Some sets and spaces on ideal N-topological spaces

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Abstract. The present paper introduces and develops the relations between various kinds of locally closed sets in ideal N-topological spaces. Moreover, some ideal N-topological spaces with such properties are derived.

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Key words: Ideal N-topological spaces; $N\tau I$-open sets; locally $(N\tau)_I$-closed sets; locally $N\tau\Lambda_I$-closed sets, $N\tau I$-T_i-spaces and $N\tau I$-D_i-spaces.

1 Introduction

The concept of ideal topological space plays a significant role in all the branches of mathematics and quantum physics. This concept was first introduced by Kuratowski [5]. Later on, Jankovic and Hamlett [2, 3] studied the topologies and the compatiblity of ideal topological spaces. Following them, Abd El-Monsef et al. [1] revealed that the collection of all I-open sets needs not to form a topology. Keskin et al. [4] further investigated the decomposition of continuity in ideal topological spaces. Thereafter Noiri et al. [8] studied the properties of separation axioms in classical topological spaces. Recently the concept of N-topology was derived by Lellis Thivagar et al. [6]. Moreover, Lellis Thivagar et al. [7] introduced $N\tau I$-open sets in ideal N-topological spaces and established the properties of $N\tau\Lambda_I$-sets and $N\tau V_I$-sets. This paper presents the properties of locally $N\tau\Lambda_I$-closed sets, locally $N\tau\Lambda_I^*$-closed sets, locally $(N\tau)_I$-closed sets and locally $(N\tau)_I^*$-closed sets. Furthermore, illustrative $N\tau I$-T_i-spaces and $N\tau I$-D_i-spaces for $i = 0, 1, 2$ are obtained.

2 Preliminaries

We recall first some familiar properties of N-topological spaces and ideal N-topological spaces, which are used in the following sections.

Definition 2.1. [6] Let X be a non empty set, and let τ_1, τ_2, ... , τ_N be N-arbitrary topologies defined on X; the collection

$$N\tau = \{S \subseteq X : S = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i), \quad A_i, B_i \in \tau_i\}$$

is called an N-topology on X if the following axioms are satisfied:

(i) $X, \varnothing \in N\tau$.

(ii) $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{\infty} \in N\tau$.

(iii) $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{n} \in N\tau$.

Then $(X, N\tau)$ is called an N-topological space on X. The elements of $N\tau$ are known as $N\tau$-open sets on X, and its complement is called $N\tau$-closed on X. We denote by $N\tau O(X, x)$ the set of all $N\tau$-open sets containing x on X.

Definition 2.2. [6] The interior and the closure of a subset A of $(X, N\tau)$ are respectively defined as

(i) $N\tau$-$int(A) = \cup\{G : G \subseteq A$ and G is $N\tau$-open$\}$.

(ii) $N\tau$-$cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau$-closed$\}$.

Definition 2.3. [7] A non empty collection I of subsets of an N-topological space $(X, N\tau)$ is an ideal on X if the heredity and finite additivity are satisfied, as follows:

(i) $B \in I$, when $A \in I$ and $B \subset A$.

(ii) $A \cup B \in I$, when $A \in I$ and $B \in I$.

Then a non empty set X equipped with an N-topology $N\tau$ and an ideal I defined on X form an ideal N-topological space. We denote it as $(X, N\tau, I)$.

Definition 2.4. [7] The local function of a subset A of a space X is defined as $A^*(I, N\tau) = \{x \in X : U \cap A \notin I$ for every $U \in N\tau O(X, x)\}$. We shall further denote A^* instead of $A^*(I, N\tau)$. The collection

$$(N\tau)_I^* = \{U \subseteq X : N\tau - cl_I^*(X - U) = X - U\}$$

is a finer topology of $N\tau$. We call it $(N\tau)_I^*$-topology, where $N\tau$-$cl_I^*(A) = A \cup A^*(I, N\tau)$.

Definition 2.5. [7] A subset A of a space X is said to be $(N\tau)_I^*$-closed if $A^* \subseteq A$; then its complement is called $(N\tau)_I^*$-open. We denote by $N\tau$-$int_I^*(A)$ and $N\tau$-$cl_I^*(A)$ the interior and the closure of A in $(N\tau)_I^*$-topology respectively. We denote $(N\tau)_I^*C(X)$ (resp. $(N\tau)_I^*O(X)$) as the set of all $(N\tau)_I^*$-closed sets (resp. $(N\tau)_I^*$-open sets).

Definition 2.6. [7] A subset A of a space X is said to be $N\tau I$-open if $A \subseteq N\tau$-$int(A^*)$; then its complement is called $N\tau I$-closed. We denote by $N\tau IO(X)$ (resp. $N\tau IC(X)$) the set of all $N\tau I$-open sets (resp. $N\tau I$-closed sets). The collection $N\tau IO(X)$ needs not to be a topology on X.

Definition 2.7. [7] Let A be a subset of an ideal N-topological space X; then we define $N\tau\Lambda_I(A)$ of X as the intersection of all $N\tau I$-open sets containing A.

Definition 2.8. [7] A subset A of a space X is called an $N\tau\Lambda_I$-set if $A = N\tau\Lambda_I(A)$.

3 Some ideal N-topological sets

We further provide properties of ideal N-topological locally closed sets. Given the space X, we consider the ideal N-topological space $(X, N\tau, I)$ without assuming separation axioms, unless explicitly stated.

Definition 3.1. Let L and F be subsets of a space X. Then a set $L \cap F \subseteq X$ is called:

(i) locally $N\tau\Lambda_I$-closed (shortly $\mathcal{L}(N\tau)\Lambda_I$-closed), if L is $N\tau\Lambda_I$-set and F is $N\tau$-closed;

(ii) locally $N\tau\Lambda_I^*$-closed (shortly $\mathcal{L}(N\tau)\Lambda_I^*$-closed), if L is $N\tau\Lambda_I$-set and F is $(N\tau)_I^*$-closed;

(iii) locally $(N\tau)_I$-closed (shortly $\mathcal{L}(N\tau)_I$-closed), if L is $N\tau I$-open and F is $N\tau$-closed;

(iv) locally $(N\tau)_I^*$-closed (shortly $\mathcal{L}(N\tau)_I^*$-closed), if L is $N\tau I$-open and F is $(N\tau)_I^*$-closed.

Taking $N = 1$ in the above definition, we get (see Noiri et al. [8]) the derivations of $\mathcal{L}\Lambda_I$-closed, $\mathcal{L}\Lambda_I^*$-closed, $\mathcal{L}_I$-closed and $\mathcal{L}_I^*$-closed sets.

Example 3.2. For $N = 5$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{c\}\}$, $\tau_3 O(X) = \{\varnothing, X, \{a, c\}\}$, $\tau_4 O(X) = \{\varnothing, X, \{a\}, \{a, c\}\}$ and $\tau_5 O(X) = \{\varnothing, X\}$. Then $5\tau O(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$ is a 5-topology on X. Consider the ideal $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ on X; then $(X, 5\tau, I)$ is an ideal 5-topological space. Then the $\mathcal{L}5\tau\Lambda_I$-closed, $\mathcal{L}5\tau\Lambda_I^*$-closed, $\mathcal{L}(5\tau)_I$-closed and $\mathcal{L}(5\tau)_I^*$-closed sets are $\varnothing$ and $\{c\}$ only. If $N = 1$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \tau O(X) = \{\varnothing, X\}$, and the ideal $I = \{\varnothing, \{a\}\}$ on X. Then (X, τ, I) is an ideal topological space. Here the set $\{a\}$ is $\mathcal{L}\Lambda_I$-closed, $\mathcal{L}\Lambda_I^*$-closed and $\mathcal{L}_I^*$-closed, but not $\mathcal{L}_I$-closed.

Lemma 3.1. In a space X, a subset A of X is $\mathcal{L}(N\tau)_I^*$-closed(resp. $\mathcal{L}N\tau\Lambda_I$-closed, $\mathcal{L}N\tau\Lambda_I^*$-closed, $\mathcal{L}N\tau\Lambda_I$-closed) if A is $\mathcal{L}(N\tau)_I$-closed (resp. $\mathcal{L}(N\tau)_I^*$-closed, $\mathcal{L}N\tau\Lambda_I$-closed,$N\tau\Lambda_I$) set.

Proof. The proof directly follows from Definition 3.1. □

Lemma 3.2. Let $\{A_i\}_{i\in\Omega}$ be $\mathcal{L}N\tau\Lambda_I$-closed subsets of a space X. Then

(i) $\bigcup_{i\in\Omega_o} A_i$ is $\mathcal{L}N\tau\Lambda_I$-closed for any finite subset Ω_o of Ω;

(ii) $\bigcap_{i\in\Omega} A_i$ is $\mathcal{L}N\tau\Lambda_I$-closed.

Proof.

(i) Assume for all $i \in \Omega_o$, A_i is $\mathcal{L}N\tau\Lambda_I$-closed; then for all $i \in \Omega_o$, there exists an $N\tau\Lambda_I$-set L_i and an $N\tau$-closed set F_i such that $A_i = L_i \cap F_i$. Then by the operator of union, $\bigcup_{i\in\Omega_o} A_i = \bigcup_{i\in\Omega_o} (L_i \cap F_i) = (\bigcup_{i\in\Omega_o} L_i) \cap (\bigcup_{i\in\Omega_o} F_i)$. Since $\bigcup_{i\in\Omega_o} L_i$ is an $N\tau\Lambda_I$-set and $\bigcup_{i\in\Omega_o} F_i$ is an $N\tau$-closed set, then $\bigcup_{i\in\Omega_o} A_i$ is $\mathcal{L}N\tau\Lambda_I$-closed.

(ii) Assume that for all $i \in \Omega$, A_i is $\mathcal{L}N\tau\Lambda_I$-closed; then there exists an $N\tau\Lambda_I$-set L_i and an $N\tau$-closed set F_i such that $A_i = L_i \cap F_i$ for all $i \in \Omega$. Then by the operator of intersection, we have $\bigcap\limits_{i\in\Omega} A_i = \bigcap\limits_{i\in\Omega}(L_i \cap F_i) = (\bigcap\limits_{i\in\Omega} L_i) \cap (\bigcap\limits_{i\in\Omega} F_i)$. Since $\bigcap\limits_{i\in\Omega} L_i$ is an $N\tau\Lambda_I$-set and $\bigcap\limits_{i\in\Omega} F_i$ is an $N\tau$-closed set, then $\bigcap\limits_{i\in\Omega} A_i$ is $\mathcal{L}N\tau\Lambda_I$-closed.

□

Lemma 3.3. Let $\{A_i\}_{i\in\Omega}$ be $\mathcal{L}N\tau\Lambda^*_I$-closed subsets of a space X. Then

(i) $\bigcup\limits_{i\in\Omega_o} A_i$ is $\mathcal{L}N\tau\Lambda^*_I$-closed for any finite subset Ω_o of Ω;

(ii) $\bigcap\limits_{i\in\Omega} A_i$ is $\mathcal{L}N\tau\Lambda^*_I$-closed.

Proof. We can similarly prove this Lemma similar to Lemma 3.4. □

Lemma 3.4. The following properties are equivalent for a subset A of a space X:

(i) A is $\mathcal{L}N\tau\Lambda^*_I$-closed;

(ii) the set A can be written as the intersection of $N\tau\Lambda_I$-set L and $N\tau\text{-}cl^*_I(A)$;

(iii) the set A can be written as the intersection of $N\tau\Lambda_I(A)$ and $N\tau\text{-}cl^*_I(A)$.

Proof. **(i)$\Longrightarrow$(ii):** Assume that A is $\mathcal{L}N\tau\Lambda^*_I$-closed; then for $(N\tau)^*_I$-closed set F and $N\tau\Lambda_I$-set L, $A = L \cap F$. This implies $A^* \subset F^* \subset F$ and $N\tau\text{-}cl^*_I(A) \subset F$. Then $A = L \cap F \supset L \cap N\tau\text{-}cl^*_I(A) \supset A$ and so $A = L \cap N\tau\text{-}cl^*_I(A)$.
(ii)$\Longrightarrow$ (iii): Assume that for an $N\tau\Lambda_I$-set L, $A = L \cap N\tau\text{-}cl^*_I(A)$; then $N\tau\Lambda_I(A) \subset N\tau\Lambda_I(L) = L$. As well, then $A = L \cap N\tau\text{-}cl^*_I(A) \supset N\tau\Lambda_I(A) \cap N\tau\text{-}cl^*_I(A) \supset A$. Thus the set A can be written as the intersection of $N\tau\text{-}cl^*_I(A)$ and $N\tau\Lambda_I(A)$.
(iii)$\Longrightarrow$ (i): Clearly, $N\tau\Lambda_I(A)$ is an $N\tau\Lambda_I$-set and $(N\tau\text{-}cl^*_I(A))^* = (A \cup A^*)^* = A^* \cup (A^*)^* \subset A^* \subset N\tau\text{-}cl^*_I(A)$. Thus the set $N\tau\text{-}cl^*_I(A)$ is $(N\tau)^*_I$-closed. □

The proof of the following lemmas can be derived as similar to the above lemma.

Lemma 3.5. The following are equivalent for a given subset A of a space X:

(i) A is $\mathcal{L}N\tau\Lambda_I$-closed;

(ii) the set A can be written as the intersection of $N\tau\Lambda_I$-set L and $N\tau\text{-}cl(A)$;

(iii) the set A can be written as the intersection of $N\tau\Lambda_I(A)$ and $N\tau\text{-}cl(A)$.

Lemma 3.6. Let A be a subset of a space $(X, N\tau, I)$. Then

(i) A is $\mathcal{L}(N\tau)_I$-closed if and only if it can be written as the intersection of $N\tau I$-open set L and $N\tau\text{-}cl(A)$;

(ii) A is $\mathcal{L}(N\tau)^*_I$-closed if and only if it can be written as the intersection of $N\tau I$-open set L and $N\tau\text{-}cl^*_I(A)$.

4 Some ideal N-topological spaces

By using $N\tau I$-open sets, we shall develop various separation axioms in an ideal N-topological space X.

Definition 4.1. A space X is said to be:

(i) $N\tau I$-T_0-space, if for every pair of different points x, y of X, there exists an $N\tau I$-open set U of X containing x but not y or containing y but not x;

(ii) $N\tau I$-T_1-space, if for every pair of different points x, y of X, there exists an $N\tau I$-open set U of X containing x but not y and an $N\tau I$-open set V containing y but not x;

(iii) $N\tau I$-T_2-space, if for every pair of different points x, y of X, there exist two disjoint $N\tau I$-open sets U and V such that $x \in U$ and $y \in V$.

Particularly, if $N = 1$, then τI-T_0-spaces, τI-T_1-spaces and $N\tau I$-T_2-spaces, which are respectively represented as I-T_0-spaces, I-T_1-spaces and I-T_2-spaces, which are defined by Noiri et al. [8].

Example 4.2. For $N = 3$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$ and $\tau_3 O(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$, then $3\tau O(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let us take the minimal ideal $I = \{\varnothing\}$ of X; then $3\tau_I^* O(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $3\tau IO(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Here the space $(X, 3\tau, I)$ is $3\tau I$-T_0-space, $3\tau I$-T_1-space and $3\tau I$-T_2-space.

Lemma 4.1. If a space X is an $N\tau I$-T_1-space, then it is an $N\tau I$-T_0-space.
Proof The proof is obvious by Definition 4.1.

Example 4.3. The converse of the above Lemma needs not to be true. If $N = 2$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$ and $\tau_2 O(X) = \{\varnothing, X, \{a, b\}\}$, then $2\tau O(X) = \{\varnothing, X, \{a\}, \{a, b\}\}$. If we take the ideal $I = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$ of X, then $2\tau IO(X) = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$. Here the ideal 2-topological space X is $2\tau I$-T_0-space, but it is not $2\tau I$-T_1-space and not $2\tau I$-T_2-space.

Lemma 4.2. The following assertions are equivalent in a space X:

(i) the space X is $N\tau I$-T_1-space;

(ii) the set $\{x\}$ is $N\tau\Lambda_I$-set for all $x \in X$;

(iii) the set $\{x\}$ is $N\tau I$-closed set for all $x \in X$.

Proof. **(i)$\Longrightarrow$(ii):** Let $x, y \in X$ and $y \neq x$; then there exists an $N\tau I$-open set U such that $x \in U$ and $y \notin U$. Then $\{x\} \subset U$ and hence $y \notin N\tau\Lambda_I(\{x\})$. This shows that $N\tau\Lambda_I(\{x\}) \subset \{x\}$. Since $\{x\} \subset N\tau\Lambda_I(\{x\})$, then $\{x\}$ is an $N\tau\Lambda_I$-set.
(ii)$\Longrightarrow$(iii): If $x \in X$ and for any $y \in (X - \{x\})$, $\{y\} = N\tau\Lambda_I(\{y\})$, then there is a $U_y \in N\tau IO(X)$ such that $x \notin U_y$ and $y \in U_y$. Then $y \in U_y \subset (X - \{x\})$ and $X - \{x\} = \bigcup\{U_y : y \in (X - \{x\})\}$. Thus $\{x\}$ is $N\tau I$-closed.
(iii)$\Longrightarrow$(i): Let $x, y \in X$ and $y \neq x$; then both $\{x\}$ and $\{y\}$ are $N\tau I$-closed. Then their complements are $N\tau I$-open sets which contains either x or y but not both, and so the space X is an $N\tau I$-T_1-space. $\square$

Lemma 4.3. The following are equivalent in a space X:

(i) the space X is $N\tau I$-T_1-space;

(ii) if A is a subset of X, then A is $N\tau\Lambda_I$-set;

(iii) if A is $N\tau I$-closed subset of X, then A is $N\tau\Lambda_I$-set.

Proof. **(i)$\Longrightarrow$(ii):** Assume that X is $N\tau I$-T_1-space and that A is a subset of X; then by Lemma 4.5, $\{x\}$ is an $N\tau\Lambda_I$-set for each $x \in A$. Since the union of $N\tau\Lambda_I$-set is an $N\tau\Lambda_I$-set (see Lellis Thivagar et al. (2017)), the subset A is an $N\tau\Lambda_I$-set.
(ii)$\Longrightarrow$(iii): It is obvious.
(iii)$\Longrightarrow$(i): Assume that A is an $N\tau I$-closed subset of X such that $x \in A$ and $y \notin A$; then $A = N\tau\Lambda_I(A) = \cap\{U \subseteq X : A \subseteq U, U \in N\tau IO(X)\}$ and so $x \in U$ and $y \notin U$. Conversely, assume that A is an $N\tau I$-closed set; then A^c is $N\tau I$-open such that $y \in A^c$ and $x \notin A^c$. Hence X is an $N\tau I$-T_1-space. □

Lemma 4.4. In a space X, every $N\tau I$-T_2-space is an $N\tau I$-T_1-space.
Proof The proof follows from Definition 4.1.

Example 4.4. The converse of the above lemma needs not to be true. If $N = 1$ and $\mathbb{N} = \{1, 2, ..., n, ...\}$, consider $\tau O(\mathbb{N}) = \{\varnothing, \mathbb{N}, \mathbb{N} - \{1\}, \mathbb{N} - \{2\}, ..., \mathbb{N} - \{1, 2\}, \mathbb{N} - \{1, 3\}, ..., \mathbb{N} - \{2, 3\}, ..., \mathbb{N} - \{1, 2, 3\}, ...\}$. If we take the ideal $I = \{\varnothing\}$ on $\mathbb{N}$, then $\tau IO(X) = \{\varnothing, \mathbb{N}, \mathbb{N} - \{1\}, \mathbb{N} - \{2\}, ..., \mathbb{N} - \{1, 2\}, \mathbb{N} - \{1, 3\}, ..., \mathbb{N} - \{2, 3\}, ..., \mathbb{N} - \{1, 2, 3\}, ...\}$. Clearly, the space $(\mathbb{N}, \tau, I)$ is I-T_1-space but not I-T_2-space.

Definition 4.5. A subset A of a space X is called $N\tau I$-D-set if there exist two $N\tau I$-open sets U_1 and U_2 such that $A = U_1 \setminus U_2$ and $U_1 \neq X$. We denote $N\tau I$-$D(X)$ as the set of all $N\tau I$-D-sets.

Example 4.6. If $N = 2$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$ and $\tau_2 O(X) = \{\varnothing, X, \{a, b\}\}$; then $2\tau O(X) = \{\varnothing, X, \{a\}, \{a, b\}\}$. If we take the ideal $I = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$ of X, then $2\tau IO(X) = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $2\tau I$-$D(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

Theorem 4.5. In a space X, if a set $A \subset X$ is $N\tau I$-open, then it is $N\tau I$-D-set.

Proof. Assume that A is $N\tau I$-open; then for $N\tau I$-open sets U_1 and $U_2 = \varnothing$, we have $A = U_1 \setminus U_2$. □

Example 4.7. The converse of the above theorem need not be true. In Example 4.6, by taking two $2\tau I$-open sets $U = \{a, c\} \neq X$ and $V = \{a\}$, we get $A = U \setminus V = \{c\}$, which is a $2\tau I$-D-set but not a $2\tau I$-open set in $(X, 2\tau, I)$.

Definition 4.8. A space X is said to be:

(i) $N\tau I$-D_0-space, if for every pair of different points x, y of X, there exists an $N\tau I$-D-set containing either x or y but not both;

(ii) $N\tau I$-D_1-space, if for every pair of different points x, y of X, there exist two $N\tau I$-D-sets G and H such that $x \in G, x \notin H$ and $y \in H, y \notin G$;

(iii) $N\tau I$-D_2-space, if for every pair of different points x, y of X, there exist two disjoint $N\tau I$-D-sets G and H such that $x \in G$ and $y \in H$.

The proof of the following remark is straightforward from Definition 4.1 and Definition 4.8.

Remark 4.9. (i) A space X is an $N\tau I$-D_i-space, then it is $N\tau I$-D_{i-1}-space ($i = 1, 2$).

(ii) A space X is an $N\tau I$-T_i-space, then it is $N\tau I$-D_i-space ($i = 0, 1, 2$).

Theorem 4.6. A space X is an $N\tau I$-T_0-space if and only if X is an $N\tau I$-D_0-space.

Proof. Assume that X is an $N\tau I$-T_0-space; by Remark 4.9(ii), X is $N\tau I$-D_0. Conversely, if X is $N\tau I$-D_0-space, then for every $x, y \in X, x \neq y$, there is an $N\tau I$-D-set S of X such that $x \in S$ but $y \notin S$. Then $S = U_1 \setminus U_2$ such that $U_1, U_2 \in N\tau IO(X)$ and $U_1 \neq X$. Therefore, either $y \notin U_1$, or $y \in U_1$ and U_2.
Case i: Suppose $y \notin U_1$. Since $x \in U_1$ and $y \notin U_1$, then U_1 is our required $N\tau I$-open set.
Case ii: Suppose $y \in U_1$ and U_2, U_2 is $N\tau I$-open such that $y \in U_2$ and $x \notin U_2$.

Thus, in both cases, $(X, N\tau, I)$ is an $N\tau I$-T_0-space. □

Theorem 4.7. If a space X is an $N\tau I$-D_1-space, then X is an $N\tau I$-T_0-space.

Proof. $N\tau I$-$D_1 \Rightarrow N\tau I$-D_0 and $N\tau I$-$D_0 \Rightarrow N\tau I$-T_0, by Remark 4.9 and Theorem 4.6. □

Example 4.10. The converse of the above theorem needs not to be true. For example, if $N = 4$ and $X = \{a, b, c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}, \{a, b\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{b\}, \{b, c\}\}$, $\tau_3 O(X) = \{\varnothing, X, \{c\}, \{b, c\}\}$ and $\tau_4 O(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, then $4\tau O(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}\{a, c\}, \{b, c\}\}$. If we take the ideal $I = \{\varnothing, \{a\}\}$ of X, then $4\tau IO(X) = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$ and $4\tau I$-$D(X) = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$. Here the 4-topological space X is $4\tau I$-T_0-space, but it is not $4\tau I$-D_1-space. Also $(X, 4\tau, I)$ is $4\tau I$-D_0-space, but not $4\tau I$-D_2-space, not $4\tau I$-T_1-space and $4\tau I$-T_2-space.

Theorem 4.8. A space X is $N\tau I$-D_1-space if and only if X is $N\tau I$-D_2-space.

Proof. $N\tau I$-$D_2 \Rightarrow N\tau I$-D_1 is obvious. Suppose X is $N\tau I$-D_1. Then for every pair of different points x, y of X, we have $N\tau I$-D-sets S_1 and S_2 such that $x \in S_1$ and $x \notin S_2$ and $y \in S_2$, $y \notin S_1$. Then $S_2 = U_3 \setminus U_4$ and $S_1 = U_1 \setminus U_2$ such that $U_1, U_2, U_3, U_4 \in N\tau IO(X)$ and $U_1, U_3 \neq X$. Since $x \notin S_2$, we have either $x \in U_3$ and $x \in U_4$ or $x \notin U_3$.
Case i: Suppose $x \notin U_3$. Since $y \notin S_1$, then either $y \notin U_1$ or $y \in U_1$ and $y \in U_2$.
Sub case (a): Suppose $y \notin U_1$. Since $y \in S_2 = U_3 \setminus U_4$ and $x \in S_1 = U_1 \setminus U_2$, then $y \in U_3 \setminus (U_1 \cup U_4)$ and $x \in U_1 \setminus (U_2 \cup U_3)$. Also $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \varnothing$. Also $U_1 \cup U_4$ and $U_2 \cup U_3$ are $N\tau I$-open sets.
Sub case (b): Suppose y is not an element of the $N\tau I$-open sets U_1 and U_2; then the disjoint pair of $N\tau I$-D sets $U_1 \setminus U_2$ and U_2 satisfy our requirement.
Case ii: $x \in U_3$ and $x \in U_4$. Here, our required sets are $U_3 \setminus U_4$ and U_4; because

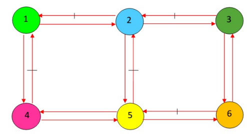

Figure 1: Relationship between the separation axioms
(1) $N\tau I$-T_2, (2) $N\tau I$-T_1, (3) $N\tau I$-T_0, (4)$N\tau I$-D_2, (5) $N\tau I$-D_1, (6) $N\tau I$-D_0.

$x \in U_4$ and $y \in U_3 \setminus U_4$, then $(U_3 \setminus U_4) \cap U_4 = \varnothing$. Therefore, in all the cases, $(X, N\tau, I)$ is an $N\tau I$-D_2-space. $\square$

The results of the above theorems, lemmas, remarks and examples are represented in the above diagram.

Conclusion. Topology is a core subject in mathematics. Many researchers contributed to the growth of this field. We emphasize besides the theoretical aspect of the developed topic, the numerous applications. We hope that the ideal N-topological sets and spaces will represent a consistent addition to topology. As well, extensions might enhance the research fields of Fuzzy topology, Nano topology, Supra topology, etc .

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