On the temperature equation in classical irreversible thermodinamics

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Abstract. In this paper, by using a procedure of classical irreversible thermodynamics with internal variables, some possible interactions among heat conduction and viscous-elastic flows for rheological media are studied. By introducing as an vectorial internal variable $\boldsymbol{\xi}$, which influences thermal and diffusion phenomena, phenomenological equation for these variables are derived. A general vector, $\boldsymbol{J}$, is introduced which assumes the role of heat flux and it is shown that, in isotropic media, $\boldsymbol{J}$ can be composed of two parts and this allows to obtain a heat equation that generalizes both the Fourier equation and the Maxwell-Cattaneo-Vernotte (M-C-V) equation. A general temperature equation and the energy balance equation for viscoelastic media are obtained.

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1 Introduction

In literature, for classical irreversible thermodynamic with internal variables (CIT-IV) one means a general approach to the study of different interactions among irreversible thermodynamic processes using classical exstensive variables.
After the pioneering works of Lars Onsager ([40, 41]), many excellent scientists: C.Eckart ([18, 19, 20, 21]), L.Prigogine ([42]), S.R. De Groot-P.Mazur ([17]), J.Meixner-H.G.Reik ([38]), B.D.Colemann-M.E.Gurtin ([16]), C.Truesdell ([45]), Kluitenberg G.A. ([26, 27, 28, 29, 30]), G.Maugin-W.Muschik ([35, 36]), developed the basic of the theory (CIT-IV) that is characterized by the introduction of thermodynamic variables, called *internal* (or *hidden*) variables.

The flexibility of the methodologies used in (CIT-IV) consists in the fact that *"a priori"* the physical meaning of the internal thermodynamical variables is not specified but only their influence on particular types of occurring phenomena in the considered medium is assumed. For this reason, in the following, we will call these thermodynamical variables: *dynamical variables* since they have also been used, successfully, for the study of problems in dielectric and magnetic relaxation phenomena ([33],[4],[13],[10]),

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diffusion ([11]) and anelastic deformation ([34],[31],[32],[8], [9],[11],[12],[14],[15]). It should be noted that the many theoretical results have been confirmed by the experimental data ([2],[3] [5], [6], [7]).

Parallel approaches were worked out in the frame of *Extended Irreversible Thermodynamics* (EIT) ([25]) and in *Rational Extended Thermodynamics* (RET) ([39]), [44]) which, with consideration suggested by kinetic theory, considers an entropy depending on the fluxes besides on the classical variables. It was shown ([14],[15],[12]) that by using the usual procedure of (CIV-IV) it is possible to describe the relaxation of thermal phenomena thus obtaining results which, generally, are justified by additional hypothesis suggested by kinetic theory.

In this paper we show that thermoconduction phenomena in visco-elastic media may be studied by using the systematic procedure of (CIT-IV).

In Sect.2, introducing a vectorial dynamical variable and a stress field $\tau^{(eq)}_{\alpha\beta}$ which is of a thermoelastic nature, an explicit form for the entropy production is derived.

In Sects.3,4,5 and 6 the phenomenological equations, which generalize the Stockes-Navier's law, are obtained.

In Sect 7, by virtue of the dynamical variable, influencing the thermal transport phenomena, the heat equation is obtained. In particular, in the isotropic case, when the medium has symmetry properties that, under orthogonal transformations, are invariant with respect to all rotations and inversions of the frame of axes, it was obtained that the heat flux can be split in two parts: a first contribution $\boldsymbol{J}^{(0)}$, governed by Fourier law, and a second contribution $\boldsymbol{J}^{(1)}$, obeying Mawell-Cattaneo-Vernotte equation (MCV) ([1], [23], [37], [47]) in which a relaxation time is present.

In Sect.8 we obtain a general temperature equation which generalizes the analogous equations of Fourier and MCV.

Finally, in Sect.9 the balance equation of energy for thermo-viscous-elastic media which generalizes the Umov's law ([46]).

2 The balance equation of entropy

In the contest of irreversible processes an important role is played by the flow of heat which, classically, is not considered to be a state variable. Therefore we will suppose that the specific entropy s, depends not only on the the specific internal energy u and the strain $\varepsilon_{\alpha\beta}$ but also on a vectorial dynamic variable, $\boldsymbol{\xi}$, that is odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium.

$$s = s(u, \varepsilon_{\alpha\beta}, \xi_\alpha)\,, \tag{2.1}$$

where ξ_α $(\alpha = 1, 2, 3)$ is the α-component of the vector $\boldsymbol{\xi}$.

Theorem 2.1 (Gibbs relation). *By using* (2.1) *we can obtain the following Gibbs relation*

$$T ds = du - \nu\, \tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} - \nu\, j_\alpha\, d\xi_\alpha\,, \tag{2.2}$$

where ν is the specific volume, $\tau^{(eq)}_{\alpha\beta}$ is the equilibrium stress tensor, T is the absolute temperature and j_α is the α-component of the vector $\boldsymbol{j}$ conjugate to the internal variable $\boldsymbol{\xi}$.

Proof. We define the *absolute temperature*

$$T^{-1} \stackrel{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{\alpha\beta}, \xi_\alpha)\,, \tag{2.3}$$

the *the equilibrium-stress tensor*

$$\tau^{(eq)}_{\alpha\beta} \stackrel{\text{def}}{=} -\varrho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \xi_\alpha)\,, \tag{2.4}$$

and the vector $\boldsymbol{j}$ conjugate to the internal vector variable $\boldsymbol{\xi}$

$$j_\alpha \stackrel{\text{def}}{=} -\varrho T \frac{\partial}{\partial \xi_\alpha} s(u, \varepsilon_{\alpha\beta}, \xi_\alpha)\,, \tag{2.5}$$

where $\varrho \stackrel{\text{def}}{=} \nu^{-1}$ is the mass density.
By using equations (2.3)-(2.5) from (2.2) we obtain the differential ds of s:

$$T ds = du - \nu\, \tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} - \nu\, j_\alpha\, d\xi_\alpha\,, \tag{2.6}$$

The relation (2.6) is usually called *Gibbs relation*, in which the usual summation convection for dummy is used. □

From (2.6) we have:

$$\varrho T \frac{ds}{dt} = \varrho \frac{du}{dt} - \tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} - j_\alpha \frac{d\xi_\alpha}{dt}\,. \tag{2.7}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \bigtriangledown \tag{2.8}$$

is the substantial derivative respect to time ($\boldsymbol{v}$ is the velocity field).
To analyze phenomena due to viscous flows (analogous to those which occurs during flows in ordinary viscous liquids and gases) we introduce the following viscous stress tensor

$$\tau^{(vi)}_{\alpha\beta} \stackrel{\text{def}}{=} \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}\,, \tag{2.9}$$

where $\tau_{\alpha\beta}$ is the mechanical stress tensor which occurs in the equation of motion

$$\varrho \frac{dv_\alpha}{dt} = \varrho\, F_\alpha + \frac{\partial \tau_{\alpha\beta}}{\partial x_\beta},, \tag{2.10}$$

and in the first law of thermodynamics

$$\varrho \frac{du}{dt} = -\nabla \cdot \boldsymbol{J}^{(q)} + \tau_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt},, \tag{2.11}$$

In (2.10) the force F_α is the volume force per unit of mass and in (2.11) the vector $\boldsymbol{J}^{(q)}$ is the heat flux.

Theorem 2.2 (**Entropy production**). *.By using the first law of thermodynamic* (2.11) *can be obtained the balance equation of the entropy*

$$\varrho\frac{ds}{dt} = -div\Big(\frac{\boldsymbol{J}^{(q)}}{T}\Big) + \sigma^{(s)} \tag{2.12}$$

and the entropy production

$$\sigma^{(s)} = T^{-1}\Big[\boldsymbol{J}^{(q)}\cdot\Big(-T^{-1}grad\,T\Big) + \tau^{(vi)}_{\alpha\beta}\,\frac{d\,\varepsilon_{\alpha\beta}}{dt} - j_\alpha\,\frac{d\xi_\alpha}{dt}\Big] \geq 0, \tag{2.13}$$

Proof. The equation (2.11),by virtue of (2.9) becomes

$$\varrho\frac{du}{dt} = -\nabla\cdot\boldsymbol{J}^{(q)} + \Big(\tau^{(eq)}_{\alpha\beta} + \tau^{(vi)}_{\alpha\beta}\Big)\frac{d\varepsilon_{\alpha\beta}}{dt},, \tag{2.14}$$

which substituted in the equation (2.7) gives

$$\varrho\,\frac{ds}{dt} = T^{-1}\Big(-\nabla\cdot\boldsymbol{J}^{(q)} + \tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} - j_\alpha\frac{d\xi_\alpha}{dt}\Big). \tag{2.15}$$

Using the following identity

$$T^{-1}\,\nabla\dot{\boldsymbol{J}}^{(q)} = \nabla\cdot\Big(\frac{\boldsymbol{J}^{(q)}}{T}\Big) - T^{-2}\,\boldsymbol{J}^{(q)}\cdot\nabla T\,, \tag{2.16}$$

the equation (2.15) takes the form

$$\varrho\frac{ds}{dt} = -\mathrm{div}\Big(\frac{\boldsymbol{J}^{(q)}}{T}\Big) + \sigma^{(s)} \tag{2.17}$$

where

$$\sigma^{(s)} = T^{-1}\Big[\boldsymbol{J}^{(q)}\cdot\Big(-T^{-1}\nabla\,T\Big) + \tau^{(vi)}_{\alpha\beta}\,\frac{d\,\varepsilon_{\alpha\beta}}{dt} - j_\alpha\,\frac{d\xi_\alpha}{dt}\Big] \geq 0, \tag{2.18}$$

The equation (2.17) is *the balance equation of entropy* and (2.18) is *the entropy production.* □

It is seen that the entropy production is due to three types of phenomena: the first term on the right-hand of (2.18) gives the contribution of the heat conduction phenomena, the second sum is the contribution of viscous phenomena, the last sum is the contribution of the variation of the dynamical variable. Each term is a production of flux ($\boldsymbol{J}^{(q)}$, $\tau^{(vi)}_{\alpha\beta}$, j_α) of the process and an affinity conjugate to it : $T^{-1}\mathrm{grad}\,T$, $d\,\varepsilon_{\alpha\beta}/dt$, $d\xi_\alpha/dt$, respectively.

3 Phenomenological equations

According to the usual procedure of non-equilibrium thermodynamics, by virtue of the form (2.18) for the entropy production, we have for anisotropic media the following phenomenological equations:

$$J^{(q)}_\alpha = L^{(q)(q)}_{\alpha\beta}\Big(-T^{-1}\frac{\partial T}{\partial x^\beta}\Big) + L^{(q)(0)}_{\alpha(\mu\nu)}\,\frac{d\varepsilon_{\mu\nu}}{dt} - L^{(q)(\xi)}_{\alpha\beta}\,\frac{d\xi_\beta}{dt}\,, \tag{3.1}$$

$$\tau^{(vi)}_{\alpha\beta} = L^{(0)(q)}_{(\alpha\beta)\nu}\left(-T^{-1}\frac{\partial T}{\partial x^{\nu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} - L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt}\,, \tag{3.2}$$

$$j_{\alpha} = L^{(\xi)(q)}_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^{\beta}}\right) + L^{(\xi)(0)}_{\alpha(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} - L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_{\beta}}{dt}\,. \tag{3.3}$$

The tensors L are called phenomenological tensors and the indices of these tensors enclosed in round brackets mean that they are symmetrical because the tensors $\varepsilon_{\mu\nu}$ and $\tau^{(vi)}_{\alpha\beta}$ are symmetric. The first of these equations may be regarded as a generalization of Fourier's law. The equation (3.2) describes the viscous flow phenomenon and it may be considered to be a generalization of Stockes-Navier's law. Finally, the equation (3.3) is the phenomenological equation for the irreversible process of the dynamic degrees of freedom.
The phenomenological tensors :

$$L^{(q)(0)}_{\alpha(\mu\nu)},\ L^{(q)(\xi)}_{\alpha\beta},\ L^{(0)(q)}_{(\alpha\beta)\nu},\ L^{(0)(\xi)}_{(\alpha\beta)\mu},\ L^{(\xi)(q)}_{\alpha\beta},\ L^{(\xi)(0)}_{\alpha(\mu\nu)},$$

represent possible cross effects among the irreversible phenomena mentioned above. Substituting the (3.1)-(3.3) into (2.18) one has

$$\begin{aligned} T\,\sigma^{(s)} =& T^{-2}\,L^{(q)(q)}_{\alpha\beta}\frac{\partial T}{\partial x^{\alpha}}\frac{\partial T}{\partial x^{\beta}} + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\varepsilon_{\mu\nu}}{dt} - L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_{\alpha}}{dt}\frac{d\xi_{\beta}}{dt} + \\ &+\left[L^{(q)(0)}_{\alpha(\mu\nu)} + L^{(0)(q)}_{(\mu\nu)\alpha}\right]\frac{d\varepsilon_{\mu\nu}}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\alpha}}\right) + \\ &+\left[L^{(q)(\xi)}_{\alpha\beta} - L^{(\xi)(q)}_{\beta\alpha}\right]\left(T^{-1}\frac{\partial T}{\partial x^{\alpha}}\right)\frac{d\xi_{\beta}}{dt} + \\ &+\left[L^{(0)(\xi)}_{(\alpha\beta)\mu} + L^{(\xi)(0)}_{\mu(\alpha\beta)}\right]\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\xi_{\mu}}{dt}\,. \end{aligned} \tag{3.4}$$

This form for the entropy production ($\sigma^{(s)} \geq 0$) is a useful detail if isotropic media are considered (see Sec.5)

4 Symmetric relations

Since the time derivative $d\varepsilon_{\mu\nu}/dt$, the heat flow $J^{(q)}_{\alpha}$ and j_{α} are odd functions of the microscopic particle velocities and the time derivative of $\boldsymbol{\xi}$, $\tau^{(eq)}_{\alpha\beta}$ and the temperature gradient are even functions of these velocities, the Onsager-Casimir reciprocity relations read ([17])

$$L^{(q)(q)}_{\alpha\beta} = L^{(q)(q)}_{\beta\alpha} \quad,\quad L^{(0)(0)}_{(\alpha\beta)(\mu\nu)} = L^{(0)(0)}_{(\mu\nu)(\alpha\beta)} \quad,\quad L^{(\xi)(\xi)}_{\alpha\beta} = L^{(\xi)(\xi)}_{\beta\alpha}\,, \tag{4.1}$$

$$L^{(q)(0)}_{\alpha(\mu\nu)} = L^{(q)(0)}_{(\mu\nu)\alpha} \quad,\quad L^{(0)(q)}_{(\mu\nu)\alpha} = L^{(0)(q)}_{\alpha(\mu\nu)}\,, \tag{4.2}$$

$$L^{(q)(0)}_{\alpha(\mu\nu)} = -L^{(0)(q)}_{(\mu\nu)\alpha}\,, \tag{4.3}$$

$$L^{(q)(\xi)}_{\alpha\beta} = L^{(q)(\xi)}_{\beta\alpha} \quad,\quad L^{(\xi)(q)}_{\alpha\beta} = L^{(\xi)(q)}_{\beta\alpha}\,, \tag{4.4}$$

$$L^{(0)(\xi)}_{(\alpha\beta)\mu} = L^{(0)(\xi)}_{\mu(\alpha\beta)} \quad , \quad L^{(\xi)(0)}_{\mu(\alpha\beta)} = L^{(\xi)(0)}_{(\alpha\beta)\mu} \,. \tag{4.5}$$

The equations (4.1)-(4.5) reduce the number of indipendent components of the phenomenological tensors.

5 Phenomenological equations for isotropic media

In this section we consider perfect isotropic media for which the symmetry properties are invariant under orthogonal transformations with respect to all rotations and to inversion of the frame of axes.
In this case, it can be shown ([17],[24]) that the phenomenological tensors have the following form

$$L^{(q)(q)}_{\alpha\beta} = T\lambda^{(q,q)}\delta_{\alpha\beta} \quad , \quad L^{(q)(\xi)}_{\alpha\beta} = -\lambda^{(q,\xi)}\,\delta_{\alpha\beta}\,, \tag{5.1}$$

$$L^{(0)(0)}_{(\alpha\beta)(\mu\nu)} = \frac{\eta_s^{(0,0)}}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) + \frac{1}{3}(\eta_v^{(0,0)} - \eta_s^{(0,0)})\delta_{\alpha\beta}\delta_{\mu\nu}\,, \tag{5.2}$$

$$L^{(\xi)(q)}_{\alpha\beta} = T\lambda^{(\xi,q)}\delta_{\alpha\beta} \quad , \quad L^{(\xi,\xi)}_{\alpha\beta} = -\lambda^{(\xi,\xi)}\delta_{\alpha\beta}\,, \tag{5.3}$$

while the tensors of the third-order vanish.
By using (5.1)-(5.3) the equations (3.1)-(3.3) become

$$J^{(q)}_\alpha = -\lambda^{(q,q)}\frac{\partial T}{\partial x^\alpha} + \lambda^{(q,\xi)}\,\frac{d\xi_\alpha}{dt}\,, \tag{5.4}$$

$$\tau^{(vi)}_{\alpha\beta} = \eta_s^{(0,0)}\frac{d\varepsilon_{\alpha\beta}}{dt} + (\eta_v^{(0,0)} - \eta_s^{(0,0)})\,\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta}\,, \tag{5.5}$$

$$j_\alpha = -\lambda^{(\xi,q)}\,\frac{\partial T}{\partial x^\alpha} + \lambda^{(\xi,\xi)}\,\frac{d\xi_\alpha}{dt}\,. \tag{5.6}$$

In (5.5) we indicate $\frac{d\varepsilon}{dt} = \frac{1}{3}\frac{d}{dt}(\varepsilon_{\mu\nu}\,\delta_{\mu\nu})$. The scalar quantities λ, $\eta_s^{(0,0)}$ and $\eta_v^{(0,0)}$ are called phenomenological coefficients. In particular $\eta_s^{(0,0)}$ and $\eta_v^{(0,0)}$ may be called the shear viscosity and the volume viscosity, respectively. These coefficients also occur in the theory of ordinary (Stokes-Navier) viscous fluids.
By substituting (5.4)-(5.6) in (3.4) we have

$$\begin{aligned} T\,\sigma^{(s)} = {} & T^{-1}\,\lambda^{(q,q)}\Big(\frac{\partial T}{\partial x^\alpha}\Big)^2 + \eta_s^{(0,0)}\Big(\frac{d\varepsilon_{\alpha\beta}}{dt}\Big)^2 + \\ & 3\Big(\eta_v^{(0,0)} - \eta_s^{(0,0)}\Big)\Big(\frac{d\varepsilon}{dt}\Big)^2 + \lambda^{(\xi,\xi)}\Big(\frac{d\xi_\alpha}{dt}\Big)^2 + \\ & + \Big(T^{-1}\,\lambda^{(q,\xi)} + \lambda^{(\xi,q)}\Big)\,\frac{\partial T}{\partial x^\alpha}\,. \end{aligned} \tag{5.7}$$

By virtue of the positive definite character of the entropy production, several inequalities for the phenomenological coefficients may be derived.
For example we have

(5.8) $$\lambda^{(q,q)} \geq 0 \quad , \quad \eta_s^{(0,0)} \geq 0 \quad , \quad \lambda^{(\xi,\xi)} \geq 0 \,,$$

(5.9) $$\eta_v^{(0,0)} - \eta_s \geq 0 \quad , \quad T^{-1}\,\lambda^{(q,\xi)} + \lambda^{(\xi,q)} \geq 0 \,.$$

The coefficients $\lambda^{(q,\xi)}$ and $\lambda^{(\xi,q)}$ represent possibile cross effects between the heat flux and the irreversible process of the dynamic degrees of freedom.

6 Linear equations of state

Let f the specific free energy of the medium

(6.1) $$f = u - T\,s$$

With the aid of Gibbs relation (2.6) we have:

(6.2) $$df = \nu\,\tau_{\alpha\beta}^{(eq)} d\varepsilon_{\alpha\beta} + \nu\, j_\alpha\; d\xi_\alpha - sdT \,,$$

where

(6.3) $$\nu\,\tau_{\alpha\beta}^{(eq)} = \frac{\partial f}{\partial \varepsilon_{\alpha\beta}}, \qquad \nu\, j_\alpha = \frac{\partial f}{\partial \xi_\alpha}, \qquad s = -\frac{\partial f}{\partial T}.$$

Let us choose a configuration $\Sigma^{(0)}$ with uniform temperature T_0 and in which s_0 is the specific entropy, ν_0 is the specific volume and $(\tau_{\alpha\beta}^{(eq)})_0\,(j_\alpha)_0\,,(\varepsilon_{\alpha\beta})_0$ and $(j_\alpha)_0$ are zero.

Let us suppose that the deviation from $\Sigma^{(0)}$ are sufficiently small (with $\varrho \approx \varrho_0$) and the free energy for isotropic medium can be written in the following form

(6.4) $$\begin{aligned} f \,=\, \nu_0 \Big\{ \frac{1}{2}\Big[a_{\alpha\beta\mu\nu}^{(0)(0)}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu} \,+\, a_{\alpha\beta}^{(\xi)(\xi)}\,\xi_\alpha\,\xi_\beta \Big] \,+\, a_{\alpha\beta\mu}^{(0)(\xi)}\varepsilon_{\alpha\beta}\,\xi_\mu \,+ \\ (T-T_0)\Big[a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_\alpha^{(\xi)(T)}\,\xi_\alpha \Big] \Big\} \,-\, \Psi \end{aligned}$$

where Ψ is some function of the temperature.

In (6.4) the tensors $a_{\alpha\beta\mu\nu}^{(0)(0)}$, $a_{\alpha\beta}^{(\xi)(\xi)}$, $a_{\alpha\beta\mu}^{(0)(\xi)}$, the vector $a_\alpha^{(\xi)(T)}$, are constants (i.e. they do not depend on temperature and the strains) and are determined by the physical proprieties of the medium in the reference state.

From $(6.3)_3$ we have

(6.5) $$s \,=\, -\,\nu_0\Big(a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_\alpha^{(\xi)(T)}\,\xi_\alpha \Big) \,+\, \frac{d\,\Psi}{dT}\,.$$

and as $u = f + T\,s$ we obtain

(6.6) $$\begin{aligned} u \,=\nu_0 \Big\{ \frac{1}{2}\Big[a_{\alpha\beta\mu\nu}^{(0)(0)}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu} \,+\, a_{\alpha\beta}^{(\xi)(\xi)}\,\xi_\alpha\,\xi_\beta \Big] \,+\, a_{\alpha\beta\mu}^{(0)(\xi)}\varepsilon_{\alpha\beta}\,\xi_\mu \,+ \\ -\,T_0\Big[a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_\alpha^{(\xi)(T)}\,\xi_\alpha \Big] \Big\} \,+\, T\,\frac{d\Psi}{dT} \,-\, \Psi\,. \end{aligned}$$

The specific heat at constant deformation, $c_{(\varepsilon)}$, may be defined by

(6.7) $$c_{(\varepsilon)} \,=\, \frac{\partial\, u}{\partial T}\,,$$

and from (6.6) it obtains

$$c_{(\varepsilon)} \,=\, T\,\frac{d^2\,\Psi}{dT^2}\,, \tag{6.8}$$

By integrating the equation (6.8) one has

$$\Psi \,=\, c_{(\varepsilon)}\log\left(\frac{T}{T_0}\right) \,+\, s_0, T - c_{(\varepsilon)}(T-T_0) - u_0\,. \tag{6.9}$$

if $c_{(\epsilon)}$ is constant where s_0 and u_0 are integration constants that are the specific entropy and the specific energy in the reference state, respectively.
From $(6.3)_{1,2}$ we obtain

$$\nu_0\,\tau^{(eq)}_{\alpha\beta} = a^{(0)(0)}_{\alpha\beta\mu\nu}\,\varepsilon_{\mu\nu} \,+\, a^{(0)(\xi)}_{\alpha\beta\mu}\,\xi_\mu \,+\, a^{(0)(T)}_{\alpha\beta}\,(T-T_0)\,, \tag{6.10}$$

$$\nu_0\,j_\alpha = a^{(\xi)(\xi)}_{\alpha\beta}\,\xi_\beta \,+\, a^{(0)(\xi)}_{\mu\nu\alpha}\,\varepsilon_{\mu\nu} \,+\, a^{(\xi)(T)}_{\alpha}\,(T-T_0)\,. \tag{6.11}$$

Now, we consider that the medium is isotropic in the reference state the even order tensors have the following forms

$$a^{(0)(0)}_{\alpha\beta\mu\nu} = \frac{\nu_0}{2}\,a^{(0,0)}(\delta_{\alpha\mu}\,\delta_{\beta\nu} + \delta_{\alpha\beta}\,\delta_{\beta\mu}) + \frac{\nu_0}{3}(b^{(0,0)} - a^{(0,0)})\delta_{\alpha\beta}\,\delta_{\mu\nu}, \tag{6.12}$$

$$a^{(\xi)(\xi)}_{\alpha\beta} = -\nu_0\;a^{(\xi,\xi)}\delta_{\alpha\beta}, \quad a^{(0)(T)}_{\alpha\beta} \,=\, \nu_0\,a^{(0,T)}\delta_{\alpha\beta}, \quad a^{(0)(\xi)}_{\alpha\beta\mu} = a^{(\xi)(T)}_{\alpha} = 0\,. \tag{6.13}$$

Using (6.12) and (6.13) the equations (6.10),(6.11) becomes

$$\tau^{(eq)}_{\alpha\beta} = a^{(0,0)}\,\varepsilon_{\alpha\beta} \,+ (b^{(0,0)} - a^{(0,0)})\varepsilon\,\delta_{\alpha\beta} \,+\, a^{(0,T)}\,(T-T_0)\,\delta_{\alpha\beta}\,, \tag{6.14}$$

$$j_\alpha = -\,a^{(\xi,\xi)}\,\xi_\alpha\,, \tag{6.15}$$

where $\varepsilon \,=\, 1/3\,\Sigma^3_{\gamma=1}\,\varepsilon_{\gamma\gamma}$ .The equation (6.14) is generalization of the Duhamel-Neumann law for phenomena in isotropic media. If $a^{(0,T)}$ vanishes or if one considers isothermal processes at the temperature T_0 is analogous to Hooke's law for isotropic media ([43]). The equation of state (6.15) is specific for the theory developed in this paper.

6.1 Stress-strain equations

In the following the deviator $\tilde{A}_{\alpha\beta}$ and the scalar part A of arbitrary tensor field $A_{\alpha\beta}$ are defined by

$$\begin{cases} \tilde{A}_{\alpha\beta} \,=\, A_{\alpha\beta} - A\,\delta_{\alpha\beta}\,, \\ A = \frac{1}{3}\,\Sigma^3_{\gamma=1}\,A_{\gamma\gamma}\,, \end{cases} \tag{6.16}$$

From (6.14), using (6.16), we have

$$\tilde{\tau}^{(eq)}_{\alpha\beta} \,=\, a^{(0,0)}\,\tilde{\varepsilon}_{\alpha\beta}\,, \tag{6.17}$$

and

$$\tau^{(eq)} = b^{(0,0)}\,\varepsilon + a^{(0,T)}\,(T - T_0)\,. \tag{6.18}$$

The equations (6.17) and (6.18) are generalization of the Duhamel-Neuman law for thermoelastic phenomena in isotropic media.
If $a^{(0,T)}$ vanishes or if one considers isothermal process at temperature T_0, the relations (6.17) and (6.18) are analogous to the Hooke's law.
From the total stress, $\tau_{\alpha\beta} = \tau^{(eq)}_{\alpha\beta} + \tau^{(vi)}_{\alpha\beta}$, by virtue of (6.14) and (5.5) we have

$$\begin{aligned}\tau_{\alpha\beta} = {} & a^{(0,0)}\,\varepsilon_{\alpha\beta} + \Big(b^{(0,0)} - a^{(0,0)}\Big)\varepsilon\,\delta_{\alpha\beta} + a^{(0,T)}(T - T_0)\,\delta_{\alpha\beta} + \\ & \eta_s^{(0,0)}\,\frac{d\,\varepsilon_{\alpha\beta}}{dt} + \Big(\eta_v^{(0,0)} - \eta_v^{(0,0)}\Big)\,\frac{d\,\varepsilon}{dt}\,\delta_{\alpha\beta}\,.\end{aligned} \tag{6.19}$$

and

$$\tilde{\tau}_{\alpha\beta} = a^{(0,0)}\,\tilde{\varepsilon}_{\alpha\beta} + \eta_s^{(0,0)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}\,, \tag{6.20}$$

$$\tau = b^{(0,0)}\,\varepsilon + \eta_v^{(0,0)}\,\frac{d\,\varepsilon}{dt} + a^{(0,T)}\Big(T - T_0\Big)\,. \tag{6.21}$$

The equations (6.20) and (6.21) are the stress-strain equations for viscoelastic media and we note that (6.20) is a relation of Kelvin (Voigt) type ([29]).

6.2 Particular case.

If $a^{(0,0)} = 0$ and $a^{(0,T)} = 0$ (or $T = T_0$) the equation (6.14) becomes

$$\tau^{(eq)}_{\alpha\beta} = b^{(0,0)}\,\varepsilon\,\delta_{\alpha\beta}\,, \tag{6.22}$$

Indicating with

$$p = b^{(0,0)}\,\varepsilon\,, \tag{6.23}$$

the equation (6.22) takes the form

$$\tau^{(eq)}_{\alpha\beta} = p\,\delta_{\alpha\beta}\,, \tag{6.24}$$

The equation (6.24) shows that the medium is a newtonian fluid. In this case the equation (6.19) gives

$$\tau_{\alpha\beta} = p\,\delta_{\alpha\beta} + \eta_s^{(0,0)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} + \eta_v^{(0,0)}\,\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta}\,, \tag{6.25}$$

which is analogous to the Stokes-Navier's law.

7 Heat flows in visco-elastic processes

Using the phenomenological equations (Section 5) and the linear equations of state (Section 6) we obtain the following results:

Theorem 7.1 (Decomposition of the heat flow). . *By using the equation* (5.4) *end* (5.6) *the heat flow* $\boldsymbol{J}^{(q)}$ *can be split in two parts*

$$\boldsymbol{J}^{(q)} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)}\,, \tag{7.1}$$

where $\boldsymbol{J}^{(0)}$ *satisfies the Fourier's law and* $\boldsymbol{J}^{(1)}$ *obeys the Maxwell-Cattaneo-Vernotte law.*

Proof. Assuming $\lambda^{(\xi)(\xi)} \neq 0$, by virtue of (6.15), from the equation (5.6) we obtain

$$\frac{d\xi_\alpha}{dt} = -\frac{a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\xi_\alpha + \frac{\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\frac{\partial T}{\partial x^\alpha}\,. \tag{7.2}$$

Substituting the equation (7.2) into (5.4) one obtains

$$\boldsymbol{J}^{(q)} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)}\,, \tag{7.3}$$

where

$$\boldsymbol{J}^{(0)} = -\lambda^{(0)}\,\nabla T\,, \tag{7.4}$$

with

$$\lambda^{(0)} = \lambda^{(q,q)} - \frac{\lambda^{(q,\xi)}\,\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\,, \tag{7.5}$$

and

$$\boldsymbol{J}^{(1)} = -\frac{\lambda^{(q,\xi)}\,a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\,\boldsymbol{\xi}\,. \tag{7.6}$$

By virtue of (7.2), we see that $\boldsymbol{J}^{(1)}$ satisfies the following equation:

$$t_r\,\frac{d\boldsymbol{J}^{(1)}}{dt} + \boldsymbol{J}^{(1)} = -\frac{\lambda^{(\xi,q)}\,\lambda^{(q,\xi)}}{\lambda^{(\xi,\xi--*)}}\,\nabla T\,. \tag{7.7}$$

where

$$t_r = \frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}\,, \tag{7.8}$$

is the relaxation time.

□

From (7.3) we can see that the heat current is split into two parts: the first one, $\boldsymbol{J}^{(0)}$, is governed by Fourier's law (7.4) while the second part, $\boldsymbol{J}^{(1)}$, is governed by the MCV type constitutive equation (7.7).

8 Temperature equation

We consider an non-deformable medium at rest, i.e.,

$$\frac{d}{dt} = \frac{\partial}{\partial t} \tag{8.1}$$

and we assume that the specific internal energy u is related to the temperature by

$$du = c_{(v)} dT\,, \tag{8.2}$$

being $c_{(v)}$ the heat capacity per unit of mass at constant volume.

Theorem 8.1 (Generalization of temperature equation.). *By using* (7.4) *and the first law of thermodynamic we obtain the following temperature equation*

$$t_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \triangle T \,+\, t_r\, \eta' \frac{\partial}{\partial t} \triangle T \tag{8.3}$$

where α' and η' are parameters expressed through the phenomenological and state coefficients.

Proof. By virtue of (8.1) and (8.2) the first law of thermodynamics (2.11) becomes

$$\varrho\, c_{(v)} \frac{\partial T}{\partial t} \,+\, \nabla \cdot \left(\boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)} \right) = 0 \tag{8.4}$$

and from (8.4) we obtain

$$\varrho\, c_{(v)}\, t_r \frac{\partial^2 T}{\partial t^2} \,+\, t_r\, \nabla \cdot \left[\frac{\partial \boldsymbol{J}^{(0)}}{\partial t} + \frac{\partial \boldsymbol{J}^{(1)}}{\partial t} \right] = 0\,. \tag{8.5}$$

Summing (8.4) and (8.5), by using (7.4) and (7.7) we have the following temperature equation

$$t_r \frac{\partial^2 T}{\partial t^2} \,+\, \frac{\partial T}{\partial t} \,=\, \alpha' \triangle T \,+\, t_r\, \eta' \frac{\partial \triangle T}{\partial t} \tag{8.6}$$

where

$$\begin{cases} \alpha' \;=\; \dfrac{\lambda^{(0)} \lambda^{(\xi,\xi)} + \lambda^{(\xi,q)} \lambda^{(q,\xi)}}{\varrho\, c_{(v)}\, \lambda(\xi,\xi)} \\[2ex] \eta' = \dfrac{\lambda^{(0)}}{\varrho\, c_v}\,. \end{cases} \tag{8.7}$$

□

The equation (8.6) generalizes the temperature equation with analoguos equations of Fourier and Maxwell-Cattaneo-Vernotte.

8.1 Special cases

- If the relaxation time t_r (7.8) can be neglected with respect to the heat propagation time t ($t_r \ll t$) the equation (8.6) becomes

$$\frac{\partial T}{\partial t} = \alpha' \triangle T \,. \tag{8.8}$$

i.e. the Fourier equation.

- If the phenomenological coefficients satisfy the following condition

$$\lambda^{(q,q)} = \frac{\lambda^{(q,\xi)}, \lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}} \,, \tag{8.9}$$

from (7.5) one has: $\lambda^{(0)} = 0$ and the equations (8.7) become

$$\begin{cases} \alpha' = \dfrac{\lambda^{(\xi,q)}\lambda^{(q,\xi)}}{\varrho\, c_{(v)}\, \lambda(\xi,\xi)} \\ \eta' = 0 \end{cases} \,, \tag{8.10}$$

and the equation (8.6) becomes

$$t_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \alpha' \triangle T = 0 \,. \tag{8.11}$$

where α' is $(8.10)_1$.
The (8.11) is the temperature equation of (MCV) which has interest in nanotechnology for high-frequences processes ([25]).

9 Balance equation of energy

From the equation of motion (2.10) we can obtain

$$\varrho \frac{d}{dt}\left(\frac{1}{2} v^2\right) = \varrho\, v_\alpha\, F_\alpha + \frac{\partial}{\partial x^\beta}(\tau_{\alpha\beta}\, v_\alpha) - \tau_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} \,. \tag{9.1}$$

Adding this equation with the first law of thermodynamic (2.11) we have

$$\varrho \frac{d}{dt}\left(u + \frac{1}{2} v^2\right) = -\operatorname{div} \boldsymbol{J}^{(q)} + \varrho\, v_\alpha\, F_\alpha + \frac{\partial\, \varphi_\beta}{\partial x^\beta} \,. \tag{9.2}$$

where

$$\varphi_\beta = \tau_{\alpha\beta}\, v_\alpha = \tau^{(eq)}_{\alpha\beta}\, v_\alpha + \tau^{(vi)}_{\alpha\beta}\, v_\alpha \,. \tag{9.3}$$

By virtue of following identity with f a scalar function

$$\varrho \frac{df}{dt} = \frac{\partial}{\partial t}(\varrho\, f) + \operatorname{div}(\varrho\, f\, \boldsymbol{v}) \tag{9.4}$$

the equation (9.2) can be written in the following form

$$\frac{\partial}{\partial t}\left[\varrho\,(u+\frac{1}{2})\,v^2)\right] \;=\; -\mathrm{div}\,\boldsymbol{S}\,+\,\boldsymbol{v}\cdot\boldsymbol{F}\,, \tag{9.5}$$

where

$$\boldsymbol{S}\;=\;\varrho\left(u\;+\;\frac{1}{2}\,v^2\right)\boldsymbol{v}\;+\;\boldsymbol{J}^{(q)}\;-\;\boldsymbol{\varphi}\,, \tag{9.6}$$

is the vector of the density of energy flow.
From (9.6) it is evident that the total flow of energy is formed by contribution of the following flows

1. a flow, $\varrho\left(u\;+\;\frac{1}{2}\,v^2\right)\boldsymbol{v}$ due to transport of mass,
2. a flow, $\boldsymbol{J}^{(q)}$ due the propagation of heat,
3. a flow due to the total mechanical stress (see the relation (9.3)).

The vector $\boldsymbol{S}$ generalized the Umov's vector ([46]).

10 Remarks and conclusion

- The Fourier equation (8.8) is a parabolic equation and therefore the heat flux can be considered of infinite speed. Several scientists view this result as a paradox. In 1992 G.Fichera ([22]) states that this claim is unfounded as Fourier's theory is not correctly interpreted. In our opinion the Fourier result is justified by the value of the phenomenological parameters of the medium and therefore it cannot be said that Fourier's theory leads to a paradox.
- The Maxwell-Cattaneo-Vernotte law leads to a hyperbolic equation obtained with intuitive hypotheses but without justifying it in the context of a general theory.Various authors, for instance ([39, 25, 44]), have proposed several theories suggested by kinetic theory in order to justify the Maxwell-Cattaneo-Vernotte equation by assuming that entropy depends on fluxes instead of state variables.

In this paper, without *a priori* hypotheses and the important role of phenomenological and state coefficients, we obtain an general equation for the temperature which generalizes the Fourier and MCV temperature equations. Furthermore, the generalization of some stress-strain laws and of the energy flow vector are obtained.

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