial_x^{k-q}w)(\partial_t^n\partial_x^{q+1}w)\right| \\
&\le \sum_{q=0}^{k}\frac{k!}{q!(k-q)!}C^{k-q+1}((k-q)!)^\sigma M^n C^{n+q+1+1}((3n+q+1)!)^\sigma \\
&\le C^{n+k+3}M^n\sum_{q=0}^{k}\frac{(k!)^\sigma}{(q!(k-q)!)^\sigma}((k-q)!)^\sigma((3n+q+1)!)^\sigma \\
&\le \frac{1}{3}C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n\frac{C}{2^\sigma}.
\end{aligned}
\tag{4.10}
$$

For the fourth term in (4.6), we have

$$
\begin{aligned}
&\left|\sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n}{m}\binom{k}{q}(\partial_t^{n-m}\partial_x^{k-q}u)(\partial_t^m\partial_x^{q+1}w)\right| \\
&\le \sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n+k}{m+q}C^{n-m+k-q+1}((3(n-m)+k-q)!)^\sigma M^{n-m}C^{m+q+1+1}((3m+q+1)!)^\sigma M^m \\
&\le C^{n+k+3}M^n\sum_{m=1}^{n-1}\sum_{q=0}^{k}\frac{(n+k)!}{(m+q)!(n+k-m-q)!}((3(n-m)+k-q)!)^\sigma((3m+q+1)!)^\sigma \\
&\le C^{n+k+3}M^n\sum_{m=1}^{n-1}\sum_{q=0}^{k}\frac{((n+k)!)^\sigma}{((m+q)!(n+k-m-q)!)^\sigma}((3(n-m)+k-q)!)^\sigma((3m+q+1)!)^\sigma \\
&\le \frac{1}{3}C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n\frac{C}{2^\sigma}.
\end{aligned}
$$

To obtain the proof in detail see proof of Lemma 4.2 in [15]. Finally by using (4.6), (4.8), (4.9),(4.10) and the last inequality we obtain

$$|\partial_t^n\partial_x^k u(x,t)| \le C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x\in\mathbb{T}_\gamma,$$

taking $k=0$ we obtain

$$|\partial_t^n u(x,t)| \le C^{n+1}((3n)!)^\sigma M^n \le L^{n+1}(n!)^{3\sigma}, \quad for\ L>0, \forall x\in\mathbb{T}_\gamma,$$

i.e. $u\in\mathcal{G}^{3\sigma}([-T,T])$ in time variable. We have also

$$|\partial_t^n\partial_x^k w(x,t)| \le C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x\in\mathbb{T}_\gamma,$$

taking $k=0$ we obtain

$$|\partial_t^n w(x,t)| \le C^{n+1}((3n)!)^\sigma M^n \le L^{n+1}(n!)^{3\sigma}, \quad for\ L>0, \forall x\in\mathbb{T}_\gamma,$$

i.e. $w\in\mathcal{G}^{3\sigma}([-T,T])$ in time variable t. □

4.2 Failure of Gevrey-d regularity in time

The next Lemma will de useful in order to estimate the higher-order derivatives of solution with respect to t.

Lemma 4.3. *[15] If (u, w) is a solution to (4.1) then for every $n \in \{1, 2, ...\}$ we have*

$$\partial_t^n u = \partial_x^{3n} u + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} w), \tag{4.11}$$

and

$$\partial_t^n w = \partial_x^{3n} w + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} w). \tag{4.12}$$

Definition 4.1. Let $\{\omega_k\}$ be a sequence of positive numbers. We denote by $\mathcal{C}(\omega_k)$ the class of all functions $h(x)$, infinitely differentiable on $[-1, 1]$, for each of which there is an $C > 0$ such that

$$|h^{(k)}(x)| \leq C^{k+1} \omega_k, \quad x \in [-1, 1] \text{ and } k = 0, 1, 2, ... \tag{4.13}$$

Lemma 4.4. *([14]) For any $\sigma > 1$ and any sequence of complex numbers $\{\varphi_k\}$, satisfying*

$$|\varphi_k| \leq C_1^{k+1} k^{k\sigma}, C_1 > 0, \tag{4.14}$$

there exists a function $h(x) \in \mathcal{C}(k^{k\sigma})$ for which $h^{(k)}(0) = \varphi_k$.

This result will be used for the sequence of real numbers

$$|h^{(k)}(x)| \leq C^{k+1} k^{k\sigma} \leq C^{k+1} (k!)^\sigma e^{k\sigma}, \quad k = 0, 1, 2, ... \tag{4.15}$$

where $h(x) \in \mathcal{C}(k^{k\sigma})$ such that $h^{(k)}(0) = \varphi_k = (k!)^\sigma$.
We choose $u_0, w_0 \in \mathcal{G}_c^\sigma(-2, 2)$ such that

$$\begin{cases} \theta(x) = 1 \ for \ \mid x \mid \leq 1 \\ and \\ \theta(x) = 0 \ for \ \mid x \mid > 2, \end{cases} \tag{4.16}$$

by modifying $h(x)$ to become having a compact support in $(-1, 1)$.
If u_0 and w_0 are extension of θh, then we have $u_0, w_0 \in \mathcal{G}^\sigma([-T, T])$. We have then the relation by $h(x)$

$$u_0^{(k)}(0) = h^{(k)}(0) = (k!)^\sigma \text{ and } w_0^{(k)}(0) = h^{(k)}(0) = (k!)^\sigma. \tag{4.17}$$

Theorem 4.5. *Let $s \geq \min\{1, s_0+\}$ for $0 < \beta < 1$, $\delta > 0$, $\sigma \geq 1$. The real-valued solution to (4.1) with real-valued initial data $(u_0, w_0) \in \mathcal{G}^{\sigma,\delta,s} \times \mathcal{G}^{\sigma,\delta,s}$ may not be in $\mathcal{G}^d([-T, T]) \times \mathcal{G}^d([-T, T])$, with $1 \leq d < 3\sigma$, in the time variable t.*

Proof. By using (4.11) and (4.12) we get

$$\begin{aligned}\partial_t^n u(0,0) &= \partial_x^{3n} u(0,0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\ &= u_0^{3n}(0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} w_0(0)) \\ &\geq u_0^{3n}(0) = ((3n)!)^\sigma \geq (n!)^{3\sigma},\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\partial_t^n w(0,0) &= \partial_x^{3n} w(0,0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} w(0,0)) \\ &= v_0^{3n}(0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} w_0(0)) \\ &\geq w_0^{3n}(0) = ((3n)!)^\sigma \geq (n!)^{3\sigma},\end{aligned} \tag{4.19}$$

we have proved that $(u(0,\cdot), w(0,\cdot)) \notin \mathcal{G}^d([-T,T]) \times \mathcal{G}^d([-T,T])$ for $1 \leq d < 3\sigma$ and for t near 0. □

Conclusion

In the past few years, periodic KdV systems has been actively studied both quantitatively and qualitatively. The importance of the present research lies mainly with KdV systems of Majda-Biello type, the complete study regarding to solution in analytic functions spaces makes it easy for applications in the sciences. The novelty to this article are mainly:

1. A new minimal conditions on the exponents s, β, σ, δ and the relationship between them are used.

2. We showed the existence of unique solution in analytic an appropriate Gevrey spaces which improves the existing results in [1, 25], obtained in the classical Sobolev spaces.

3. The regularity in time is given, in addition we established the failure of Gevrey-d regularity.

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