Probability distributions characterisations on a homogeneous cone

I. Boutouria and I. Bouzida

Abstract. In this paper, we attributed a notion of generalized power function $x \mapsto \Delta_\chi(x)$, for a multiplier χ defined on the homogeneous cone $\mathcal{P}$ of a Vinberg algebra $\mathcal{A}$. We then extended the famous Gindikin result to $\mathcal{A}$. Specifically, we determined the set of multipliers χ such that the map $\theta \mapsto \Delta_\chi(\theta^{-1})$, defined on $\mathcal{P}^*$, is the Laplace transform of a positive measure R_χ.

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1 Introduction

It is well known that the Wishart distributions on the cone of (r, r) positive symmetric matrices (see Letac [7]) or on the symmetric cone Ω of any Euclidean Jordan algebra E of rank r are the elements of the natural exponential family[1, 2] generated by the measures μ_p such that its Laplace transform is defined on Ω by

$$L_{\mu_p}(\theta) = (\det(\theta^{-1}))^p,$$

with p in $\{\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{r-1}{2}\} \cup]\frac{r-1}{2}, +\infty[$. The measure μ_p is absolutely continuous when $p \in]\frac{r-1}{2}, +\infty[$ and it is singular concentrated on the boundary of the cone, when $p \in \{\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{r-1}{2}\}$. In 2001, Hassairi and Lajmi introduced the Riesz distribution on Ω as an extension of the Wishart distribution. The Riesz distribution is based on the notion of generalized power in the space real symmetric matrices $r \times r$. An important result due to Gindikin says that the generalized power $\Delta_s(\theta^{-1})$ defined on $-\Omega$ is the Laplace transform of a positive measure R_s if and only if $s = (s_1, \cdots, s_r)$ belongs to a subset E of $\mathbb{R}^r$. The generalized power $\Delta_s(x)$ is a power function of the principal minors of $x \in \Omega$ which is reduced to $(\det(x))^p$ in the particular case where $s_1 = s_2 = \cdots = s_r = p$. In this case, the measure $R_s = \mu_p$. It is worth reminding here

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that Ishi [9] gave a more detailed description of the Gindikin set Ξ based on the orbit structure of $\overline{\Omega}$ under the action of some Lie groups. He also explicitly attributed the measure R_s for each s in Ξ. In all these works, the definition of the Riesz measure R_s and in particular the Riesz probability distribution is based on the choice of a totally ordered Jordan frame which allows the definition of the principal minors and of the generalized power of an element in the algebra. Since the order is total, it is a fundamental condition not only for the definition of the distribution but also for the proof of many results. To define models in which some specified conditional independencies, usually given by a graph, are taken into account, there focused on probability distributions on the homogeneous cone of a Vinberg algebra. For instance, Andersson and Wojnar [3] defined a class of absolutely continuous " Wishart" distributions on an homogeneous cone. These distributions were characterized by Boutouria [4]. They have also been characterized by Boutouria and Hassairi in the Olkin and Rubin way (see [5]) and in the Bobecka and Wesolowski way (see [6]). Some distributions are absolutely continuous with respect to the Lebesgue measure and some are singular concentrated on the boundary of the cone. Within the same framework, the Riesz distribution on the symmetric cone of a Jordan algebra may be seen as the particular one corresponding to the particular directed graph with a vertex set $\{1, \cdots, r\}$ and edges defined by the usual order on integers. We first define two kinds of principal minors for an element of an homogeneous cone. Minors which are said strict and minors which are said large.n Then, we define for a multiplier χ, a notion of a generalized power function $x \mapsto \Delta_\chi(x)$. One of our main results is the determination of the set of multipliers χ such that the map $\theta \mapsto \Delta_\chi(\theta^{-1})$ is the Laplace transform of a positive measure R_χ. It is a generalization of Gindikin result with a more elaborated proof adapted to the properties of the Vinberg algebra and the graph.

2 Vinberg algebras and homogeneous cones

In this section, we presented some notations and reviewed some basic concepts concerning Vinberg algebras and their homogeneous cones. We also introduced a useful decomposition of an element of the cone.

Throughout the paper, I denotes a partially ordered finite set equipped with a relation denoted $\preceq$. We consider $i \prec j$ if $i \preceq j$ and $i \neq j$. We assume that I satisfies the following condition

(F): for any two points i and j in I whether $i \prec j$ or $j \prec i$,

the path on the Hasse diagram of I between i and j is unique.

For all pairs $(i, j) \in I \times I$ with $j \prec i$, let E_{ij} denote $n_{ij} = \dim(E_{ij}) > 0$. Let

$$\mathcal{A}_{ij} = \begin{cases} E_{ij} & \text{for} \quad j \succ i \quad \text{or} \quad j \prec i \\ \{0\} & \text{otherwise.} \end{cases}$$

and $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$. An element $A \equiv (a_{ij},\ i, j \in I)$ of $\mathcal{A}$ may be seen as a matrix and so we define the trace $\mathrm{tr}A = \sum_{i\in I} a_{ii}$. We also define

$$(2.1)\quad n_{i.} = \sum_{\mu<i} n_{i\mu}, n_{.i} = \sum_{i<\mu} n_{\mu i},\ n_i = 1 + \frac{1}{2}(n_{i.} + n_{.i}), i \in I \text{ and } n_{.} = \sum_{i\in I} n_i.$$

Let $f_{ij} : E_{ij} \to E_{ij}$, $i \succ j$, be an involutional linear mapping, i.e., $f_{ij}^{-1} = f_{ij}$. They induce an involutional mapping ($A \mapsto A^*$) of $\mathcal{A}$ given as follows: $A^* = (a^*_{ij}|(i,j) \in I \times I)$, where

$$a^*_{ij} = \begin{cases} a_{ii} & \text{for} \quad i = j \\ f_{ij}(a_{ji}) = a^*_{ij} & \text{for} \quad j \prec i \quad \text{or} \quad i \prec j \\ \{0\} & \quad \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_u = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I : i \not\preceq j \Rightarrow a_{ij} = 0\}$, $\mathcal{T}_l = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I : j \not\preceq i \Rightarrow a_{ij} = 0\}$ and $\mathcal{H} = \{A \in \mathcal{A},\ A^* = A\}$ denote respectively the set of upper triangular matrices, the lower triangular matrices and the Hermitian matrices.

The sets of upper and lower triangular matrices in $\mathcal{P}$ with positive diagonal elements are respectively denoted by $\mathcal{T}_u^+$ and $\mathcal{T}_l^+$. The sets of diagonal matrices and diagonal matrices with positive entries are denoted by $\mathcal{D}$ and $\mathcal{D}^+$, respectively.

The space $\mathcal{A}$ is equipped with a bilinear map called multiplication and denoted by $(A,B) \mapsto AB$, using bilinear mappings $\mathcal{A}_{ij} \times \mathcal{A}_{jk} \to \mathcal{A}_{ik}$, denoted by $(a_{ij}, b_{jk}) \mapsto a_{ij}b_{jk}$, such that $AB = C \equiv (c_{ij}|(i,j) \in I \times I)$ with $c_{ij} = \sum_{\mu\in I} a_{i\mu}b_{\mu j}$.

Multiplication is needed to respond to the following properties:

$$\begin{aligned}
&i)\ \ \forall A \in \mathcal{A};\ A \neq 0 \Rightarrow \mathrm{tr}(AA^*) > 0\\
&ii)\ \ \forall A, B \in \mathcal{A};\ (AB)^* = B^*A^*\\
&iii)\ \ \forall A, B \in \mathcal{A};\ \mathrm{tr}(AB) = \mathrm{tr}(BA)\\
&iv)\ \ \forall A, B, C \in \mathcal{A};\ \mathrm{tr}(A(BC)) = \mathrm{tr}((AB)C)\\
&v)\ \ \forall U, S, T \in \mathcal{T}_l;\ (ST)U = S(TU)\\
&vi)\ \ \forall U, S, T \in \mathcal{T}_l;\ T(UU^*) = (TU)U^*.
\end{aligned}$$

An algebra $\mathcal{A}$ with the above structure and properties is called a Vinberg algebra (For more details, we can refer to Andersson and Wojnar [3]). We define the inner product $(.,.)_{ij}$ on E_{ij}, $i \succ j$ by $\| a_{ij} \|^2_{ij} = a_{ij}f_{ij}(a_{ij})$, $a_{ij} \in E_{ij}$. Thus instead of specifying the bilinear form $(a_{ij}, b_{ji}) \mapsto a_{ij}b_{ji}$ on E_{ij} one can specify an inner product $(.,.)_{ij}$ on E_{ij}, $i \succ j$. It can be deduced that the following two conditions hold:

1. $\forall\ a_{ij} \in E_{ij},\ b_{jk} \in E_{jk} : \| a_{ij}b_{jk} \|^2_{ik} = \| a_{ij} \|^2_{ij} \| b_{jk} \|^2_{jk},\ k \prec j \prec i,$
2. If $a_{ik} \in E_{ik},\ b_{jk} \in E_{jk}$, with $k \prec j \prec i$ and $(a_{ik}, c_{ij}b_{jk})_{ik} = 0$ for all $c_{ij} \in E_{ij}$, then $(d_{li}a_{ik}, c_{lj}b_{jk})_{lk} = 0$ for all $l \in I$ with $i \prec l$, and all $c_{lj} \in E_{lj}$, and $d_{li} \in E_{li}$.

We consider the element $(a_{ij},\ \ (i,j) \in I \times I)$ of $\mathcal{D}$ such that $a_{ii} = 1,\ \forall\ i \in I$ as the unit element of $\mathcal{A}$ and we denote it by e. We also define $E_k = (d_{ij})_{i,j\in I} \in \mathcal{D}$ with $d_{kk} = 1$ and $d_{jj} = 0\ \forall j \neq k$. It is clear that $\sum_{k\in I} E_k = e$.

Vinberg [10] proved that the subset $\mathcal{P} = \{TT^* \in \mathcal{A},\ T \in \mathcal{T}_l^+\} \subset \mathcal{H} \subset \mathcal{A}$ forms a homogeneous cone, that is the action of its automorphism group is transitive. Let G be the connected component of the identity in $\mathrm{Aut}(\mathcal{P})$; the group of linear transformations leaving $\mathcal{P}$ invariant. We recall that $\chi : G \mapsto \mathbb{R}_+$ is said to be a multiplier on the group G if it is continuous, $\chi(e) = 1$ and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$.

Consider the map $\pi\ :\ T \in \mathcal{T}_l^+ \mapsto \pi(T) \in \pi(\mathcal{T}_l^+) \subset G$ such that for $X = VV^* \in \mathcal{P}$, $V \in \mathcal{T}_l^+$

$$\pi(T)(X) = (TV)(V^*T^*). \tag{2.2}$$

Andersson and Wojnar [3] proved that the restriction of a multiplier χ to the (lower) triangular group $\mathcal{T}_l^+$, i.e., $\chi \circ \pi : T_l^+ \to \mathbb{R}_+$ is in one to one correspondence with the set of $(\lambda_i, i \in I) \in \mathbb{R}^I$. We then describe a multiplier χ by its corresponding point in $\mathbb{R}^I$ and we denote $\mathcal{X} = \{\chi : T_l^+ \to \mathbb{R}_+\}$.

If $\preceq$ is the opposite ordering on the index set I, i.e., $i \preceq j \Leftrightarrow j \preceq i$. The Vinberg algebra $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$, where

$$\mathcal{A}_{ij} = \begin{cases} \mathbb{R} & \text{for} \quad i = j \\ E_{ij} & \text{for} \quad j \succ i \quad \text{or} \quad j \prec i \\ \{0\} & \text{otherwise,} \end{cases}$$

differs from the Vinberg algebra $\mathcal{A}$ only in the ordering of the index set I. Vinberg [10] proved that $\mathcal{P}_{\preceq} = \{T^*T \in \mathcal{A},\ T \in \mathcal{T}_l^+\}$ is the dual cone of $\mathcal{P}$. The inner product $(A, B) \to \mathrm{tr}(AB)$ on H identifies H with its dual H^*, i.e.,

$$\begin{array}{ccc} H & \leftrightarrow & H^* \\ A & \mapsto & (B \mapsto \mathrm{tr}(AB)), \end{array}$$

and this isomorphism identifies $\mathcal{P}_{\preceq}$ with the dual cone $\mathcal{P}^*$ of $\mathcal{P}$.

Now, for $i \in I$, we denote $I_{\preceq i} = \{j \in I;\ j \preceq i\}$ and $I_{i\preceq} = \{j \in I;\ i \preceq j\}$ and we state that j separates i_1 and i_2 if $j \in I_{i_1 \preceq} \cap I_{i_2 \preceq}$ and $j \notin \{i_1,\ i_2\}$. In this case, j is called a separator. We also denote $S_i = \{j \in I_{i\preceq};\ j \text{ is a separator and } \forall\ k \neq j,\ k \not\prec j\}$, $\mathfrak{S} = \bigcup_{i \in I} S_i$ and $S = \{i \in \mathfrak{S}, \forall\ j \neq i,\ j \not\prec i\}$.

If $T = (t_{ij})_{i,j \in I}$ is in $\mathcal{T}_l$, we define the element $T_{i\preceq}$ of $\mathcal{T}_l$ by

$$T_{i\preceq} = (t'_{ij})_{i,j \in I}, \text{ with } t'_{jk} = t_{jk} \text{ if } i \preceq j,\ k \text{ and } t'_{jk} = 0 \text{ otherwise.} \tag{2.3}$$

We also define the element $T_{i\prec}$ of $\mathcal{T}_l$ by

$$T_{i\prec} = (t'_{ij})_{i,j \in I}, \text{ with } t'_{jk} = t_{jk} \text{ if } i \prec j,\ k \text{ and } t'_{jk} = 0 \text{ otherwise.} \tag{2.4}$$

If $X = TT^*$, we denote

$$X_{i\preceq} = T_{i\preceq} T^*_{i\preceq}, \qquad X_{i\prec} = T_{i\prec} T^*_{i\prec}. \tag{2.5}$$

We also denote by $\mathcal{P}_{i\preceq}$ (resp $\mathcal{P}_{i\prec}$) the set of $X_{i\preceq} = T_{i\preceq} T^*_{i\preceq}$ (resp $X_{i\prec} = T_{i\prec} T^*_{i\prec}$) corresponding to $T \in \mathcal{T}_l^+$. It is easy to see that $\mathcal{P}_{i\preceq}$ and $\mathcal{P}_{i\prec}$ are respectively the homogeneous cones of the Vinberg subalgebras of $\mathcal{A}$ defined by $\mathcal{A}_{i\preceq} = \prod_{k,l\in I_{i\preceq}} \mathcal{A}_{kl}$ and $\mathcal{A}_{i\prec} = \prod_{k,l\in I_{i\prec}} \mathcal{A}_{kl}$. We represent by e_i and $\check{e}_i$ the unit element of $\mathcal{A}_{i\preceq}$ respectively and $\mathcal{A}_{i\prec}$. We also define the rank of $\mathcal{P}_{i\preceq}$ (resp the rank of $\mathcal{P}_{i\prec}$) the cardinal of the set $\{j \in I, i \preceq j\}$ (resp the cardinal of the set $\{j \in I, i \prec j\}$). This leads to the following decomposition of an element of $\mathcal{P}$ which will serve in our characterization result.

Proposition 2.1. *Let $Z = TT^*$ be an element of $\mathcal{P}$ with $T \in \mathcal{T}_l^+$. Consider $\wp = \{i \in I,\ I_{\prec i} = \varnothing\}$ and define, for $i \in I$,*

$$Z_{i\preceq} = T_{i\preceq}T_{i\preceq}^* \qquad \text{and} \qquad Z_i = \begin{cases} Z_{i\preceq} - \sum_{s\in S_i} Z_{s\preceq} & \text{if} \quad i \in \wp \\ Z_{i\preceq} & \text{if} \quad i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$Z = \sum_{i\in I} Z_i = \sum_{i\in\wp\cup S} Z_i. \tag{2.6}$$

Proof. In order to prove the equality (2.6), we compare the blocks of Z and $\sum_{i\in I} Z_i$ on each subalgebra $\mathcal{A}_j = \prod_{k,l\in I_{j\preceq}} \mathcal{A}_{kl}$, $j \in I$ denoted respectively by $(Z)_j$ and $(\sum_{i\in\wp\cup S} Z_i)_j$. Departing from the definition of $(Z_{j\preceq})$, we first notice that

$$(Z)_j = (Z_{j\preceq})_j, \qquad \forall\ j \in I. \tag{2.7}$$

Now, we discuss according to the position of j.
If $j \notin \wp \cup S$, then there exists a unique $i_0 \in \wp \cup S$ such that $j \in I_{i_0\preceq}$. The fact that $j \in I_{i_0\preceq}$ implies that $(Z_{i_0})_j = (Z_{j\preceq})_j$. Therefore, with reference to(2.7), we get

$$(Z)_j = (Z_{j\preceq})_j = (Z_{i_0})_j = (\sum_{i\in\wp\cup S} Z_i)_j.$$

If $j \in S$, we have $(Z_i)_j = 0$, for all $i \in \wp$. Hence

$$(\sum_{i\in\wp\cup S} Z_i)_j = (\sum_{i\in S} Z_i)_j = (Z_j)_j = (Z_{j\preceq})_j.$$

For $j \in \wp$, we have that $(\sum_{i\in\wp} Z_i)_j = Z_j = Z_{j\preceq} - \sum_{s\in S_j} Z_{s\preceq}$, and we consider separately the cases $S_j = \varnothing$ and $S_j \neq \varnothing$. If $S_j = \varnothing$, then $(\sum_{i\in S} Z_i)_j = 0$ and $(Z)_j = (Z_{j\preceq})_j = Z_j = (\sum_{i\in\wp} Z_i)_j$.

If $S_j \neq \varnothing$, $(\sum_{i\in S} Z_i)_j = (\sum_{i\in S_j} Z_i)_j = (\sum_{i\in S_j} Z_{i\preceq})_j$. Therefore $(Z)_j = (\sum_{i\in\wp\cup S} Z_i)_j$. □

Example 2.1. Consider the following poset on $I = \{1,2,3,4\}$ where $1 \prec 3,\ 1 \prec 4,\ 2 \prec 4$. Then $S = S_1 = S_2 = \{4\}$, $I_{3\preceq} = \{3\}$. Hence $Z_3 = (0)$,

$$Z_4 = Z_{4\preceq} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} t_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_{13} & 0 & t_{33} & 0 \\ t_{14} & 0 & 0 & t_{44} \end{pmatrix} \begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} - Z_4,$$

and

$$Z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t_{24} & 0 & t_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{22} & 0 & t_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} - Z_4.$$

Finally, with reference to what Andersson and Wojnar proved in [3] that if $\sigma = ZZ^*$, where $Z \in \mathcal{T}_l^+$,

$$\sigma^{-\chi} = (Z^*)^{-1}\mathrm{diag}(\lambda_i | i \in I) Z^{-1}.$$

Then, it is clear that

$$\sigma^{-1} = (Z^*)^{-1} Z^{-1}.$$

3 Riesz measures on an homogeneous cone

The definition of a Riesz measure on the symmetric cone of a Jordan algebra relies on the notion of generalized power of an element of the cone which is a power function of the so-called principal minors. In order to define a Riesz distribution on an homogeneous cone, we need to extend all these things to a Vinberg algebra.

3.1 Generalized power

We first introduce the notion of determinant. For $X = TT^*$, with $T = (t_{ij}) \in \mathcal{T}_l$, we define the following determinants

$$\det X = \prod_{i \in I} t_{ii}^2, \qquad \det{}^{\preceq} X_{\preceq i} = \prod_{j \in I_{\preceq i}} t_{jj}^2 \qquad \text{and} \qquad \det{}^{\prec} X_{\prec i} = \prod_{j \in I_{\prec i}} t_{jj}^2.$$

For $X = TT^*$, with $T \in \mathcal{T}_l$, we define the strict principal minor of order k of X as

$$\Delta_{\prec k}(X) = \begin{cases} \det{}^{\prec}(X_{\prec k}) & \text{if} \quad I_{\prec k} \neq \varnothing \\ 1 & \text{if} \quad I_{\prec k} = \varnothing \end{cases} \tag{3.1}$$

and the large principal minor of order k of X as

$$\Delta_{\preceq k}(X) = \det{}^{\preceq}(X_{\preceq k}). \tag{3.2}$$

Definition 3.1. Let $\chi = \{\lambda_i, i \in I\}$ be a multiplier and $X \in \mathcal{P}$, then the map defined by

$$X \mapsto \Delta_\chi(X) = \prod_{k\in I} (\frac{\Delta_{\preceq k}(X)}{\Delta_{\prec k}(X)})^{\lambda_k}. \tag{3.3}$$

is called the generalized power function corresponding to χ.

We also denote

$$\Delta_\chi^{(i)}(X) = \prod_{k\in I_{i\preceq}} (\frac{\Delta_{\preceq k}(X)}{\Delta_{\prec k}(X)})^{\lambda_k}. \tag{3.4}$$

Note that, if $\lambda_i = \lambda,\ i \in I$, then $\Delta_\chi(X) = (\det X)^\lambda$. It is easy to verify that $\Delta_{\chi+\chi'}(X) = \Delta_\chi(X)\Delta_{\chi'}(X)$, where $\chi + \chi' = \{\lambda_i + \lambda_i',\ i \in I\}\cdot$ We introduce the following example to explain the notion of generalized power.

Example 3.2. Let us consider $I = \{1, 2, 3, 4\}$ and the poset defined by

$$1 \prec 3,\ 1 \prec 4,\ 2 \prec 3.$$

For $X = TT^* \in \mathcal{P}$, with $T = (t_{ij}) \in \mathcal{T}_l$, we have $\Delta_{\preceq 1}(X) = \det^{\preceq}(X_{\preceq 1}) = t_{11}^2$, $\Delta_{\preceq 2}(X) = \det^{\preceq}(X_{\preceq 2}) = t_{22}^2$, $\Delta_{\preceq 4}(X) = \det^{\preceq}(X_{\preceq 4}) = t_{11}^2 t_{44}^2$ and $\Delta_{\preceq 3}(X) = \det^{\preceq}(X_{\preceq 3}) = t_{11}^2 t_{22}^2 t_{33}^2$. Hence, for $\chi = \{\lambda_i = 1,\ i \in I\}$,

$$\Delta_\chi(X) = \frac{t_{11}^2}{1}\frac{t_{22}^2}{1}\frac{t_{11}^2 t_{44}^2}{t_{11}^2}\frac{t_{11}^2 t_{22}^2 t_{33}^2}{t_{11}^2 t_{22}^2} = t_{11}^2 t_{22}^2 t_{33}^2 t_{44}^2.$$

3.2 Orbit decomposition of the closure $\overline{\mathcal{P}}$ of $\mathcal{P}$

For, $i \in \wp \cup \mathcal{S}$, we denote by ε^i the set of maps ψ defined from I into $\{0, 1\}$ as follows. If $i \in \wp$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\preceq}$ or $j \in S_i$, and if $i \in \mathcal{S}$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\preceq}$. We also denote by $\varepsilon^{\check{i}}$ the set of maps ψ defined from I into $\{0, 1\}$ as follows. If $i \in \wp$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\prec}$ or $j \in S_i$, and if $i \in \mathcal{S}$, ψ is such that $\psi(j) = 0$, when $j \notin I_{i\prec}$. Let $i \in \wp \cup \mathcal{S}$, and $\psi \in \varepsilon^i$, we define $e_\psi = \mathrm{diag}(\psi) = \mathrm{diag}(\psi(j), j \in I) \in \mathcal{D}$ and we denote by $\mathcal{T}_l^+.e_\psi = \{Te_\psi T^*,\ T \in \mathcal{T}_l^+\}$. We also consider the two elements of $\mathcal{A}$

$$E^i = \begin{cases} e_i & \text{in} \quad \mathcal{A}_{i\preceq} \\ 0 & \text{elsewhere}\ , \end{cases} \qquad \text{and} \qquad \check{E}^{\check{i}} = \begin{cases} \check{e}_{\check{i}} & \text{in} \quad \mathcal{A}_{i\prec} \\ 0 & \text{elsewhere}\ . \end{cases}$$

Next, we state and prove a fundamental result, which decomposes $\overline{\mathcal{P}}$ to orbits.

Theorem 3.1. *i)*

$$\overline{\mathcal{P}} = \sum_{i\in\wp\cup\mathcal{S}} \overline{\mathcal{P}}_{i\preceq}. \tag{3.5}$$

ii) Let $i \in \wp \cup \mathcal{S}$, then

$$\overline{\mathcal{P}}_{i\preceq} = \bigcup_{\psi\in\varepsilon^i} \mathcal{T}_l^+.e_\psi. \tag{3.6}$$

Proof. $i)$ Let $Z \in \overline{\mathcal{P}}$, then there exists a sequence $\{Z^{(n)}\}_{n\in N}$ in $\mathcal{P}$ such that $Z^{(n)} \to Z$ as $n \to \infty$. Since $\{Z^{(n)}\}_{n\in N}$ is in $\mathcal{P}$, using the decomposition (2.6) we write $Z^{(n)} = \sum_{i\in I} Z_i^{(n)}$, where $Z_i^{(n)} \in \overline{\mathcal{P}}_{i\preceq}$. Hence $Z = \sum_{i\in I} Z_i$, where $Z_i \in \overline{\mathcal{P}}_{i\preceq}$ and (3.5) is proved.
$ii)$ We will prove (3.6) by induction on the rank of the cone $\mathcal{P}_{i\preceq}$. It is obvious that (3.6) holds for $i \in \wp \cup \mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} = 1$. Suppose that (3.6) holds for any $i \in \wp\cup\mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} < l$ and let us show that it holds for i such that rank$\mathcal{P}_{i\preceq} = l$. Consider the set $M_i = \{j \in I;\ I_{\prec j} = \{i\}\}$. Then using the decomposition defined by (2.6) for an element of the cone $\mathcal{P}_{i\prec}$, we easily infer

$$\overline{\mathcal{P}}_{i\prec} = \sum_{j\in M_i} \overline{\mathcal{P}}_{j\preceq}. \tag{3.7}$$

Since rank$\mathcal{P}_{i\preceq} = l$ and rank$\mathcal{P}_{i\prec} = l-1$ then rank$\mathcal{P}_{j\preceq} \le l-1$, $\forall j \in M_i$. Using the induction hypothesis, we can write

$$\overline{\mathcal{P}}_{j\preceq} = \bigcup_{\psi\in\varepsilon^j} \mathcal{T}_l^+.e_\psi, \qquad j \in M_i.$$

Now, let $\varepsilon^{i\prec} = \sum_{j\in M_i} \varepsilon^j$, then we obtain

$$\overline{\mathcal{P}}_{i\prec} = \sum_{j\in M_i} \overline{\mathcal{P}}_{j\preceq} = \sum_{j\in M_i} \Big(\bigcup_{\psi\in\varepsilon^j} \mathcal{T}_l^+.e_\psi\Big) = \bigcup_{\psi\in\varepsilon^{i\prec}} \mathcal{T}_l^+.e_\psi \subset \bigcup_{\psi\in\varepsilon^i} \mathcal{T}_l^+.e_\psi. \tag{3.8}$$

To conclude, we verify that for $Z \in \overline{\mathcal{P}}_{i\preceq}$, there exists $\psi \in \varepsilon^i$ and $T \in \mathcal{T}_l^+$ such that $Z = T.e_\psi$. Let $Z \in \overline{\mathcal{P}}_{i\preceq}$, then there exists a sequence $\{Z^{(n)}\}_{n\in N}$ in $\mathcal{P}_{i\preceq}$ such that $Z^{(n)} \to Z$ when $n \to \infty$. As $Z^{(n)} \in \mathcal{P}_{i\preceq}$, there exists $U_{i\preceq}^{(n)} = (u_{jk}^{(n)})_{j,k\in I}$ in $\mathcal{T}_l$ such that $Z^{(n)} = U_{i\preceq}^{(n)}(U_{i\preceq}^{(n)})^*$ (see (2.3) and (2.5)). In particular, we have

$$z_{kk}^{(n)} = (u_{kk}^{(n)})^2 + \sum_{j\prec k} \| u_{kj}^{(n)} \|_{kj}^2, \tag{3.9}$$

for $k \in I_{i\preceq}$. This implies that the sequences $(u_{kk}^{(n)})_{n\in\mathbf{N}}$ and $(u_{kj}^{(n)})_{n\in\mathbf{N}}$ are bounded. Therefore there exists a subsequence of positive integers (n_m) such that $(u_{kk}^{(n_m)})_m$ and $(u_{kj}^{(n_m)})_m$ converge. Let $\widetilde{u}_{kk} = \lim_{m\to+\infty} u_{kk}^{(n_m)}$ and $\widetilde{u}_{kj} = \lim_{m\to+\infty} u_{kj}^{(n_m)}$. Then $\lim_{m\to+\infty} U_{i\preceq}^{(n_m)} = \widetilde{U}_{i\preceq}$, so that $Z = \widetilde{U}_{i\preceq}\widetilde{U}_{i\preceq}^* = (z_{kj})_{k,j\in I}$. As $z_{ii} \ge 0$, we will consider the case $z_{ii} = 0$ and the case $z_{ii} > 0$ separately.
Suppose that $z_{ii} = 0$. Then $\widetilde{u}_{ii} = (z_{ii})^{1/2} = 0$, so that $z_{ki} = \widetilde{u}_{ii}\widetilde{u}_{ki} = 0,\ i \prec k$. Thus $Z = Z_{i\prec} \in \overline{\mathcal{P}}_{i\prec}$, and the result follows according to (3.8).

If $z_{ii} > 0$, we consider the elements of $\mathcal{A}_{i\preceq}$

$$\widetilde{u}^i = \begin{pmatrix} \widetilde{u}_{ii} & 0 \\ \widetilde{C}_i & e_{\check{i}} \end{pmatrix} \text{ and } \widetilde{u}_i = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{U}_{i\prec} \end{pmatrix},$$

where $\widetilde{C}_i = \sum_{i\prec j} \widetilde{u}_{ji}$ and $\widetilde{U}_{i\prec} = (\widetilde{u}_{jk})_{j,k\in I_{i\prec}}$. Then $\widetilde{u}_{i\preceq} = \widetilde{u}^i\widetilde{u}_i$ and $\widetilde{u}_{ii} = (z_{ii})^{1/2} > 0$.
Let T_1 and T_2 in $\mathcal{T}_l^+$, such that $T_1 = \widetilde{u}^i$ in $\mathcal{A}_{i\preceq}$ and $T_2 = \widetilde{u}_i$ in $\mathcal{A}_{i\preceq}$, we have

$T_2.\check{E}^{\check{i}} \in \overline{\mathcal{P}}_{i\prec}$. By induction hypothesis, there exists a unique $\psi_1 \in \varepsilon^i$ such that $\psi_1(i) = 0$ and there exists $\tilde{T}_2 \in \mathcal{T}_l^+$ such that $T_2.\check{E}^{\check{i}} = \tilde{T}_2.e_{\psi_1}$. Let $\psi_2 \in \varepsilon^i$, such that $\psi_2(i) = 1$ and $\psi_2(j) = 0\ \forall j \neq i$ and put $\psi = \psi_1 + \psi_2 \in \varepsilon^i$ and $T = T_1\tilde{T}_2 \in \mathcal{T}_l^+$. Then we have

$$\begin{aligned} Z &= T_1T_2.(E_i + \check{E}^{\check{i}}) \\ &= T_1.E_i + T_2.\check{E}^{\check{i}} \\ &= T_1.E_i + \tilde{T}_2.e_{\psi_1} \\ &= (T_1\tilde{T}_2).e_\psi \\ &= T.e_\psi. \end{aligned}$$

Hence (3.6) is proved. □

3.3 Gamma functions

We use the generalized power function to introduce a generalized gamma function on an homogeneous cone.

For $i \in \wp \cup \mathcal{S}$, $\psi \in \varepsilon^i$ and $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$, we set

$$\mathcal{X}(\psi) = \{\chi_i \in \mathcal{X} \mid \lambda_j = 0, \text{ for all } j \in I_{i\preceq} \text{ such that } \psi(j) = 0\}. \tag{3.10}$$

For every $\chi_i \in \mathcal{X}(\psi)$, we define a generalized power function on $\mathcal{T}_l^+.e_\psi$ by

$$\Delta^{\psi}_{\chi_i}(T.e_\psi) = \Delta^{(i)}_{\chi_i}(TT^*), \qquad \forall T \in \mathcal{T}_l^+, \tag{3.11}$$

where $\Delta^{(i)}_{\chi_i}(TT^*)$ is defined by (3.4). We also define $n^\psi = (n^i_{j.}, j \in I)$ by

$$n^i_{j.} = \sum_{k\prec j} \psi(k) n_{kj} \qquad \forall j \in I_{i\preceq}. \tag{3.12}$$

When $\psi \neq 0$, we introduce the measure ν_ψ on $\mathcal{T}_l^+.e_\psi$ defined by

$$\nu_\psi(d(T.e_\psi)) = \Delta^{(i)}_{\dot{\chi}_i^\psi}(TT^*) \prod_{\substack{i \preceq j \preceq k \\ \psi(j)=1}} dt_{kj}, \tag{3.13}$$

where $\dot{\chi}_i^\psi = \{\lambda_j \in \mathbb{R}, j \in I,$ such that $\lambda_j = -\psi(j)(1 + n^i_{j.})/2,$ if $j \in I_{i\preceq}$ and $\lambda_j = 0$ if $j \notin I_{i\preceq}\}$, and $T = (t_{jk})_{j,k\in I} \in \mathcal{T}_l^+$. Finally, we denote by ν_0 the Dirac measure at 0.

Theorem 3.2. *Let $i \in \wp \cup \mathcal{S}$ and $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\} \in \mathcal{X}(\psi)$. The integral*

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \int_{\mathcal{T}_l^+.e_\psi} \exp\{-tr(Z)\} \Delta^{\psi}_{\chi_i}(Z) \nu_\psi(dZ) \tag{3.14}$$

converges if and only if $\chi_i \in \mathcal{X}(\psi)$ satisfies the following condition:

$$\lambda_j > \frac{n^i_{j.}}{2} \qquad \forall j \in I \ \text{ such that } \ \psi(j) = 1. \tag{3.15}$$

Moreover, under this condition, one has

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = 2^{-|\psi|}\pi^{-|n^\psi|/2} \prod_{\substack{j \in I \\ \psi(j)=1}} \Gamma(\lambda_j - \frac{n^i_{j.}}{2}), \tag{3.16}$$

where $|\psi| = \sum_{j \in I_{i\preceq}} \psi(j)$ *and* $|n^\psi| = \sum_{j \in I_{i\preceq}} n^i_{j.}$.

Proof. If $\psi = 0$, the integral (3.14) is reduced to 1. Thus (3.15) and (3.16) hold trivially. If $\psi \neq 0$, then writing $Z = U.e_\psi$, where $U = (u_{jk})_{j,k \in I} \in \mathcal{T}_l^+$, as matter of fact, the integral (3.14) can be written as follows

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \int_{\mathcal{T}_l^+ e_\psi} \exp\{-(\sum_{\psi(j)=1}(u_{jj}^2 + \sum_{j \prec k} \| u_{kj} \|^2_{kj}))\} \prod_{\substack{i \preceq j \prec k \\ \psi(j)=1}} u_{jj}^{2\lambda_j - n^i_{j.} - 1} du_{jj} du_{kj}.$$

For $j \in I_{i\preceq}$, let

$$C_j = (s_{ln})_{l,n \in I} \in \mathcal{C}^j = \sum_{j \prec k} E_{kj}, \tag{3.17}$$

with $s_{lj} = u_{lj}$, if $j \prec l$, and $s_{ln} = 0$, otherwise. It is clear that $\| C_j \|^2 = \sum_{j \prec k} \| u_{kj} \|^2_{kj}$ and $dC_j = \prod_{j \prec k} du_{kj}$. Hence

$$\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \prod_{\substack{i \preceq j \\ \psi(j)=1}} \int_0^{+\infty} \exp^{-u_{jj}^2} u_{jj}^{2\lambda_j - n^i_{j.} - 1} du_{jj} \prod_{\substack{i \preceq j \\ \psi(j)=1}} \int_{\mathcal{C}^j} \exp^{\{-\|C_j\|^2\}} dC_j.$$

Therefore the convergence condition is reduced to the one of the ordinary gamma functions, that is $\lambda_j > \frac{n^i_{j.}}{2}$ $\forall j \in I$ such that $\psi(j) = 1$. □

Remark 3.3. Referring to Theorem 3.2, we have, for $i \in I$, a relation between $\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i)$ and $\Gamma_{\mathcal{P}}(\chi)$. In fact, if we denote by $\mathbf{1}_i$ the element of ε^i, such that $\psi(j) = 1$, $\forall\, j \in I_{i\preceq} \setminus S_i$, if $i \in \wp$ and such that $\psi(j) = 1$, $\forall\, j \in I_{i\preceq}$, if $i \in \mathcal{S}$, then it is clear that

$$\mathcal{P} = \sum_{i \in \wp \cup \mathcal{S}} \mathcal{T}_l^+.e_{\mathbf{1_i}}.$$

Morever, using (2.1), we have for $\chi = \sum_{i \in I} \chi_i \in \mathcal{X}$, where $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$

$$\prod_{i \in \wp \cup \mathcal{S}} \Gamma_{\mathcal{T}_l^+ . e_{1_i}}(\chi_i) = 2^{-|I|} \pi^{\frac{n_{.} - |I|}{2}} \prod_{i \in I} \Gamma(\lambda_i - \frac{n_{i.}}{2}) = 2^{-|I|} \Gamma_{\mathcal{P}}(\chi) \cdot$$

3.4 Riesz measures

For the definition of the Riesz distribution, we need to introduce some other notations. Let $i \in \wp \cup \mathcal{S}$ and $\chi_i = \{\lambda_j,\ j \in I;\ \lambda_j = 0, \forall j \notin I_{i\preceq}\}$ and introduce for $\psi \in \varepsilon^i$ and $\omega \in \varepsilon^{\check{i}}$, the following sets

$$\text{(3.18)} \qquad \mathcal{B}(\rangle, \psi) = \left\{ \chi_\rangle \in \mathcal{X};\ \lambda_| = \prime, \text{for } | \notin \mathcal{I}_{\rangle\preceq}, \lambda_| = \frac{\backslash^{\rangle}_{|.}}{\in} \right\},$$

when $j \in I_{i\preceq}$, and $\psi(j) = 0$,

$$\text{(3.19)} \qquad \check{\mathcal{B}}(\check{i}, \omega) = \left\{ \chi_i \in \mathcal{X}; \upsilon_j = 0, \text{for } j \notin I_{i\prec}, \vartheta_j = \frac{n^i_{j.}}{2} \right\},$$

when $j \in I_{i\prec}$ and $\omega(j) = 0$,

$$\text{(3.20)} \quad \Xi(i,\ \psi) = \left\{ \chi_i\ \in \mathcal{B}(i,\ \psi),\ \lambda_j > \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\preceq} \text{ and } \psi(j) = 1 \right\},$$

$$\text{(3.21)} \qquad \check{\Xi}(\check{i},\ \omega) = \left\{ \chi_i\ \check{\mathcal{B}}(\check{i}, \omega)\ \lambda_j > \frac{n^i_{j.}}{2} \text{ when } j \in I_{i\prec} \text{ and } \omega(j) = 1 \right\},$$

$$\text{(3.22)} \Xi(i) = \bigcup_{\psi \in \varepsilon^i} \Xi(i,\ \psi), \qquad \check{\Xi}(\check{i}) = \bigcup_{\omega \in \varepsilon^{\check{i}}} \check{\Xi}(\check{i},\ \omega), \qquad \Xi = \sum_{i \in \wp \cup \mathcal{S}} \Xi(i) \cdot$$

For every $\chi_i = \{\lambda_j,\ j \in I\} \in \mathcal{X}$, let

$$\text{(3.23)} \qquad \widetilde{\chi}_i = \{\lambda_j - (1 - \psi(j)) \frac{n^i_{j.}}{2}, \text{when } j \in I_{i\preceq}, \text{ and } 0 \text{ if } j \notin I_{i\preceq}\}.$$

It is clear that if $\chi_i \in \mathcal{B}(i,\ \psi)$, then $\widetilde{\chi}_i \in \mathcal{X}(\psi)$.

In what follows, we define the Laplace transform of a positive measure μ on the cone $\mathcal{P}$ by

$$\text{(3.24)} \qquad L_\mu(\theta) = \int_{\mathcal{P}} \exp\{-\mathrm{tr}(\theta Z)\} \mu(dZ), \qquad \theta \in \mathcal{P}^*.$$

Theorem 3.3. *There exists a positive measure R_χ with Laplace transform defined on $\mathcal{P}^*$ by $\Delta_\chi(\theta^{-1})$ if and only if $\chi \in \Xi$.*

The proof for Theorem 3.3 relies on the following proposition.

Proposition 3.4. *Let $i \in \wp \cup \mathcal{S}$. Then there exists a positive measure R_{χ_i} such that the Laplace transform is defined on $\mathcal{P}^*$ and is equal to $\Delta^{(i)}_{\chi_i}(\theta^{-1})$ if and only if $\chi_i \in \Xi(i)$.*

Proof. $\Leftarrow$) Let $\chi_i \in \Xi(i)$. Then there exists $\psi \in \varepsilon^i$ such that $\chi_i \in \Xi(i,\ \psi)$. It is clear that $\widetilde{\chi}_i$ defined by (3.23) satisfies (3.15). Here after, we are going to prove that the Laplace transform of the measure

$$R_{\chi_i}(dZ) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\Delta^{\psi}_{\widetilde{\chi}_i}(Z)\mathbf{1}_{\mathcal{T}_l^+.e_\psi}(Z)\nu_\psi(dZ)$$

defined on $\mathcal{P}^*$ and is given by

$$L_{R_{\chi_i}}(\theta) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi} \exp\{-\mathrm{tr}(\theta Z)\}\Delta^{\psi}_{\widetilde{\chi}_i}(Z)\nu_\psi(dZ) = \Delta^{(i)}_{\chi_i}(\theta^{-1}).$$

In fact, as $\theta \in \mathcal{P}^*$, then $\theta_{i\preceq}$ defined by (2.5) is in the dual cone $\mathcal{P}^*_{i\preceq}$ of $\mathcal{P}_{i\preceq}$, and there exists T in $\mathcal{T}_l^+$ such that $\theta_{i\preceq} = T^*.E^i$. Let $Y = \pi(T)(Z)$, where π is defined by (2.2). As $Z \in \mathcal{T}_l^+.e_\psi$, there exists $S \in \mathcal{T}_l^+$ such that $Z = S.e_\psi$. This with (3.13) implies that

$$\nu_\psi(dY) = \nu_\psi(d(\pi(T)(Z))) = \Delta^{(i)}_{\ddot{\chi}_i^\psi}(\theta)\nu_\psi(dZ), \tag{3.25}$$

where $\ddot{\chi}_i^\psi = \{\lambda_j \in \mathbb{R}, j \in I \text{ such that } \lambda_j = (1-\psi(j))\frac{n^i_{j.}}{2},\ \text{if } j \in I_{i\preceq} \text{ and } 0 \text{ if } j \notin I_{i\preceq}\}$. Since $\widetilde{\chi}_i \in \mathcal{X}(\psi)$, then

$$\begin{aligned}\Delta^{\psi}_{\widetilde{\chi}_i}(Z) &= \Delta^{\psi}_{\widetilde{\chi}_i}(\pi^{-1}(T)(Y))\\ &= \Delta^{\psi}_{\widetilde{\chi}_i}(T^{-1}.e_\psi)\Delta^{\psi}_{\widetilde{\chi}_i}(Y)\\ &= \Delta^{(i)}_{\widetilde{\chi}_i}(\theta^{-1})\Delta^{\psi}_{\widetilde{\chi}_i}(Y).\end{aligned} \tag{3.26}$$

Based on (3.23), (3.25) and (3.26), we get

$$\Delta^{\psi}_{\widetilde{\chi}_i}(Y)\nu_\psi(dY) = \Delta^{(i)}_{\chi_i}(\theta)\Delta^{\psi}_{\widetilde{\chi}_i}(Z)\nu_\psi(dZ). \tag{3.27}$$

Then

$$\begin{aligned}L_{R_{\chi_i}}(\theta) &= \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi} \exp\{-\mathrm{tr}(\theta\pi^{-1}(T)(Y))\}\Delta^{(i)}_{\chi_i}(\theta^{-1})\Delta^{\psi}_{\widetilde{\chi}_i}(Y)\nu_\psi(dY)\\ &= \Delta^{(i)}_{\chi_i}(\theta^{-1})\frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)}\int_{\mathcal{T}_l^+.e_\psi} \exp\{-\mathrm{tr}Y\}\Delta^{\psi}_{\widetilde{\chi}_i}(Y)\nu_\psi(dY)\\ &= \Delta^{(i)}_{\chi_i}(\theta^{-1}).\end{aligned}$$

$\Rightarrow$) Suppose that there exists a positive measure R_{χ_i} such that the Laplace transform is defined on $\mathcal{P}^*$ and is equal to $\Delta^{(i)}_{\chi_i}(\theta^{-1})$. Our objective is to confirm that $\chi_i \in \Xi(i)$.

For this χ_i and a ψ in ε^i, consider the generalized positive Riesz measure R_{χ_i} defined for φ in the Schwartz space $\mathcal{S}(\mathcal{A})$ of rapidly decreasing functions on $\mathcal{A}$ by

$$R_{\chi_i}(\varphi) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_\psi}(\widetilde{\chi}_i)} \int_{\mathcal{T}_l^+.e_\psi} \varphi(Z)\Delta^{\psi}_{\widetilde{\chi}_i}(Z)\nu_\psi(dZ). \tag{3.28}$$

We will prove by induction on the rank of the cone $\mathcal{P}_{i\preceq}$ that $\chi_i \in \Xi(i,\ \psi)$. Suppose that rank$\mathcal{P}_{i\preceq} = 1$. Then, we have either cardinality of $\wp$ equal to 1 or cardinality of $\mathcal{S}$ equal to 1. Thus R_{χ_i} coincides with the Riesz measure ρ_λ on $]0,+\infty[$ given by

$$\rho_\lambda(\varphi) = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} \varphi(u)u^{\lambda-1}du. \tag{3.29}$$

This implies that $\chi_i \equiv \lambda$ and $\lambda > 0$ which means that the result is true when rank$\mathcal{P}_{i\preceq} = 1$. Now, suppose that the claim holds for any $i \in \wp \cup \mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} \le k-1$, and let us explain that it also holds for i such that rank$\mathcal{P}_{i\preceq} = k$. Consider $i \in \wp\cup\mathcal{S}$ such that rank$\mathcal{P}_{i\preceq} = k$. Then rank$\mathcal{P}_{i\prec} = k-1$ so rank$\mathcal{P}_{j\preceq} \le k-1$, $\forall j \in M_i$. Using the induction hypothesis, we have $\forall j \in M_i$, $\xi_j = \{\beta_l \in \mathbb{R},\ l \in I,\ \beta_l = 0;\ \forall l \notin I_{j\preceq}\}$ is in $\Xi(j)$. Let R_{ξ_j} be the Riesz measure defined in (3.28) on the cone $\mathcal{P}_{j\preceq}$ for some ψ in ε^j and let $\check{\xi}_{\check{i}} = \sum_{j\in M_i} \xi_j = \{\beta_j \in \mathbb{R}, j \in I; \beta_j = 0 \text{ for } j \notin I_{i\prec}\}$. Then, departing from (3.7), the measure $\check{R}_{\check{\xi}_{\check{i}}} = \prod^*_{j\in M_i} R_{\xi_j}$, where $\prod^*$ represents the convolution product, is concentrated on $\mathcal{P}_{i\prec}$. Consider the sets

$$\check{\chi}_{\check{i}} = \{\lambda_j \in \mathbb{R}, j \in I; \lambda_j = 0 \text{ for } j \notin I_{i\prec}\},$$

$$\check{n}^i = \{\beta_k \in \mathbb{R},\ k \in I; \beta_k = n_{ki} \text{ for } k \in I_{i\prec} \text{ and } \beta_k = 0 \text{ for } k \notin I_{i\prec}\},$$

and

$$M(\lambda_i) = \{\alpha_k,\ k \in I\},$$

where

$$\alpha_k = \begin{cases} \lambda_i & \text{for} \quad k = i \\ \frac{n_{ki}}{2} & \text{for} \quad k \in I_{i\prec} \\ 0 & \quad \text{otherwise.} \end{cases}$$

Then, it is easy to verify that $M(\lambda_i) \in \mathcal{B}(i,\psi_1)$, where $\psi_1 \in \varepsilon^i$ such that $\psi_1(i) = 1$, and $\psi_1(j) = 0\ \forall j \neq i$. Also, we have $\chi_i - M(\lambda_i) = \check{\chi}_{\check{i}} - \frac{\check{n}^i}{2} \in \check{\mathcal{B}}(\check{i},\omega)$, where $\check{\mathcal{B}}(\check{i},\omega)$ is defined by (3.19).

Using the Laplace transforms, we obtain

$$R_{\chi_i} = R_{M(\lambda_i)} * \check{R}_{\check{\chi}_{\check{i}} - \frac{\check{n}^i}{2}}. \tag{3.30}$$

Proceeding as in the proof of Theorem 3.2, and using (3.16), we get

$$R_{M(\lambda_i)}(\varphi) = \frac{2\pi^{-\frac{\dim \mathcal{C}^i}{2}}}{\Gamma(\lambda_i)} \int_{\mathcal{T}_l^+ e_{\psi_1}} \varphi(U.e_{\psi_1})u_{ii}^{2\lambda_i-1}du_{ii}dC_i,$$

where $U \in \mathcal{T}_l^+$, $U.e_{\psi_1} = (u_{jk})_{j,k\in I}$ and C_i is defined by (3.17). On the other hand, it is easy to verify that

$$U.e_{\psi_1} = u_{ii}^2 E_i + u_{ii}C_i + \sum_{i\prec j} \| u_{ji} \|_{ji}^2 \, E_j.$$

Hence, if we define

$$\begin{array}{rcl} \mathcal{Q}^i \; : \; \mathcal{C}^i \times \mathcal{C}^i & \to & \mathcal{A}_{i\prec} \\ (C_i, C_i) & \mapsto & \mathcal{Q}^i(C_i, C_i) = \sum_{i\prec j} \| u_{ji} \|_{ji}^2 \, E_j, \end{array}$$

then we have

$$U.e_{\psi_1} = u_{ii}^2 E_i + u_{ii}C_i + \mathcal{Q}^i(C_i, C_i).$$

Setting $u_{ii} = \sqrt{v}$, we get

$$R_{M(\lambda_i)}(\varphi) = \frac{2\pi^{-\frac{\dim \mathcal{C}^i}{2}}}{\Gamma(\lambda_i)} \int_0^{+\infty} \int_{\mathcal{C}^i} \varphi(vE_i + \sqrt{v}C_i + \mathcal{Q}^i(C_i, C_i))dC_i v^{\lambda_i - 1}dv.$$

Using (3.29), it becomes

$$R_{M(\lambda_i)}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\int_{\mathcal{C}^i} \varphi(vE_i + \sqrt{v}C_i + \mathcal{Q}^i(C_i, C_i))dC_i)_v. \tag{3.31}$$

As for $\lambda_i = 0$, ρ_0 is the Dirac measure at $v = 0$, we get

$$R_{M(0)}(\varphi) = R_{\frac{\check{n}^i}{2}}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \int_{\mathcal{C}^i} \varphi(\mathcal{Q}^i(C_i, C_i))dC_i. \tag{3.32}$$

Using (3.30) and (3.31), we obtain

$$\begin{aligned} &R_{\chi_i}(\varphi) = \\ &\quad \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\int_{\mathcal{C}^i} \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}(\varphi(vE_i + \sqrt{v}C_i + \mathcal{Q}^i(C_i, C_i) + Y))_Y dC_i)_v, \end{aligned} \tag{3.33}$$

Denote by C_c^∞ the set of C^∞ functions with compact support and consider the functions of the form

$$\varphi(Z) = \varphi_1(z_{11})\varphi_2(Z_{i\prec})$$

where $Z = (z_{ij})_{i,j\in I} \in \mathcal{A}$, $Z_{i\prec} \in \mathcal{A}_{i\prec}$, $\varphi_1 \in C_c^\infty(\mathbb{R})$ and $\varphi_2 \in C_c^\infty(\mathcal{A}_{i\prec})$. Then by (3.32) and (3.33), we have

$$\begin{aligned} R_{\chi_i}(\varphi) &= \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\varphi_1) \int_{\mathcal{C}^i} \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}(\varphi_2(\mathcal{Q}^i(C_i, C_i) + Y))_Y dC_i \\ &= \rho_{\lambda_i}(\varphi_1)(\check{R}_{\frac{\check{n}^i}{2}} * \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}})(\varphi_2) \\ &= \rho_{\lambda_i}(\varphi_1)\check{R}_{\check{\chi}_i}(\varphi_2). \end{aligned} \tag{3.34}$$

For a suitable choice of non-negative $\varphi_1 \in C_c^\infty(\mathbb{R})$, we have $\rho_{\lambda_i}(\varphi_1) > 0$. If $\varphi_2 \geq 0$, then using (3.33) and the positivity of R_{χ_i}, we get $\check{R}_{\check{\chi}_{\check{i}}}(\varphi_2) = (\rho_{\lambda_i}(\varphi_1))^{-1} R_{\chi_i}(\varphi) \geq 0$. Thus $\check{R}_{\check{\chi}_i}$ is positive and the induction hypothesis ensures that $\check{\chi}_{\check{i}} \in \check{\Xi}(\check{i},\ \omega)$.

Now, fix a non-negative φ_2 such that $\check{R}_{\check{\chi}_{\check{i}}}(\varphi_2)$ is strictly positive. Then using (3.33) again, we get $\rho_{\lambda_i}(\varphi_1) \geq 0$ for any $\varphi_1 \geq 0$. Therefore ρ_{λ_i} is positive and we deduce that $\lambda_i \geq 0$. If $\lambda_i = 0$, then choosing a ψ in ε^i such that $\psi(i) = 0$, we get $\chi_i \in \Xi(i,\ \psi)$. To study the case $\lambda_i > 0$, we first observe that the map

$$\begin{array}{rcl}]0,+\infty[\times \mathcal{C}^i \times \mathcal{A}_{i\prec} & \to & \{X \in \mathcal{A}, x_{11} > 0\} \\ (v, C_i, Y) & \mapsto & vE_i + \sqrt{v}C_i + \mathcal{Q}^i(C_i, C_i) + Y \end{array}$$

is a diffeomorphism whose inverse is given by

$$x \mapsto (x_{11},\ x_{11}^{-1/2} \sum_{1\prec k} X_{k1},\ x_{i\prec} - \frac{1}{x_{11}} \mathcal{Q}^i(\sum_{1\prec k} X_{k1}, \sum_{1\prec k} X_{k1}).$$

For a functions the functions $\varphi \in C_c^\infty(\mathcal{A})$ of the form

$$\varphi(Z) = \begin{cases} \varphi_1(v)\varphi_2(C_i)\varphi_3(Y) & (z_{11} > 0), \\ 0 & (z_{11} \leq 0), \end{cases}$$

with $\varphi_1 \in C_c^\infty(]0,+\infty[)$, $\varphi_2 \in C_c^\infty(\mathcal{C}^i)$ and $\varphi_3 \in C_c^\infty(\mathcal{A}_{i\prec})$, by (3.33), we have that

$$R_{\chi_i}(\varphi) = \pi^{-\frac{\dim \mathcal{C}^i}{2}} \rho_{\lambda_i}(\varphi_1) \check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}(\varphi_3) \int_{\mathcal{C}^i} \phi_2(C_i) dC_i.$$

Since $\lambda_i > 0$, the positivity assumption of R_{χ_i} yields that $\check{R}_{\check{\chi}_i - \frac{\check{n}^i}{2}}$ is positive. By the induction hypothesis this implies that $\check{\chi}_i - \frac{\check{n}^i}{2} \in \check{\Xi}(\check{i},\ \omega)$. Finally, choose a ψ in ε^i such that

$$\psi(j) = \begin{cases} \omega(j) & \text{for} \quad \forall j \neq i \\ 1 & \text{for} \quad j = i, \end{cases}$$

where $\omega \in \varepsilon^{\check{i}}$. Then, as $\check{\chi}_i - \frac{\check{n}^i}{2} \in \check{\Xi}(\check{i},\ \omega)$, for $j \in I_{i\prec}$, we have that $\lambda_j = \frac{n^i_{j.}}{2}$ if $\omega(j) = 0$ and $\lambda_j > \frac{n^i_{j.}}{2}$ if $\omega(j) = 1$. As $\lambda_i > 0$ and $n^i_{i.} = 0$, then $\lambda_i > \frac{n^i_{i.}}{2}$. This means that for such a ψ, we have $\chi_i \in \Xi(i,\ \psi)$. Hence $\chi_i \in \Xi(i)$ and Proposition 3.4 is proved. □

Proof. **of Theorem 3.3** (⇐) Let $\chi = \sum_{i \in \wp \cup \mathcal{S}} \chi_i \in \Xi$, where $\chi_i \in \Xi(i)$ and let

$$R_\chi = \prod_{i \in \wp \cup \mathcal{S}}^{*} R_{\chi_i}, \tag{3.35}$$

R_{χ_i} is the positive measure defined from Proposition 3.4. Then, for $\theta \in -\mathcal{P}^*$

$$\begin{aligned} L_{R_\chi}(\theta) &= \prod_{i\in\wp\cup\mathcal{S}} L_{R_{\chi_i}}(\theta) \\ &= \prod_{i\in\wp\cup\mathcal{S}} \Delta^{(i)}_{\chi_i}(\theta^{-1}) \\ &= \Delta^{(i)}_{\sum\limits_{i\in\wp\cup\mathcal{S}}\chi_i}(\theta^{-1}) \\ &= \Delta_\chi(\theta^{-1}). \end{aligned}$$

($\Rightarrow$) We have $L_{R_\chi}(\theta) = \Delta_\chi(\theta^{-1})$, then using the fact $\prod\limits_{i\in\wp\cup\mathcal{S}} \Delta^{(i)}_{\chi_i}(\theta^{-1}) = \Delta_\chi(\theta^{-1})$, Proposition 3.4, and putting $R_\chi = \prod\limits^{*}_{i\in\wp\cup\mathcal{S}} R_{\chi_i}$, such that for $\theta \in \mathcal{P}^*$, $L_{R_{\chi_i}}(\theta) = \Delta^{(i)}_{\chi_i}(\theta^{-1})$, we get $\chi_i \in \Xi(i)$. Therefore $\chi = \sum\limits_{i\in\wp\cup\mathcal{S}} \chi_i \in \Xi$. □

Following the terminology used in [8] in the case of symmetric matrices, we call the measures R_χ defined above in terms of their Laplace transforms Riesz measures on the homogeneous cones. These measures are divided into two classes according to the position of $\chi \in \Xi$. A class of measures which are absolutely continuous with respect to the Lebesgue measure on $\mathcal{P}$ and a class concentrated on the boundary $\partial\mathcal{P}$ of $\mathcal{P}$.

Proposition 3.5. *Let $\chi = \{\lambda_i,\ i \in I\} \in \mathcal{X}$. Then R_χ is absolutely continuous if and only if $\lambda_i > \frac{n_{i.}}{2}$, $i \in I$. In this case*

$$R_\chi(dZ) = \frac{1}{\Gamma_{\mathcal{P}}(\chi)} \Delta_{\chi+\ddot{\chi}}(Z) 1_{\mathcal{P}}(Z) dZ, \tag{3.36}$$

where $\ddot{\chi} = \{-n_i,\ i \in I\}$ and $\Gamma_{\mathcal{P}}(\chi) = \pi^{\frac{n_{.}-|I|}{2}} \prod\limits_{i\in I} \Gamma(\lambda_i - \frac{n_{i.}}{2})$.

Proof. $\Rightarrow$) We have $\mathcal{P} = \sum\limits_{i\in\wp\cup\mathcal{S}} \mathcal{T}_l^+.e_{\mathbf{1_i}}$, then $\widetilde{\chi}_i$ defined by (3.23) is equal to χ_i.

From (3.35), we have $R_\chi = \prod\limits^{*}_{i\in\wp\cup\mathcal{S}} R_{\chi_i}$. Writing $Z = UU^* \in \mathcal{P}$, $U \in \mathcal{T}_l^+$, then $Z = \sum\limits_{i\in I} Z_i$ where Z_i is in $\mathcal{T}_l^+.e_{\mathbf{1_i}}$, and using the proof of Proposition 3.4, we have for $Z_i = U^{\mathbf{1}}_{i\preceq} U^{\mathbf{1}*}_{i\preceq}$, $U^{\mathbf{1}}_{i\preceq} = (u_{kj})_{k,j\in I}$

$$R_{\chi_i}(dZ_i) = \frac{1}{\Gamma_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}(\chi_i)} \Delta_{\chi_i}(Z_i)\nu(dZ_i),$$

where $\Delta^{\mathbf{1_i}}_{\chi_i} = \Delta_{\chi_i}$ and $\nu(dZ_i) = \nu_{\mathbf{1_i}}(dZ_i) = \Delta^{(i)}_{\dot{\chi}_i^{\mathbf{1_i}}}(Z) \prod\limits_{\substack{i\preceq j\preceq k \\ \psi(j)=1}} du_{kj}$.

Then $\nu(dZ) = \Delta_{\dot\chi}(Z)dU$, where

$$\dot\chi = \{-(\frac{1+n_{j\cdot}}{2}),\ j \in I\} \text{ and } dU = \prod_{i\in\wp\cup\mathcal{S}} \prod_{\substack{i \preceq j \preceq k \\ \psi(j)=1}} du_{kj}.$$

Using the fact that the mapping $U \in \mathcal{T}_l^+ \mapsto UU^* \in \mathcal{P}$ is a diffeomorphism, we have $dU = 2^{-|I|}\Delta_{\hat\chi}(Z)dZ$, where $\hat\chi = \{-\frac{1+n_{\cdot j}}{2},\ \forall j \in I\}$. As $n_j = 1 + \frac{1}{2}(n_{\cdot j} + n_{j\cdot})$, we have

$$R_\chi(dZ) = \frac{1}{\Gamma_\mathcal{P}(\chi)}\Delta_\chi(Z)\Delta_{\dot\chi}(Z)\Delta_{\hat\chi}(Z)dZ = \frac{1}{\Gamma_\mathcal{P}(\chi)}\Delta_{\chi+\ddot\chi}(Z)dZ,$$

where $\ddot\chi = \{-n_j,\ j \in I\}$ and $\Gamma_\mathcal{P}(\chi) = 2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\Gamma_{\mathcal{T}_l^+.e_\psi}(\chi_i) = \pi^{\frac{n_{\cdot\cdot}-|I|}{2}}\prod_{i\in I}\Gamma(\lambda_i - \frac{n_{i\cdot}}{2})$.
Moreover, the condition $\lambda_i > \frac{n_{i\cdot}}{2},\ i \in I$ is easily deduced from Theorem 3.2.
$\Leftarrow$) It suffices to verify that for χ such that $\lambda_i > \frac{n_{i\cdot}}{2},\ i \in I$, the Laplace transform of the measure

$$\frac{1}{\Gamma_\mathcal{P}(\chi)}\Delta_{\chi+\ddot\chi}(Z)1_\mathcal{P}(Z)dZ$$

is equal to $\Delta_\chi(\theta^{-1})$, $\forall\theta \in \mathcal{P}^*$. In fact, let $\theta \in \mathcal{P}^*$, then there exists $T = (t_{ij})$ in $\mathcal{T}_l^+$ such that $\theta = T^*T$.
Let $Y = \pi(T)(Z)$, where π is defined by (2.2). Then $dZ = \det\pi^{-1}(T)dY$ and

$$\Delta_{\chi+\ddot\chi}(Z) = \Delta_{\chi+\ddot\chi}(\pi^{-1}(T)Y) = \Delta_{\chi+\ddot\chi}(\theta^{-1}Y) = \Delta_{\chi+\ddot\chi}(\theta^{-1})\Delta_{\chi+\ddot\chi}(Y).$$

From Andersson and Wojnar [3], we have

$$\det\pi^{-1}(T) = \prod_{i\in I} t_{ii}^{-2n_i} = \Delta_{-\ddot\chi}(\theta^{-1}),$$

where $-\ddot\chi = \{n_i, i \in I\}$. Writing $Y = \sum_{i\in I} Y_i$, where Y_i is in $\mathcal{T}_l^+.e_{\mathbf{1_i}}$, then as

$$\begin{aligned}\Gamma_\mathcal{P}(\chi) = 2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\Gamma_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}(\chi_i) &= 2^{|I|}\prod_{i\in\wp\cup\mathcal{S}}\int_{\mathcal{T}_l^+.e_{\mathbf{1_i}}}\exp\{-\mathrm{tr}Y_i\}\Delta_{\chi_i}^{\mathbf{1_i}}(Y_i)\nu_{\mathbf{1_i}}(dY_i) \\ &= \int_\mathcal{P}\exp-\{\mathrm{tr}Y\}\Delta_{\chi+\ddot\chi}(Y)dY,\end{aligned}$$

we obtain that

$$\begin{aligned}\frac{1}{\Gamma_\mathcal{P}(\chi)}\int_\mathcal{P}\exp\{-\mathrm{tr}(\theta Z)\}\Delta_{\chi+\ddot\chi}(Z)dZ &= \Delta_\chi(\theta^{-1})\frac{1}{\Gamma_\mathcal{P}(\chi)}\int_\mathcal{P}\exp-\{\mathrm{tr}Y\}\Delta_{\chi+\ddot\chi}(Y)dY \\ &= \Delta_\chi(\theta^{-1}).\end{aligned}$$

□

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Authors' addresses:

Imen Boutouria
Laboratory of Probability and Statistics,
Sfax University, B.P. 1171, Tunisia.
E-mail: imen.boutouria@fss.rnu.tn

Imed Bouzida
College of Education Al Ain University Abu Dhabi Campus,
P.O.Box: 112612 Abu Dhabi, UAE.
E-mail: imed.bouzida@gmail.com , imed.bouzida@aau.ac.ae