Geometry on the surface of revolution with first approximate slope metric

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Abstract. The motive of this article is to study globally defined slope metrics on surface of revolution. An introduction to first approximate slope metric and its geodesic behaviour on the surface of revolution is also discussed.

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1 Introduction

As stated by Chern [10], Finsler geometry is just Riemannian geometry without quadratic restrictions. In Riemannian geometry, tangent spaces at different points are linearly isometric to each other since restriction of metric to tangent space is an inner product. But the case is not same in Finsler geometry since tangent space may vary from point to point. Most importantly all Finsler geometric quantities not only depend on the point \( x \in M \) but also on the direction \( y \in T_x M \) such that \((x, y) \in TM\) are canonical coordinates of tangent bundle \( TM \). For instance, in anistropic medium speed of light depends on its direction of travel.

Time minimizing problems have been an appealing research area for the geometers since very long time [7]. Beginning with Zermelo’s navigation problem, which focused on finding solutions of geodesic in a Riemannian manifold \((M, h)\) under the influence of wind or current \( W \), such that \( ||W|| < 1 \), (in case of weak wind) are exactly geodesics of Randers metric \([6, 21]\), whereas when \( ||W|| = 1 \), (in case of critical wind) are geodesics of Kropina metric \([23]\). This particular area fascinates because of its applications to the theoretical approach \([15, 12]\) and real world problems \([14, 13]\).

Also, problem on reversible geodesics with \((\alpha, \beta)\)-metrics have been worked out in \([19]\). In \([20]\), authors have constructed weighted quasi-metrics associated with some Finsler metrics.

The Zermelo navigation problem was also worked out on Hermitian manifold, where solution to this were metrics of special type in complex geometry, i.e., complex Randers and complex Kropina metrics. Further, in \([1]\), it has been proved that solution of Zermelo navigation problem on Hermitian manifold \((M, h)\), under action of weak wind and with variable space-dependent ship’s dependent relative speed \( ||u_h|| \) are \( \mathbb{R} \).
complex Finsler metrics. It is quite obvious to observe that one’s walking speed depends strongly on the slope of the surface, to be precise on the direction of travel. Hence, a particular interesting part of this time minimizing problem was proposed by Finsler to Matsumoto in 1969.

The problem can be stated as “Suppose a person walking on a horizontal plane with velocity \( c \), while the gravitational force is acting perpendicularly on this plane. The person is almost ignorant of the action of this force. Imagine the person walks now with same velocity on the inclined plane of angle \( \epsilon \) to the horizontal sea level. Under the influence of gravitational forces, what is the trajectory the person should walk in the center to reach a given destination in the shortest time?” Refer [17, 18] for intrinsic geometry behind slope metric.

Some problems can be formulated further like “In the case of drought or hikings across any geographical region with complex area like forests, hillside or grass fields etc. On what kind of surface this problem can be formulated as variational problem of slope metric or to be precise first approximate slope metric.” In case of wildfire [16], slope metric have been a great tool for predicting, controlling and fighting wildlife fire. Also, we know shortest distance problems are a part of calculus of variations. Let us recall that Finsler metric is non-degenerate when Hessian metric is regular metric and extremal paths are minimizing when metric is positive definite, and this happens in the case when unit circle in tangent space is \( T_xM \), is strongly convex. Refer [4] for details. This paper covers the following objectives:

1. We discuss first approximate slope metric and construct some surfaces with globally defined slope metric.
2. We study the geometry of a surface of revolution endowed with first approximate slope metric.

Here is outline of the paper. In section 3, we introduce first approximate slope metric. Next, we calculate Hessian metric for the same. Further, we recall necessary and sufficient condition for surfaces to admit globally defined slope metric, hence we construct examples of the surfaces which admit strongly convex slope metric.

Section 4 basically concerns with surface of revolution. In subsection (4.1), we study the geometry of Riemannian surface of revolution, which includes finding the geodesic equation for this surface. In subsection (4.2), we find slope metric on the surface of revolution using Okubo’s method [2]. In the end, we find the geodesics of surface of revolution endowed with first approximate slope metric.

2 Preliminaries

Definition 2.1. Let \( V \) be an \( n \)-dimensional real vector space endowed with smooth norm \( F \) on \( V \setminus \{0\} \) satisfying the following conditions:

1. \( F(u) \geq 0 \ \forall \ \ u \in V \).
2. \( F(\lambda u) = \lambda F(u) \ \forall \ \lambda > 0 \), i.e., \( F \) is positively homogeneous,
3. Let \( \{u_1, u_2, ..., u_n\} \) be the basis of \( V \) such that \( y = y^1u_1 + y^2u_2 + ... + y^nu_n \).
   Then the Hessian matrix
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\[(g_{ij}) := \left( \frac{1}{2} F^2 \right)_{y^i y^j}, \]

is positive definite at every point of \( V \setminus \{0\} \). The pair \((V, F)\) is called Minkowski space and \(F\) is called Minkowski norm.

**Definition 2.2.** Let \(M\) be a connected (smooth) manifold. A Finsler metric \(F\) on \(M\) is a function \(F : TM \to [0, \infty)\), which satisfies:

1. \(F\) is smooth on the slit tangent bundle \(TM \setminus \{0\}\),
2. The restriction of \(F\) to any \(T_x M, x \in M\) is a Minkowski norm.

The space \((M, F)\) is called Finsler space.

**Definition 2.3.** Let \(\gamma : [0, 1] \to M\), be \(C^1\)-curve. Then the Finsler length is defined as

\[L(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t)) dt.\]

Further, the Finsler distance between two points \(p, q \in M\) is defined as

\[d_F(p, q) = \inf_{\gamma} L(\gamma),\]

where the infimum is taken over all piecewise \(C^1\)-curves joining \(p\) and \(q\).

**Definition 2.4.** Christoffel symbols of a Riemannian metric \((g_{ij})\) are expressed as:

\[
\Gamma^m_{ij} = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km},
\]

where \(g^{ij}\) is inverse of \(g_{ij}\).

**Definition 2.5.** Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. The function

\[g_{ij} := \left( \frac{1}{2} F^2 \right)_{y^i y^j} = FF_{y^i y^j} + F_{y^i} F_{y^j} = h_{ij} + \ell_i \ell_j\]

is called fundamental tensor of the metric \(F\).

**Definition 2.6.** Let \(F = \alpha \phi(s) ; s = \beta/\alpha\), where \(\phi\) is a smooth function on an open interval \((-b_0, b_0)\), \(\alpha = \sqrt{a_{ij}(x)y^i y^j}\) is a Riemannian metric, \(\beta = b_i(x) y^i\) is a 1-form on an \(n\)-dimensional manifold \(M\) with \(||\beta|| < b_0\). Then \(F\) is Finsler metric if and only if following conditions are satisfied:

\[\phi(s) > 0, \ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \ \forall \ |s| \leq b < b_0.\]

**Definition 2.7.** The slope metric (Matsumoto metric) on a Finsler manifold \(M\) can be defined as

\[
F = \frac{\alpha^2}{\alpha - \beta},
\]

where

\[
\alpha = \sqrt{(1 + f_x^2)\dot{x}^2 + 2f_x f_y \dot{x} \dot{y} + (1 + f_y^2)\dot{y}^2}, \ \beta = f_x \dot{x} + f_y \dot{y}.
\]
Definition 2.8. The general formula for the geodesic spray coefficients of an \((\alpha, \beta)\)-metric \(F\) is

\[
G^i = G^i_\alpha + \alpha Qs_0 + \Theta\{-2Q\alpha s_0 + r_{00}\} \frac{y^j}{\alpha} + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i,
\]

where \(G^i_\alpha\) are the spray coefficients of \(\alpha\).

3 First approximate slope metric

It is well known that the Matsumoto (slope) metric is defined as \(F(\alpha, \beta) = \frac{\alpha^2}{\alpha - \beta}\), where

\[
\alpha = \sqrt{(1 + f_x^2)x^2 + 2f_xf_yx\dot{y} + (1 + f_y^2)y^2}, \quad \beta = f_x \dot{x} + f_y \dot{y}.
\]

We shall refer to [22] for the geometry of the slope metric. The Matsumoto metric can be expressed as

\[
F = \alpha \left[ \sum_{k=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^k \right]
\]

for \(|\beta| < |\alpha|\). If we neglect all the powers of \(b_i(x) > 2\), we get Randers metric and if we neglect all powers of \(b_i(x) > 3\) then, \(F\) is the first approximate of the Matsumoto metric.

We can easily prove that the first approximate slope metric \(F(\alpha, \beta)\), defined as

\[
F(\alpha, \beta) = \alpha + \beta + \frac{\beta^2}{\alpha}
\]

belongs to the class of \((\alpha, \beta)\)-metrics, where \(\alpha = \sqrt{a_{ij}y^iy^j}\) is the Riemannian metric and \(\beta = b_i(x)y^i\) is a 1-form in \(TM\).

We rewrite \(F(\alpha, \beta) = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\) and \(\phi(s) = 1 + s + s^2\). Simple calculations show that the \(F(\alpha, \beta)\) metric satisfies the following relations for \(s < \frac{1}{2}\)

\[
\phi(s) > 0,
\phi(s) - s\phi'(s) > 0,
\phi''(s) \geq 0, \text{ for } s < b.
\]

Hence \(F(\alpha, \beta)\) is a strongly convex \((\alpha, \beta)\)-metric.

Next, we calculate the Hessian metric for \(F(\alpha, \beta)\) defined as \(g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}\).

\[
g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_j + b_j \alpha_i) - \rho \rho_1 \alpha_i \alpha_j,
\]
where
\[ \alpha_i := \frac{\partial \alpha}{\partial y^i}, \rho = \phi^2 - s \phi \rho', \rho_0 = \phi \phi + \phi' \phi' \rho, \rho_1 = -s(\phi \phi'' + \phi' \phi') + \phi' \phi'. \]

For the metric defined in equation (3.2), we have
\[ \rho = 1 - s - s^3 - s^4 = \frac{\alpha^4 + \alpha^3 \beta - \alpha \beta^3 - \beta^4}{\alpha^4}, \]
\[ \rho_0 = 3 + 6s + 6s^2 = \frac{3(\alpha^2 + 2\alpha \beta + 2\beta^2)}{\alpha^2}, \]
\[ \rho_1 = 1 - 3s^2 - 4s^3 = \frac{\alpha^3 - 3\alpha \beta^2 - 4\beta^3}{\alpha^3}. \]

By replacing these values in (3.3), we get the Hessian metric \( g_{ij} \), as follows:
\[
g_{ij} = \left( \frac{\alpha^4 + \alpha^3 \beta - \alpha \beta^3 - \beta^4}{\alpha^4} \right) a_{ij} + 3 \left( \frac{\alpha^2 + 2\alpha \beta + 2\beta^2}{\alpha^2} \right) b_i b_j \\
+ \left( \frac{\alpha^3 - 3\alpha \beta^2 - 4\beta^3}{\alpha^3} \right) [b_j \alpha_j + b_j \alpha_i] \\
- \left( \frac{\alpha^7 + \alpha^6 \beta - 3\alpha^5 \beta^2 - 8\alpha^4 \beta^3 - 5\alpha^3 \beta^4 + 3\alpha^2 \beta^5 + 7\alpha \beta^6 + 4\beta^7}{\alpha^7} \right) \alpha_i \alpha_j.
\]

### 3.1 Examples of slope metrics

In [5], we can easily see that there are many examples of surfaces that locally admit slope metrics. For example the paraboloid of revolution \( f(x, y) = 81 - x^2 - y^2 \) is strongly convex only in the circular vicinity of its hilltop, and not globally. In order to construct surfaces which admit globally defined slope metric, we recall the following Lemmas:

**Lemma 3.1.** [9] A surface \( M \rightarrow \mathbb{R}^3, (x, y) \rightarrow (x, y, z = f(x, y)) \) admits a strongly convex slope metric \( F = \frac{\alpha^2}{\alpha - \beta} \), where \( \alpha, \beta \) are given in (3.1) if and only if
\[
f_x^2 + f_y^2 < \frac{1}{3},
\]
where \( f_x, f_y \) are the partial derivatives of \( f \).

**Example 3.1.** Based on Lemma (3.1), we may construct surfaces which admit strongly convex slope metric, as follows:

1. \( f(x, y) = \frac{1}{2\sqrt{6}} e^{-(x+y)} \).
2. \( f(x, y) = \frac{1}{6} \ln(x + y + 1) \).
3. \( f(x, y) = \frac{1}{2\sqrt{6}} \arctan(x + y + 1) \).
Some of the above examples are plotted in Figures 1-3, respectively.

Indeed, using elementary calculus we can check that the above surfaces globally admit strongly convex slope metrics, with \( f : \mathbb{R}^2 \to \mathbb{R} \). The graphs of the first two examples are provided in the subsequent pictures:

\begin{align*}
\text{Fig. 1. } f(x, y) &= \frac{1}{2\sqrt{6}} e^{-(x+y)}.
\text{Fig. 2. } f(x, y) &= \frac{1}{6} \log(x + y + 1).
\text{Fig. 3. } f(x, y) &= \frac{1}{2\sqrt{6}} \arctan(x + y + 1).
\end{align*}

Further, it was observed that the surfaces of revolution are good candidates for the study of the slope metric, further emphasized by our next section.

**Lemma 3.2.** A surface of revolution \( M \to \mathbb{R}^3 \), \((u, v) \to (m(u)\cos v, m(u)\sin v, u)\) admits a strongly convex slope metric \( F = \frac{\alpha^2}{\alpha - \beta} \), with \( \alpha, \beta \) given by \( \alpha = \sqrt{1 + m'(u)^2 \dot{u}^2 + m''(u)^2 \dot{v}^2} \), \( \beta = \dot{u} \) if and only if \( (m')^2 > 3 \). Moreover, \((M, F)\) is a Finsler surface of revolution.

**Example 3.2.** Based on Lemma 3.2, the following examples can be constructed:

1. \( m(u) = u^2 + 5 \), for \( u \in \left(\frac{\sqrt{3}}{2}, \infty\right) \).
2. \( m(u) = e^u \), for \( u \in (\ln\sqrt{3}, \infty) \).

**Theorem 3.3.** [8] Let \( f : S \to \mathbb{R}^3 \) be a surface of revolution. Then the following statements are equivalent:

1. \( S \) admits a strongly convex slope metric;
2. \( [\phi']^2 < \frac{1}{3} \), where \( f : S \to \mathbb{R}^3 : (x, y, z = \phi(s)) \) and \( s = \sqrt{x^2 + y^2} \);
3. \( [m']^2 > 3 \), where \( f : S \to \mathbb{R}^3 : (u, v) \to (m(u)\cos v, m(u)\sin v, v) \).
We further present illustrative examples for this result.

**Example 3.3.**

1. Consider the surface of revolution
   \[ f(x, y) = \left( x, y, z = \frac{1}{2\sqrt{6}} \frac{1}{1 + x^2 + y^2} \right). \]
   On putting \( z = \phi(s) := \frac{1}{2\sqrt{6}} \frac{1}{1 + s^2} \), we get \( \phi'(s) = \frac{1}{\sqrt{6} (1 + s^2)^{3/2}} \). Hence the strongly convexity condition is satisfied for all \( s \in \mathbb{R} \). The graph of this surface is plotted in Figure 4.

2. Consider the surface of revolution
   \[ (x, y, z = \sin(\sqrt{x^2 + y^2}) - \sqrt{x^2 + y^2}). \]
   On putting \( z = \phi(s) := \sin s - s \), we get that the strongly convexity condition is satisfied for \( \cos^{-1}(1 - \frac{1}{\sqrt{3}}) \leq s \leq \cos^{-1}(1 + \frac{1}{\sqrt{3}}) \).

![Fig. 4.](image)

**4 Surfaces of revolution**

**4.1 Riemannian surfaces of revolution**

From the geometry of Riemannian surfaces of revolution, we recall that a surface of revolution \( \phi : M \rightarrow \mathbb{R}^3 \) can be parametrized as [11]

\[
(4.1) \quad \phi(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).
\]

The induced Riemannian metric is

\[
(a_{ij}) = \begin{pmatrix}
(f')^2 + (g')^2 & 0 \\
0 & f^2
\end{pmatrix}.
\]

The Christoffel symbols are

\[
\Gamma^1_{11} = \frac{f' f'' + g' g''}{f^2 + g^2}, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0,
\]

\[
\Gamma^1_{22} = \frac{-ff'}{f^2 + g^2}, \quad \Gamma^2_{11} = 0, \quad \Gamma^2_{12} = \frac{f'}{f}, \quad \Gamma^2_{22} = 0, \quad \Gamma^2_{21} = \frac{f'}{f}.
\]
Hence the unit-speed geodesic equations take the following form:

\[ \begin{align*}
  \frac{d^2u}{dt^2} + \frac{f'f'' + g'g''}{f'^2 + g'^2} \left( \frac{du}{dt} \right)^2 - \frac{f'}{f'^2 + g'^2} \left( \frac{dv}{dt} \right)^2 &= 0, \\
  \frac{d^2v}{dt^2} + 2\frac{f'}{f} \frac{du}{dt} \frac{dv}{dt} &= 0.
\end{align*} \]

4.2 The slope metric on surfaces of revolution

Consider the surface of revolution \( M \) with the parametrization (3.1) and its induced Riemannian metric. We notice that the orthonormal frame in \( T_pM \) at a given point \( p \in M \) is

\[ e_1 = \frac{-1}{\sqrt{(f')^2 + (g')^2}} \frac{\partial}{\partial u}, \quad e_2 = \frac{1}{f} \frac{\partial}{\partial v}. \]

Let \((X, Y)\) be the coordinates of \( T_pM \) with respect to the orthonormal frame \( \{e_1, e_2\} \) and denote the canonical coordinates as \((\dot{u}, \dot{v})\). Then

\[ X = -\sqrt{(f')^2 + (g')^2} \dot{u}, \quad Y = f \dot{v}. \]

We know that the implicit equation of the limacon is

\[ X^2 + Y^2 = c \sqrt{X^2 + Y^2 + aX}, \]

which implies

\[ (f'^2 + g'^2) \dot{u}^2 + f^2 \dot{v}^2 = c \sqrt{(f'^2 + g'^2) \dot{u}^2 + f^2 \dot{v}^2} - a \sqrt{f'^2 + g'^2} \dot{u}. \]

By taking into account that \( a = \sin\epsilon = \frac{1}{\sqrt{(f')^2 + (g')^2}} \), we obtain usual slope metric form with

\[ \alpha = \sqrt{(f'^2 + g'^2) \dot{u}^2 + f^2 \dot{v}^2}, \quad \beta = \dot{u}. \]

4.3 The geodesics of a surface of revolution with first approximate slope metric

Let us recall the formula for a geodesic spray of an arbitrary \((\alpha, \beta)\)-metric [3]

\[ G^i = G^i_\alpha + \alpha Q \delta^i_0 + \Theta\{-2Q\alpha s_0 + r_{00}\} \frac{y^i}{\alpha} + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i, \]

where \( G^i \) and \( G^i_\alpha \) denote the spray coefficients of \( F \) and \( \alpha \) respectively. Also,

\[ Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \left( \frac{\phi - s\phi'}{\phi'} - s \right) \Psi, \]
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As well, we have

\[ \Psi := \frac{\phi''}{2(\phi - s\phi' + (b^2 - s^2)\phi''')} \cdot \]

Here \( \phi(s) = 1 + s + s^2 \), and after computations we get

\[ Q = \frac{2s + 1}{1 - s^2} = \frac{\alpha(2\alpha + \beta)}{\alpha^2 - \beta^2}, \quad \Psi = \frac{1}{1 - 3s^2 + 2b^2} = \frac{\alpha^2}{\alpha^2 - 3\beta^2 + 2\alpha\beta}, \quad \Theta = \frac{1 - 4s^3 - 3s^2}{2(1 + s + s^2)(1 - 3s^2 + 2b^2)} = \frac{\alpha^4 - 3\beta^2\alpha^2 - 4\beta^3\alpha}{(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2 - 2\alpha^2\beta^2)} .\]

In the case of a surface of revolution with slope metric, using (4.2), we have

\[ \alpha = \sqrt{(f^2 + g^2)\dot{u}^2 + f^2\dot{v}^2}, \quad \beta = \dot{u}. \]

We know that

\[ b_{ij} = \frac{\partial b_i}{\partial x^j} - b_k\Gamma^k_{ij}, \quad r_{00} = \frac{\partial b_i}{\partial x^j} y^j y^i - 2b_m G^{\alpha}_m. \]

After calculations for given \( \alpha \) and \( \beta \), we obtain

\[ b_{11} = -\frac{f f'' + g g''}{f^2 + g^2}, \quad b_{22} = \frac{f f'}{f^2 + g^2}, \quad b_{12} = 0, \quad b_{21} = 0, \quad r_{00} = -2G^1_\alpha, \quad s_j = s_j = 0. \]

So, finally, replace these parameters into the geodesic spray equation, and we infer

\[ G^i = G^i_\alpha + (-2G^1_\alpha) \left[ \frac{\alpha^4 - 3\beta^2\alpha^2 - 4\beta^3\alpha}{2(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2)} \frac{y^i}{\alpha} + \frac{\alpha^2}{\alpha^2 - 3\beta^2 + 2\alpha\beta^2} b^j \right] .\]

In particular, we have

\[ G^1 = G^1_\alpha \left[ 1 - \frac{\beta(\alpha^3 - 3\beta^2\alpha - 4\beta^3)}{(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2)} + \frac{\alpha^2}{\alpha^2 - 3\beta^2 + 2\alpha\beta^2} \right], \]

\[ G^2 = G^2_\alpha - G^1_\alpha \frac{\alpha^4 - 3\beta^2\alpha - 4\beta^3}{2(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2)} y^2. \]

Finally, we are ready to give unit speed \( F \)-geodesic equation are as follows :

\[ \frac{d^2u}{ds^2} + 2G^1_\alpha \left[ 1 - \frac{\beta(\alpha^3 - 3\beta^2\alpha - 4\beta^3)}{(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2)} + \frac{\alpha^2}{\alpha^2 - 3\beta^2 + 2\alpha\beta^2} \right] = 0, \]

\[ \frac{d^2v}{ds^2} + 2G^2_\alpha - 2G^1_\alpha \frac{\alpha^3 - 3\beta^2\alpha - 4\beta^3}{2(\alpha^2 + \alpha\beta + \beta^2)(\alpha^2 - 3\beta^2 + 2\alpha\beta^2)} \frac{dv}{ds} = 0. \]

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References


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