Bifurcation control in a water resources model

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Abstract. In this paper, by introducing a mathematical model based on the system of prey and predator, we have studied the interaction of water resources and population with each other and the delayed model is analyzed for stability and bifurcation phenomena. Then, a delay feedback controller with the coefficient depending on delay is introduced to system. And by using it, we control the start time of the bifurcation and stability. Some numerical examples are given to verify the effectiveness of the delay feedback control method and the existence of crossing curve, which shows the effectiveness of the controller.

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1 Introduction

In the coming years, the challenge of water scarcity and the optimal use of these resources will probably be the most important threat to the budget of many countries. Population growth, improper use of water resources simultaneously with global warming, and reduced rainfall. All in all, these reduce water resources. This has threatened the security of agricultural products and food supply. However, any kind of prediction of the intensity and amount of water consumption and its impact on the water resources of a land is very important. Meanwhile, mathematical models have the ability to predict the intensity and distribution of water consumption in society. They are one of the most effective tools in studying patterns of change in water resources, and allow us to simulate this, to study the interaction of population and water resources with each other, to determine the maximum index for water consumption in a community and finally to predict the future of this interaction.

Predator-prey models are one of the most important mathematical tools in studying the interaction of two (or more) species with each other. One of the most well-known predator-prey models, is the Lesli-Gower model, introduced in 1948 to study population change for two types of predator and prey, where the final capacity of the predatory environment is proportional to the number of prey[1]. The general form of

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these equations is as follows:

$$\begin{cases} \dfrac{dx}{dt} = xg(x,K) - yp(x), \\ \dfrac{dy}{dt} = y(-d + cq(x)), \end{cases} \tag{1.1}$$

where $g(x,K)$ is a differentiable and continuous function which shows the growth rate of a particular species of prey in the absence of a predator, and where $g(0,K) = r > 0, g(K,K) = 0, g_x(K,K) < 0$ and $g_x(x,K) \leq 0$ (for more details, see [1] and [2]). For example, the function $g(x,K) = r(1 - \frac{x}{K})$ holds in the conditions given. This function is known as the logical growth model and is the basis of many mathematical models for studying the interaction between prey and predator (see [3], [4]-[9]). As well, the function $p(x)$ is called the feedback function, satisfying $p(0) = 0$. This function indicates the fact that in each unit of time, the amount of prey hunted by each predator depends on the density of the predator. Functional feedback depends on many factors, including the difference in prey density, the effect that the predator can find prey, time management, etc.

The Holling-type response functions are probably the most well-known types of response functions, which are used to simulate properties such as ability to defend the group prey against predators or self-safety capability (see [10],[11]-[20]). In this paper, we propose a mathematical model to study the interaction of water resources and population with each other by modeling the Leslie-Gower hunter-gatherer model. In order to increase the accuracy and efficiency, we introduce a time delay and, by studying the behavior of this model, we examine some mathematical aspects of this effect. Also, by adding a controller to the delay model, we control the model (for examples of control, see [21],[22],-[25])).

In the next chapter, we describe the conditions and assumptions of the model. The principles of the model and the rules governing its components are introduced. We also introduce the model without delay. In the third part of this paper, we present the time delay model and investigate the stability and existence of bifurcations in the controlled system. Finally, in the fourth section, numerical examples are given to confirm the efficiency of the controller.

It is useful to mention that in the prey-predator model, the word stability for the prey species and the word survival for the predator species are commonly used; that stability in a model means maintaining the population of prey and survival in a model means maintaining a predatory population in a favorable ratio. This happens in some models of prey and predators in which species of prey and predators are non-living (e.g., models related to chemical reactions, etc.). Sometimes the word stability (or survival) is used for a situation where the population of prey (or predator) does not reach zero, that is, complete extinction does not occur. However, in many sources, the term is commonly used when prey and predator populations reach an equilibrium in interaction with each other. In other words, in such an ecosystem, water resources and the population of the community are preserved and eventually they will reach a desired balance.

2 Introducing the mathematical model

In this section, we introduce a mathematical model to describe the pattern of water consumption in a hypothetical society and the interaction of population and water resources with each other. The basic assumptions governing this model are as follows:

▶ The community has a renewable water source that is constantly replacing predictable amounts of water with previously consumed amounts. And of course the amount of resource storage is a finite amount.

▶ If more water resources are available then the population will increase. Also, with the increase of population, the consumption of water resources increases.

▶ Community dependence on water resources is a vital dependency; This means that lack of access to this resource stops the growth of society. And Eventually, lack of access will destroy society. Also, the maximum population that can be expected for the community is directly related to the amount of water available.

With little reflection, we can see that at a glance that the above assumptions can be applied to any other vital resource consumed by a community. However, as mentioned earlier, we explore water consumption for a hypothetical community in this article. Suppose that $x(t)$ indicates the amount of water available to the community and $y(t)$ indicates the population of the community that uses this resource. Also assume that x_{in} and x_{out} represent the amount of water entering the source and water leaving the source in each unit of time, y_{in} and y_{out} respectively represent the amount of population entering the community and the population leaving the community in each unit of time, respectively. We can write:

$$\begin{cases} x(t+\triangle t) = x(t) + (x_{in} - x_{out}) \triangle t, \\ y(t+\triangle t) = y(t) + (y_{in} - y_{out}) \triangle t. \end{cases}$$

In the first equation, usually the amount of water entering the existing water reserves (due to annual rainfall, etc.) is obtained by a statistical study of the amount of water received in previous periods. As well, usually the amount of water is equal to the average amount of water received in previous periods, which is determined under the influence of various factors in time periods, which is reduced or increased. In other words:

$$x_{in} = B_0 + B(x,y), \qquad B_0 > 0.$$

Here, the constant B_0 represents the average amount of water inflow per unit of time that we expect and $B(x,y)$ shows changes in water inflow per unit time.Clearly, water storage is always finite, For example, suppose $K > 0$ represents the ultimate capacity for water storage. We expect as $x(t)$ approaches K The value of $B_0 + B(x,y)$ should gradually approach zero and if $x(t) > k$ then we expect that $B_0 + B(x,y) < 0$. In fact, the last inequality shows that if the amount of water in the tank is more than its capacity, Excess water spills out of the source. Statistical studies show that in the simplest possible case, $B_0 + B(x,y)$ can be approximated as a function as follows:

$$B_0 + B(x,y) = (b_0 + b(x,y))(1 - \frac{x}{K}),$$

where $b_0 + b(x,y) \geq 0$ represents the rate of water entering the tank in per unit time. Considering the average amount of water consumed by each member of the community, in addition, the amount of water that is wasted due to various factors (such as distribution network wear or other related factors). The amount of water consumed per unit of time can be calculated. If each member of the community has an average of $p(x, y) \geq 0$ water resources per unit of time, then the total amount of water consumed by people in the community in each unit of time will be equal to $yp(x, y)$. We yield:

$$x_{out} = yp(x, y) + q(x),$$

where $q(x) \geq 0$ represents the amount of water lost per unit time. The $p(x, y)$ and $q(x)$ functions are called the response function and the waste function, respectively. By replacing the above functions in the equation of $x(t)$-changes, when $\triangle t \longrightarrow 0$, we get the equation of $x(t)$-changes as follows:

$$\frac{dx}{dt} = (b_0 + b(x, y))(1 - \frac{x}{K}) - (yp(x, y) + q(x)).$$

These are border functions which apply to the following relationships:

$$\frac{\partial p}{\partial x} \geq 0, \qquad \frac{\partial p}{\partial y} \leq 0, \qquad \frac{\partial q}{\partial x} \geq 0.$$

The above conditions indicate the fact that with the increase of water resources, the possibility of increasing the amount of water consumption of each member of society per unit of time also increases, while with the increase of population, the share of water consumption per person per unit of time decreases. Finally, due to the increase of water resources, water loss will also increase. Note that in the above relations p and q can be fixed non-negative functions. In other words, each person in the community uses a constant amount of water resources per unit of time and also the rate of water loss per unit of time is a fixed amount. If the functions p and q are considered non-fixed functions in terms of x and y, they must satisfy the conditions $p(0, y) = 0$ and $q(0) = 0$, which reflect the fact that in the absence of water resources, each person's share of these resources is zero, and of course in this case there will be no waste. We now study the equation of population change. Like in many recent studies, our population growth model is based on a quasi-logical growth model. The word quasi-logic means that we assume that the ultimate capacity of society is vitally dependent on water,

$$\frac{dy}{dt} = ay(1 - \frac{y}{c(x)}).$$

where $c(x) \geq 0$ is a continuous function and bounded in relation to time, which shows the dependence of the final capacity of society on water resources, valid for $c(0) = 0$ and $\frac{\partial c}{\partial x} \geq 0$. These conditions indicate that population dependence on water resources is vital and with the increase of water resources, the final capacity of the society also increases. The $c(x)$ function is called the dependency function. Considering the above, our proposed model for studying the pattern of interaction between water resources and population on each other is summarized by the following equations:

$$\begin{cases} \frac{dx}{dt} = (b_0 + b(x, y))(1 - \frac{x}{K}) - (yp(x, y) + q(x)), \\ \frac{dy}{dt} = ay(1 - \frac{y}{c(x)}). \end{cases} \tag{2.1}$$

Due to the nature of the variables x and y in the above equation, negative values are meaningless. In fact, solutions of (2.1) as $\gamma(t) = (x(t), y(t))$, have a biological interpretation. The start moment is $t = 0$, from a point $\gamma(0)$ from the first region. The first area for this system is a invariant subset. The following Lemma ensures that the (2.1) model is compatible.

Lemma 2.1. *The first region is invariant, meaning that any answer in* (2.1) *that starts from the first region remains in this region. In addition to system solutions have boundaries in this area. The first area is called the acceptable area for this system.*

Proof. The first part is obvious. Now we show that each solution is bounded. Suppose $\gamma(t) = (x(t), y(t))$ is the solution of the system/ Because $x(t)$ and $c(x)$ are both boundaries, then it can be assumed that $c(x) \leq M$. Now from the second equation of (2.1), we conclude that $y(t)$ cannot be infinite. □

According to the statements from above, we will further consider only solutions of the system which are in the acceptable area.

3 Introduction of delay system and bifurcation, stability analysis in the controlled system

In this section, by adding time delays to the equations, the system becomes more accurate. Suppose that τ_1 is the time delay of the effect of water resources on the human population, and τ_2 is the time delay of the effect of human population on water resources. We rewrite the (2.1) system as follows:

$$(3.1) \qquad \begin{cases} \dfrac{dx}{dt} = (b_0 + b(x, y(t-\tau_2)))(1 - \dfrac{x}{K}) - (y(t-\tau_2)p(x, y(t-\tau_2)) + q(x)), \\ \dfrac{dy}{dt} = ay(1 - \dfrac{y}{c(x(t-\tau_1))}). \end{cases}$$

In the real world, it is sometimes necessary to control a population at a reasonable level, otherwise the population may lead to the increase, decrease or even extinction of some populations. In terms of biological system control, the current research focus is to achieve feedback control by changing the structure of the biological community and increasing the feeding pressure of prey. Now we can add a time-delayed force to the system like $Ae^{-d\sigma}(y(t) - y(t-\sigma))$[26], so we shall have:

$$(3.2) \qquad \begin{cases} \dfrac{dx}{dt} = (b_0 + b(x, y(t-\tau_2)))(1 - \dfrac{x}{K}) - (y(t-\tau_2)p(x, y(t-\tau_2)) + q(x)), \\ \dfrac{dy}{dt} = ay(1 - \dfrac{y}{c(x(t-\tau_1))}) + Ae^{-d\sigma}(y(t) - y(t-\sigma)). \end{cases}$$

For convenience, using techniques from [27, 28], we introduce the new variables $u(t) = x(t-\tau_1)$, $v(t) = y(t)$, $\tau = \tau_1 + \tau_2$, such that system (3.1) can be written as the following equivalent system with a single delay:

$$(3.3)\qquad \begin{cases} \dfrac{du}{dt} = (b_0 + b(u, v(t-\tau)))(1 - \dfrac{u}{K}) - \\ (v(t-\tau)p(u, v(t-\tau)) + q(u)), \\ \dfrac{dv}{dt} = av\left(1 - \dfrac{v}{c(u)}\right) + Ae^{-d\sigma}(v(t) - v(t-\sigma)). \end{cases}$$

Now we calculate the equilibrium points of the system(3.3). We put $\dfrac{du}{dt} = 0$ and $\dfrac{dv}{dt} = 0$. Therefore the possible equilibrium points $E^* = (u^*, v^*)$ satisfy:

$$v^* = c(u^*) \text{ and } (b_0 + b(u^*, c(u^*)))(1 - \tfrac{u^*}{K}) = c(u^*)p(u^*, c(u^*)) + q(u^*).$$

Let $F(u) = (b_0 + b(u, c(u)))(1 - \frac{u}{K})$ and $E(u) = c(u)p(u, c(u)) + q(u)$, then $F(0) = b_0 + b(0,0) > 0$ and $F(+\infty) = -\infty$ and also $G(0) = 0$, $G' \geq 0$. Therefore, according to the conditions mentioned, $F(u)$ and $G(u)$ have a common point, such that $0 < u^* < K$. As a result, the system (3.3) has a equilibrium point $E^* = (u^*, v^*)$. Now we linearize the system (3.3) around the point $E^* = (u^*, v^*)$. We have:

$$(3.4)\qquad \begin{cases} \dfrac{du}{dt} = A_1 u(t) - A_2 V(t-\tau) \\ \dfrac{dv}{dt} = A_3 u(t) + A(\sigma)v(t) - Ae^{-d\sigma}v(t-\sigma), \end{cases}$$

where

$$A_1 = \frac{\partial b}{\partial u}(u^*, v^*)(1 - \frac{u^*}{k}) - \left(\frac{b_0 + b(u^*, v^*)}{k}\right) - v^* \frac{\partial p}{\partial u}(u^*, v^*) - \frac{\partial q}{\partial u}(u^*).$$

$$A_2 = -\frac{\partial b}{\partial v(t-\tau)}(u^*, v^*)(1 - \frac{u^*}{k}) + p(u^*, v^*) + v^* \frac{\partial p}{\partial v(t-\tau)}(u^*, v^*)$$

$$A_3 = av^* \left(\frac{v^* \dfrac{\partial c}{\partial u}(u^*)}{(c(u))^2}\right)$$

$$A_4(\sigma) = a\left(1 - \frac{v^*}{c(u^*)}\right) - \frac{av^*}{c(u^*)} + Ae^{d\sigma}.$$

As a result, the characteristic equation of system (3.3) is around the equilibrium point as follows:

$$(3.5)\qquad (\lambda - A_1)(\lambda - A_4(\sigma)) + Ae^{-d\sigma}(\lambda - A_1)e^{-\lambda\sigma} + A_2 A_3 e^{-\lambda\tau} = 0$$

We investigate the roots of the equation (3.5). Beretta and Kuang [29] established a geometric method to determine the existence of purely imaginary roots when the coefficients contain delays.

▶: The case $\sigma = \tau$

When $\sigma = \tau$, the equation (3.5) becomes:

$$(\lambda - A_1)(\lambda - A_4(\sigma)) + [Ae^{-d\tau}(\lambda - A_1) + A_2A_3]e^{-\lambda\tau} = 0 \tag{3.6}$$

Define $P(\lambda,\tau) = (\lambda - A_1)(\lambda - A_4(\sigma))$, $Q(\lambda,\tau) = Ae^{-d\tau}(\lambda - A_1) + A_2A_3$, and suppose that $\lambda = i\omega(\omega = \omega(\tau) > 0)$ is the root of the equation (3.6). Then for all $\tau \in I_\tau$, we have:

$$\omega^2{}_\pm = \frac{1}{2}\left[-Z_1(\tau) \pm \sqrt{\bigtriangledown}\right],$$

with

$$Z_1(\tau) = {A_1}^2 + {A_4}^2(\tau) - A^2e^{-2d\tau}, \;\; Z_2(\tau) = (A_1A_4(\tau))^2 - (A_1Ae^{-d\tau} - A_2A_3)^2$$

$$\bigtriangledown = {Z_1}^2(\tau) - 4Z_1(\tau), \;\; F(\omega,\tau) = a^4 + Z_1(\tau)\omega^2 + Z_2(\tau)$$

$$I_\tau = \left\{\tau > 0 : F(\omega,\tau) = 0 \quad for \quad \omega > 0\right\}.$$

For $\tau \in I_\tau$, let $\theta(\tau) \in [0, 2\pi)$ be defined by:

$$\begin{cases} sin(\theta(\tau)) = \dfrac{\omega[A_1 + A_4(\tau)](A_1Ae^{-d\tau} - A_2A_3) - \omega Ae^{-d\tau}[A_1A_4(\tau) - \omega^2]}{(\omega Ae^{-d\tau})^2 + (A_1Ae^{-d\tau} - A_2A_3)^2} \\ \\ \cos(\theta(\tau)) = \dfrac{[A_1A_4(\tau) - \omega^2](A_1Ae^{-d\tau} - A_2A_3) - \omega^2Ae^{-d\tau}[A_1 + A_4(\tau)]}{(\omega Ae^{-d\tau})^2 + (A_1Ae^{-d\tau} - A_2A_3)^2}. \end{cases}$$

For $\tau \in I_\tau, n \in N$, we can introduce $S_n(\tau) := \tau - \dfrac{\theta(\tau) + 2n\pi}{\omega(\tau)}$. Hence, the following result holds:

Theorem 3.1. *For the system* (3.4)*, we have:*
(i) If I_τ is empty or $Sup_{\tau\in I_\tau}(S_0) \leq 0$, then E^* is locally asymptotically stable for all $\tau \geq 0$.
(ii) If I_τ is non-empty, $Sup_{\tau\in I_\tau}(S_0) > 0$ and $Sign\left\{\dfrac{dRel\lambda}{d\tau}|_{\lambda=i\omega}\right\} \neq 0$ for some $n \in N$, let $\tau^0 = min\left\{\tau : S_n(\tau) = 0\right\}$ and $\tau^1 = max\left\{\tau : S_n(\tau) = 0\right\}$, then E^* is locally asymptotically stable for all $\tau \in [0, \tau^0) \cup (\tau^1, \infty)$ and unstable for $\tau \in (\tau^0, \tau^1)$. Here τ^0 and τ^1 are the Hopf bifurcation values.

▶: The case $\sigma \neq \tau$

Suppose that $\tau = 0$ and $\sigma > 0$. As a result, the following theorem can be established.

Theorem 3.2. *For the system* (3.4)*, we have:*
(i) If I_σ is empty or not empty and $S_n(\sigma) = 0$ has no positive root in I_σ, then E^* is locally asymptotically stable for all $\sigma > 0$.
(ii) If I_σ is non-empty, $S_n(\sigma) = 0$ has positive roots in I_σ and $Sign\left\{\dfrac{dRel\lambda}{d\sigma}|_{\lambda=i\omega}\right\} \neq 0$ for some $n \in N$, let $\sigma^0 = min\left\{\sigma : S_n(\sigma) = 0\right\}$ and $\sigma^1 = max\left\{\sigma : S_n(\tau) = 0\right\}$, then E^* is locally asymptotically stable for all $\sigma \in [0, \sigma^0) \cup (\sigma^1, \infty)$ and unstable for $\sigma \in (\sigma^0, \sigma^1)$. Here σ^0 and σ^1 are the Hopf bifurcation values.

Suppose now that $\sigma = \sigma^*$ is constant. Then (3.5) becomes:

$$(\lambda - A_1)(\lambda - A_4(\sigma)) + Ae^{-d\sigma^*}(\lambda - A_1)e^{-\lambda\sigma^*} + A_2A_3e^{-\lambda\tau} = 0. \tag{3.7}$$

Suppose that $i\omega$ is the root of Equation (3.7). Therefore:

$$g(\omega) := \left\{-\omega^2 + A_1A_4(\sigma) + Ae^{-d\sigma^*}[\omega sin\omega\sigma^* - A_1 \cos\omega\sigma^*]\right\}^2 + \left\{-\omega[A_1 + A_4(\sigma)] + Ae^{-d\sigma^*}[\omega cos\omega\sigma^* + A_1 sin\omega\sigma^*]\right\}^2 - [A_2A_3]^2 = 0. \tag{3.8}$$

According to the type of roots of the above equation, the stability and occurrence of the bifurcation can be investigated. We get the following result:

Theorem 3.3. *Let $\sigma = \sigma^*$ be constant; then:*
(i) If equation (3.8) has no positive roots, then E^* is locally asymptotically stable for any $\tau \geq 0$.
(ii) If equation (3.8) has positive roots and $Sign\left\{\frac{dRel\lambda}{d\tau}|_{\lambda=i\omega}\right\} \neq 0$, then E^* is locally asymptotically stable for $\tau \in [0, \tau_0]$ and τ_0 is the first Hopf bifurcation value.

As mentioned above, we made the system parameters dependent on the control parameters, and by controlling the parameters of the controller, the system can be directed to the desired goals. In the next section, we provide a numerical example to illustrate the effect of the controller, which indicates the effectiveness of the controller.

4 Numerical examples and conclusions

We shall present numerical examples and study the system behavior in the pre-control and post-control systems. This means that, by controlling the system, its performance can be controlled and by adjusting the control parameters, which sustain that human survival can be guaranteed. According to the data of from the region of Asia in a period of one year and in units of liters, suppose that in the system (3.3),

$$\begin{cases} \frac{du}{dt} = 412 \times 10^9(1 - \frac{u}{824 \times 10^9}) - (81760\, v(t-\tau) + 107.12 \times 10^9), \\ \frac{dv}{dt} = v(1 - \frac{v}{u}) + Ae^{-d\sigma}(v(t) - v(t-\sigma)). \end{cases} \tag{4.1}$$

In the following, by the help of the shapes of this system and of the results from the previous section, we analyze the system (4.1). First we consider the system (4.1) without a controller, i.e., $A = 0$.

Figure 1 illustrates with this trend in the not too distant future. People in this area will face a serious risk of water shortage and may have to leave these areas to save their lives; also, other living things in these areas are in danger of extinction.

Figure 2 shows that if the population is controlled and the consumption pattern of the community is improved, we can hope for a better future. Moreover, with more accurate choices of control parameters, human survival in these areas can be guaranteed and society can be led to a stable state.

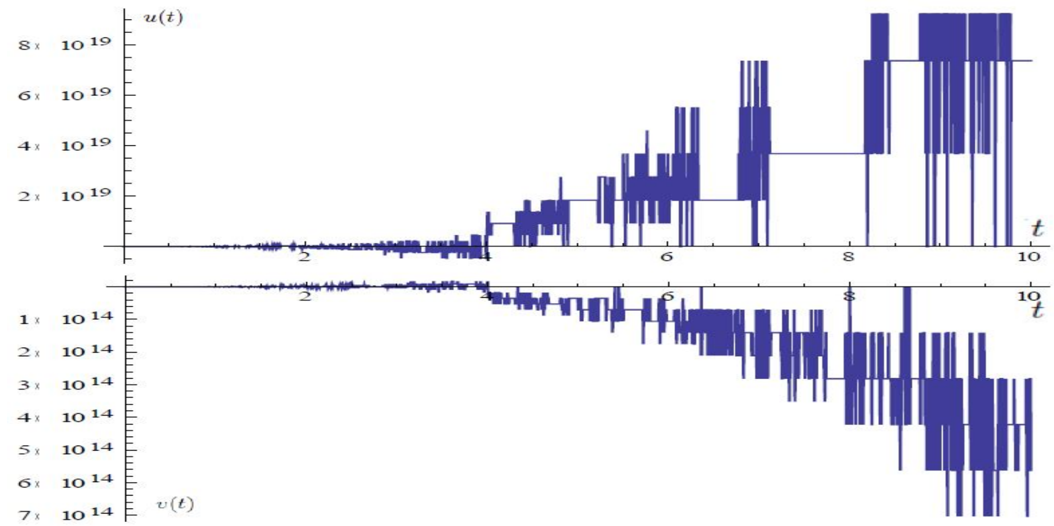

Figure 1: the numerical solution of uncontrol with corresponding to $\tau = 0.00000051$.

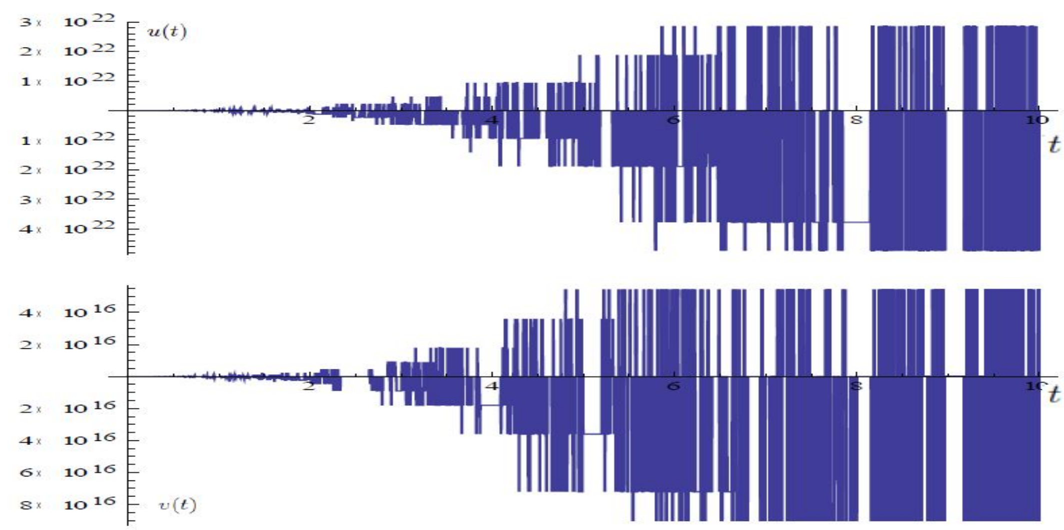

Figure 2: the numerical solution of uncontrol corresponding to $\tau = 0.00000051, A = 1, d = 1 and \sigma = 0.00000043$.

Conclusions

Rainfall and excessive consumption of water resources in some areas have endangered human survival. To get rid of this danger, the past trend must be corrected. In this article, we introduce a controlled model to preserve life in these areas. If we adjust the control parameters according to the regional conditions, we can preserve the life of the community in these areas, and population control can be effective. For future research, it is suggested to use alternative controllers to increase the efficiency of this model.

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