On N_θ and lacunary statistical derivative

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Abstract. In this paper, following a very recent and new approach, we further extend recently introduced Cesaro and statistical derivative to N_θ and lacunary statistical derivative respectively. We mainly study the interrelationship between the strong N_θ derivative and lacunary statistical derivative. In the end, we establish the relationship between the lacunary statistical derivative and statistical derivative.

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1 Introduction and background

In 1993, Fridy [11] introduced the concept of lacunary statistical convergence mainly as one of the extensions of statistical convergence (for more details on statistical convergence, one may refer to [1, 3, 6, 7, 8, 9, 12, 16]).

A lacunary sequence is an increasing integer sequence $\theta = (k_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfying $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$, as $n \to \infty$.

A real-valued sequence (x_k) is said to be lacunary statistically convergent (in short $S_\theta-$convergent) to a real number l, if for any $\varepsilon > 0$,

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : |x_k - l| \geq \varepsilon\} \mid = 0,$$

where $I_n = (k_{n-1}, k_n]$. In this case, l is called the lacunary statistical limit of the sequence (x_k) and symbolically, it is written as $S_\theta - \lim(x_k) = l$ or $x_k \to l(S_\theta)$. Further, the set of all lacunary statistical convergent sequences with regard to the lacunary sequence θ is denoted by S_θ. In [11], Fridy and Orhan established the relation between statistical convergence and lacunary statistical convergence. In [10], Fridy introduced the notion of lacunary statistically Cauchy sequences and in [7], Freedman et. al. studied the relationship between the two sequence spaces $|\sigma_1|$ and N_θ defined as follows:

$$|\sigma_1| = \left\{(x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{n}\left(\sum_{k=1}^{n}|x_k - l|\right) = 0\right\}$$

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$$\text{and} \quad N_\theta = \left\{ (x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{h_n} \left(\sum_{k \in I_n} |x_k - l| \right) = 0 \right\}.$$

For more information about of lacunary convergence and its generalizations, [5, 13, 14, 15, 17, 18, 21, 22, 23, 24, 25, 26, 27] can be addressed where many more references can be found.

The notion of limit, continuity, and differentiability of a function plays a crucial role almost in all major branches of mathematics, in particular in mathematical analysis. These concepts have wide applicability in science and engineering and therefore several researchers are exploring this area for many years. The concept of Cesaro continuity, statistical limit, and statistical continuity was not known until it was introduced in 2003 by Connor and Grosse-Erdmann [4]. In [2], strongly sequentially continuous functions were introduced and investigated. Very recently, Nuray [19] has introduced the concept of Cesaro and statistical derivative and studied some fundamental properties and implication relations between Cesaro and statistical derivatives. This work is the main motivation to introduce the concept of N_θ and lacunary statistical derivative in this paper.

2 Definitions and preliminaries

For a function $f : \mathbb{R} \to \mathbb{R}$, Newton's difference coefficient to determine the slope of a secant line of the graph of the function f is given by the formula

$$f'(a) = \lim_{k\to\infty} \frac{f(a + x_k) - f(a)}{x_k}.$$

The symmetric difference quotient to determine the slope of a chord of the graph of the function f is given by the formula

$$f'(a) = \lim_{k\to\infty} \frac{f(a + x_k) - f(a - x_k)}{2x_k}.$$

It is easy to verify that the above two formulas are equivalent. For more details regarding the above two notions one may refer to [20].

Definition 2.1. [19] A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a Cesaro derivative $x_0 \in \mathbb{R}$ at a point $a \in \mathbb{R}$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(a + x_k) - f(a)}{x_k} = x_0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.1 is as follows:

Definition 2.2. [19] A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a Cesaro derivative $x_0 \in \mathbb{R}$ at a point $a \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(a + x_k) - f(a - x_k)}{2x_k} = x_0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

Definition 2.3. [19] A function $f : \mathbb{R} \to \mathbb{R}$ is Cesaro continuous at a point a if $(C, 1) - \lim f(a + x_n) = f(a)$ holds for each sequence $(x_n) \to 0$.

Definition 2.4. [19] A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesaro derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(a + x_k) - f(a)}{x_k} - x_0 \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.4 as follows:

Definition 2.5. [19] A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesaro derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(a + x_k) - f(a - x_k)}{2x_k} - x_0 \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

Definition 2.6. [19] A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{f(a + x_k) - f(a)}{x_k} - x_0 \right| \geq \varepsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.6 as follows:

Definition 2.7. [19] A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{f(a + x_k) - f(a - x_k)}{2x_k} - x_0 \right| \geq \varepsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

3 N_θ derivative

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a N_θ derivative $x_0 \in \mathbb{R}$ at a point $a \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} \frac{f(a + x_k) - f(a)}{x_k} = x_0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 3.1 is as follows:

Definition 3.2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a N_θ derivative $x_0 \in \mathbb{R}$ at a point $a \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} \frac{f(a + x_k) - f(a - x_k)}{2x_k} = x_0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

If we take $h_n = n$, then the above two definition coincides with Definition 2.1 and Definition 2.2 of [19] respectively.

Definition 3.3. A function $f : \mathbb{R} \to \mathbb{R}$ is N_θ continuous at a point $a \in \mathbb{R}$, if $N_\theta - \lim f(a + x_n) = f(a)$ holds for each sequence $(x_n) \to 0$.

Theorem 3.1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is N_θ differentiable at a point a, then it is N_θ continuous at a.*

Proof. Let (x_n) be a sequence such that $\lim\limits_{n\to\infty} x_n = 0$. Then we must have $N_\theta - \lim x_n = 0$. Consequently, the following expression

$$f(a + x_n) - f(a) = \frac{f(a + x_n) - f(a)}{x_n} x_n$$

implies that

$$N_\theta - \lim(f(a + x_n) - f(a)) = N_\theta - \lim \frac{f(a + x_n) - f(a)}{x_n} N_\theta - \lim x_n = 0.$$

Hence, $N_\theta - \lim f(a + x_n) = f(a)$ i.e., f is N_θ continuous at point a and the proof is complete. □

Definition 3.4. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly N_θ derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} \left| \frac{f(a + x_k) - f(a)}{x_k} - x_0 \right| = 0$$

holds whenever $x_n > 0$ and $\lim\limits_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 3.4 as follows:

Definition 3.5. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly N_θ derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} \left| \frac{f(a + x_k) - f(a - x_k)}{2x_k} - x_0 \right| = 0$$

holds whenever $x_n > 0$ and $\lim\limits_{n\to\infty} x_n = 0$.

Clearly, if we take $h_n = n$, then the above two definition turns to the Definition 2.3 and Definition 2.4 of [19] respectively.

Theorem 3.2. *If a function $f : \mathbb{R} \to \mathbb{R}$ has a strongly N_θ derivative at a point a then f has N_θ derivative at a.*

Proof. Proof is straightforward so omitted. □

4 Lacunary statistical derivative

Definition 4.1. A function $f : \mathbb{R} \to \mathbb{R}$ has a lacunary statistical derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{h_n} \left| \left\{ k \in I_n : \left| \frac{f(a+x_k) - f(a)}{x_k} - x_0 \right| \geq \varepsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 4.1 as follows:

Definition 4.2. A function $f : \mathbb{R} \to \mathbb{R}$ has a lacunary statistical derivative $x_0 \in \mathbb{R}$ at a point a if

$$\lim_{n\to\infty} \frac{1}{h_n} \left| \left\{ k \in I_n : \left| \frac{f(a+x_k) - f(a-x_k)}{2x_k} - x_0 \right| \geq \varepsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

Note that if we take $h_n = n$, then the above two definition coincides with the definition of statistical derivative [19]. So, from that point of view lacunary statistical derivative is an extension of statistical derivative.

Theorem 4.1. *If a function $f : \mathbb{R} \to \mathbb{R}$ has strongly N_θ derivative at a point $a \in \mathbb{R}$, then it has lacunary statistical derivative at the point a.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ has strongly N_θ derivative at $a \in \mathbb{R}$. Then, by definition for any $\varepsilon > 0$ there exists some $x_0 \in \mathbb{R}$ such that,

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} |y_k - x_0| = 0, \textit{ where } y_k = \frac{f(a+x_k) - f(a)}{x_k}. \tag{4.1}$$

Now for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{h_n} \sum_{k\in I_n} |y_k - x_0| &= \frac{1}{h_n} \sum_{\substack{k\in I_n \\ |y_k - x_0| \geq \varepsilon}} |y_k - x_0| + \frac{1}{h_n} \sum_{\substack{k\in I_n \\ |y_k - x_0| < \varepsilon}} |y_k - x_0| \\ &\geq \frac{1}{h_n} \sum_{\substack{k\in I_n \\ |y_k - x_0| \geq \varepsilon}} |y_k - x_0| \\ &\geq \frac{1}{h_n} \mid \{k \in I_n : |y_k - x_0| \geq \varepsilon\} \mid \varepsilon. \end{aligned}$$

From (4.1) and the above inequation we have

$$\lim_{n\to\infty} \frac{1}{h_n} |\{k \in I_n : |y_k - x_0| \geq \varepsilon\}| = 0.$$

Hence, f has lacunary statistical derivative at a. □

Theorem 4.2. *If a function $f:\mathbb{R}\to\mathbb{R}$ has lacunary statistical derivative at a point $a\in\mathbb{R}$, then it has strongly N_θ derivative at a provided that $(\frac{f(a+x_k)-f(a)}{x_k})$ is bounded for each $k\in\mathbb{N}$.*

Proof. Suppose $f:\mathbb{R}\to\mathbb{R}$ has lacunary statistical derivative at a point $a\in\mathbb{R}$. Then by definition for any $\varepsilon>0$ there exists some $x_0\in\mathbb{R}$ such that,

$$\lim_{n\to\infty}\frac{1}{h_n}\left|\{k\in I_n:|y_k-x_0|\geq\varepsilon\}\right|=0,\ \textit{where}\ y_k=\frac{f(a+x_k)-f(a)}{x_k}. \tag{4.2}$$

Now since (y_k) is bounded for every $k\in\mathbb{N}$, so there exists some $B>0$ such that $\forall k\in\mathbb{N}$, $|y_k-x_0|\leq B$. Now for any $\varepsilon>0$,

$$\begin{aligned}\frac{1}{h_n}\sum_{k\in I_n}|y_k-x_0| &= \frac{1}{h_n}\sum_{\substack{k\in I_n\\ |y_k-x_0|\geq\varepsilon}}|y_k-x_0|+\frac{1}{h_n}\sum_{\substack{k\in I_n\\ |y_k-x_0|<\varepsilon}}|y_k-x_0|\\ &\leq \frac{B}{h_n}\sum_{\substack{k\in I_n\\ |y_k-x_0|\geq\varepsilon}}1+\frac{1}{h_n}\sum_{\substack{k\in I_n\\ |y_k-x_0|<\varepsilon}}|y_k-x_0|\\ &\leq \frac{B}{h_n}\mid\{k\in I_n:|y_k-x_0|\geq\varepsilon\}\mid+\frac{1}{h_n}\sum_{k\in I_n}\varepsilon.\end{aligned}$$

From (4.2) and the above inequation we have

$$\lim_{n\to\infty}\frac{1}{h_n}\sum_{k\in I_n}|y_k-x_0|=0.$$

Hence, f has strongly N_θ derivative at a. □

Theorem 4.3. *Let $\theta=(k_n)$ be a lacunary sequence satisfying $\limsup_n q_n<\infty$ where $q_n=\frac{k_n}{k_{n-1}}$ and $f:\mathbb{R}\to\mathbb{R}$ has lacunary statistical derivative at a point $a\in\mathbb{R}$. Then, f has statistical derivative at a.*

Proof. Suppose $f:\mathbb{R}\to\mathbb{R}$ has lacunary statistical derivative at a point $a\in\mathbb{R}$. Then, by definition for any $\varepsilon>0$ there exists some $x_0\in\mathbb{R}$ such that,

$$\lim_{n\to\infty}\frac{1}{h_n}\left|\{k\in I_n:|y_k-x_0|\geq\varepsilon\}\right|=0,\ \textit{where}\ y_k=\frac{f(a+x_k)-f(a)}{x_k}. \tag{4.3}$$

Now the condition $\limsup_n q_n<\infty$ implies that there exists a $B>0$ such that $q_n<B$ for all $n\in\mathbb{N}$. Let N_n denote the cardinal number of the set $\{k\in I_n:|y_k-x_0|\geq\varepsilon\}$. Then from (4.3), we can say that for given $\eta>0$, there is an $n_0\in\mathbb{N}$ such that $\forall n>n_0$, $\frac{N_n}{h_n}<\eta$. Let $M=\max\{N_1,N_2,...,N_{n_0}\}$ and let t be an integer satisfying

$k_{n-1} < t < k_n$. Then we have,

$$\begin{aligned}
\frac{1}{t} \mid \{k \leq t : |y_k - x_0| \geq \varepsilon\} \mid &\leq \frac{1}{k_{n-1}} \mid \{k \leq k_n : |y_k - x_0| \geq \varepsilon\} \mid \\
&= \frac{1}{k_{n-1}} \{N_1 + N_2 + ... + N_{n_0} + N_{n_0+1} + ... + N_n\} \\
&\leq \frac{M}{k_{n-1}} n_0 + \frac{1}{k_{n-1}} \{h_{n_0+1} \frac{N_{n_0+1}}{h_{n_0+1}} + ... + h_n \frac{N_n}{h_n}\} \\
&\leq \frac{n_0 M}{k_{n-1}} + \frac{1}{k_{n-1}} \left(\sup_{n > n_0} \frac{N_n}{h_n} \right) \{h_{n_0+1} + ... + h_n\} \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta \frac{k_n - k_{n_0}}{k_{n-1}} \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta q_n \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta B,
\end{aligned}$$

which immediately gives $\lim_{t \to \infty} \frac{1}{t} |\{k \leq t : |y_k - x_0| \geq \varepsilon\}| = 0$. Hence, f has statistical derivative at the point a. □

Theorem 4.4. *Let $\theta = (k_n)$ be a lacunary sequence satisfying $\liminf_n q_n > 1$, where $q_n = \frac{k_n}{k_{n-1}}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ has statistical derivative at a point $a \in \mathbb{R}$. Then, f has lacunary statistical derivative at a.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ has statistical derivative at a point $a \in \mathbb{R}$. Then by definition for any $\varepsilon > 0$ there exists some $x_0 \in \mathbb{R}$ such that,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |y_k - x_0| \geq \varepsilon\}| = 0, \text{ where } y_k = \frac{f(a + x_k) - f(a)}{x_k}. \tag{4.4}$$

Now since $\liminf_n q_n > 1$ holds, so for sufficiently large n, there exists an $\nu > 0$ such that $q_n > 1+\nu$ which further implies $\frac{k_n}{h_n} \leq \frac{1+\nu}{\nu}$. Now for any $\varepsilon > 0$ and for sufficiently large n, the following inequation holds

$$\begin{aligned}
\frac{1}{h_n} \mid \{k \in I_n : |y_k - x_0| \geq \varepsilon\} \mid &= \frac{k_n}{h_n} \frac{1}{k_n} \mid \{k \in I_n : |y_k - x_0| \geq \varepsilon\} \mid \\
&\leq \frac{1+\nu}{\nu} \frac{1}{k_n} \mid \{k \leq k_n : |y_k - x_0| \geq \varepsilon\} \mid .
\end{aligned}$$

Consequently, using (4.4) we obtain $\lim_{n \to \infty} \frac{1}{h_n} |\{k \in I_n : |y_k - x_0| \geq \varepsilon\}| = 0$. Hence, f has lacunary statistical derivative at the point a. □

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