Relationships between statistical convergence concepts of complex uncertain sequences

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Abstract. In this paper, a new kind of convergence of complex uncertain sequence is presented, statistically convergent in $p-$distance and statistically convergent completely. We have also establish relationships between statistically convergent completely, statistically convergent in $p-$distance, statistically convergence in measure, statistically convergence in distribution, statistically convergence uniformly almost surely and statistically convergence almost surely.

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1 Introduction

In the absence of samples required to estimate a probability distribution, some domain experts are invited to calculate or assess the belief degree of the eventuality of each event. According to some theorists, the belief degree must be modelled by subjective probability or fuzzy set theory. But there is a chance that both of these theories may lead to counterintuitive results in this case. Therefore, both their uses are inappropriate. To solve those problems, uncertainty theory was founded by Liu [3]. This theory rationally deals with personal belief degrees. In recent time, uncertainty theory has evolved as a special branch of mathematics. Up to now, uncertainty theory has been studied from different directions by You [12], You and Yan [13, 14], Zhang [15] and many others. Since the convergence of sequences plays an important part in the fundamental theory of mathematics, Liu [3] provided some convergence concepts of uncertain sequences and discussed their relationships. Peng [5] developed the new concepts of complex uncertain variables. After that, Chen *et al.* [1] introduced the convergence of complex uncertain sequence and many others, like Tripathy and Nath [11], Nath and Tripathy [4], Saha *et al.* [8, 9] and Roy *et al.* [6] developed this concept according to their different requirements of measurability. There is an extension of convergence of sequence, which is statistical convergence. The details about the statistical convergence of a sequence have been discussed in the articles of Šalát [7], Fridy [2] and Tripathy [10]. Until now, complex uncertain variables have

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been used in statistical convergence by Tripathy and Nath [11]. Motivated by this paper, we have developed some new convergence concepts and relationships between them.

2 Preliminaries

In this section, some basic concepts on uncertainty theory are introduced which will be used throughout the paper.

Definition 2.1. [3] Let $\mathcal{L}$ be a $\sigma-$algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \wedge_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain uncertainty measure of compound event, a product uncertain measure is defined by Liu [3] as follows:
Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, ...$. The product uncertainty measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$, for $k = 1, 2, ...$, respectively.

Definition 2.2. [3] An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 2.3. [11] The complex uncertain sequence (ζ_n) is said to be statistically convergent almost surely to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\}| = 0.$$

Lemma 2.1. *[Tripathy and Nath [11], Proposition 1.] Let, $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, (ζ_n) statistically converges almost surely to ζ if and only if for any $\varepsilon,\ \delta > 0$,we have*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leqslant n : \mathcal{M}(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \delta\right\}\right| = 0.$$

Definition 2.4. [11] The complex uncertain sequence (ζ_n) is said to be statistically convergent in measure to ζ if

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leqslant n:\mathcal{M}(\|\zeta_k-\zeta\|\geqslant\varepsilon)\geqslant\delta\}|=0,$$

for every $\varepsilon,\delta>0$.

Definition 2.5. [11] Let $\phi,\phi_1,\phi_2,....$ be the complex uncertain distributions of complex uncertain variables $\zeta,\zeta_1,\zeta_2,....$ respectively. We say the complex uncertain sequence (ζ_n) statistically converges in distribution to ζ if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\|\phi_k(c)=\phi(c)\|\geq\varepsilon\}|=0,$$

for all c at which $\phi(c)$ is continuous.

Lemma 2.2. *[Tripathy and Nath [11], Theorem 7.] Assume complex uncertain sequence (ζ_n) with real and imaginary parts (ξ_n) and (η_n) respectively, for $n=1,2,....$ If the uncertain sequence (ξ_n) and (η_n) statistically converge in measure to ξ and η respectively, then the complex uncertain sequence (ζ_n) statistically converges in distribution to $\zeta=\xi+i\eta$.*

Lemma 2.3. *[Tripathy and Nath [11], Propostion 2.] Let $\zeta,\zeta_1,\zeta_2,...$ be complex uncertain variables. Then, (ζ_n) statistically converges uniformly almost surely to ζ if and only if for any $\varepsilon,\ \delta>0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leqslant n:\mathcal{M}(\bigcup_{k=n}^{\infty}\|\zeta_k-\zeta\|\geqslant\varepsilon)\geqslant\delta\right\}\right|=0.$$

Lemma 2.4. *[Tripathy and Nath [11],Theorem 9.] If a complex uncertain sequence (ζ_n) statistically converges uniformly almost surely to ζ, then (ζ_n) statistically converges in measure to ζ.*

Here, we introduce two new definitions of statistical convergence of complex uncertain sequence which are statistically convergent in $p-$distance and statistically convergent completely.

Definition 2.6. Let (ζ_n) be a complex uncertain sequence. Then, the sequence (ζ_n) is said to be statistically convergent in $p-$distance to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\}=1$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leqslant n:\ (E[\|\zeta_k-\zeta\|^p])^{\frac{1}{p+1}}\geqslant\varepsilon\right\}\right|=0,$$

for every $\varepsilon>0$.

Definition 2.7. The complex uncertain sequence (ζ_n) is said to be statistically convergent completely to ζ if

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leqslant n:\sum_{k=n}^{\infty}\mathcal{M}(\|\zeta_k-\zeta\|\geqslant\varepsilon)\geqslant\delta\right\}\right|=0,$$

for every $\varepsilon,\ \delta>0$.

3 Main results

Theorem 3.1. *If the complex uncertain sequence (ζ_n) statistically converges in $p-$distance to ζ, then the sequence also statistically converges in measure to ζ.*

Proof. It follows from the Markov inequality that for any given $\varepsilon, \delta > 0$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}(\|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \delta\}| \\ &\quad \leq \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leqslant n : (\frac{E[\|\zeta_k - \zeta\|^p]}{\varepsilon^p}) \geqslant \delta\right\}\right|.\end{aligned}$$

Thus, (ζ_n) is statistically convergent in measure to ζ. □

Remark 3.1. Converse of the theorem is not true in general, i.e. statistically convergent in measure does not imply statistically convergent in $p-$distance in general.

Example 3.2. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n\in\Lambda} \dfrac{1}{n+1}, & if \ \sup\limits_{\gamma_n\in\Lambda} \dfrac{1}{n+1} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\gamma_n\in\Lambda^c} \dfrac{1}{n+1}, & if \ \sup\limits_{\gamma_n\in\Lambda^c} \dfrac{1}{n+1} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & otherwise, \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & if \ \gamma = \gamma_n, \\ 0, & otherwise, \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$, where i denotes the imaginary unit. For some small $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}(\|\zeta_n - \zeta\| \geqslant \varepsilon) \geqslant \delta\}| \\ &\quad = \lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}(\gamma : \|\zeta_n - \zeta\| \geqslant \varepsilon) \geqslant \delta\}| \\ &\quad = \lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}\{\gamma_n\} \geqslant \delta\}| \\ &\quad = 0.\end{aligned}$$

Thus, the sequence (ζ_n) is statistically convergent in measure to ζ. However, for each $n \geq 2$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\| = \|\zeta_n\|$ is

$$\phi_n(x) = \begin{cases} 0, & if \ x < 0, \\ 1 - \frac{1}{n+1}, & if \ 0 \leq x < n+1, \\ 1, & x \geq n+1. \end{cases}$$

So, for each $n \geq 2$, we have

$$\begin{aligned} E[\|\zeta_n - \zeta\|^p] &= \int_o^{n+1} \mathcal{M}\{\|\zeta_n - \zeta\|^p \geq x\}dx \\ &\geq \int_o^{n+1} \mathcal{M}\{\|\zeta_n - \zeta\| \geq x\}dx \\ &= E[\|\zeta_n - \zeta\|], \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : E[\|\zeta_n - \zeta\| - 1]\}| \\ &\quad = \left[\int_o^{n+1} 1 - (1 - \frac{1}{n+1})dx\right] - 1 = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : E[\|\zeta_n - \zeta\|^p - 1]\}| \\ &\quad \geq \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : E[\|\zeta_n - \zeta\| - 1]\}| = 0. \end{aligned}$$

Thus, the sequence is not statistically convergent in $p-$distance.

Theorem 3.2. *Let (ζ_n) be a complex uncertain sequence with real and imaginary parts (ξ_n) and (η_n) respectively, for $n = 1, 2, ...$ If the uncertain sequences (ξ_n) and (η_n) statistically converge in $p-$ distance to ξ and η respectively, then the complex uncertain sequence (ζ_n) statistically converges in distribution to $\zeta = \xi + i\eta$.*

Proof. By Theorem 3.1, statistically convergent in $p-$distance imply statistically convergent in measure. Then, it follows from Lemma 2.2 that the complex uncertain sequence (ζ_n) statistically converges in distribution to $\zeta = \xi + i\eta$. □

Remark 3.3. Converse of the theorem is not true in general, i.e. statistically convergent in distribution does not imply statistically convergent in $p-$distance in general.

Example 3.4. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$. We define a complex uncertain variable by

$$\zeta(\gamma) = \begin{cases} i, & if\ \gamma = \gamma_1, \\ -i, & if\ \gamma = \gamma_2. \end{cases}$$

Define, $\zeta_n \equiv -\zeta$, for $n = 1, 2, ...$ Then, ζ_n and ζ have the same distribution

$$\phi_n(c) = \phi_n(a + ib) = \begin{cases} 0, & if\ a < 0, -\infty < b < +\infty, \\ 0, & if\ a \geq 0, b < -1, \\ \frac{1}{2}, & if\ a \geq 0, -1 < b < 1, \\ 1, & if\ a \geq 0, b \geq 1. \end{cases}$$

Then, the sequence is statistically convergent in distribution to ζ. However, for a given $\varepsilon > 0$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}(\|\zeta_n - \zeta\| \geqslant \varepsilon) \geqslant 1\}|\\&\quad= \lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : \mathcal{M}(\gamma : \|\zeta_n - \zeta\| \geqslant \varepsilon) \geqslant 1\}| = 0.\end{aligned}$$

Since the sequence is not statistically convergent in measure, by Theorem 3.1 it is not statistically convergent in $p-$distance.

Theorem 3.3. *If the complex uncertain sequence (ζ_n) statistically converges completely to ζ, then (ζ_n) statistically converges almost surely to ζ.*

Proof. From the definition of statistically converges completely, we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leqslant n : \sum_{k=n}^{\infty}\mathcal{M}(\|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \delta\right\}\right| = 0.$$

It follows from Axiom 3 that

$$\begin{aligned}&\delta\left(\mathcal{M}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right)\right)\\&\quad\leq \delta\left(\mathcal{M}\left(\bigcup_{k=n}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right)\right)\\&\quad\leq \delta\left(\sum_{k=n}^{\infty}\mathcal{M}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right).\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ on both side of the above inequality, we obtain

$$\delta\left(\mathcal{M}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right)\right) = 0.$$

Therefore, (ζ_n) statistically converges almost surely to ζ. □

Remark 3.5. Converse of the above theorem is not true in general, i.e. statistically convergent almost surely to ζ does not imply statistically convergent completely to ζ.

Example 3.6. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n\in\Lambda}\dfrac{n}{2n+1}, & if \ \sup\limits_{\gamma_n\in\Lambda}\dfrac{n}{2n+1} < \dfrac{1}{2},\\ 1 - \sup\limits_{\gamma_n\in\Lambda^c}\dfrac{n}{2n+1}, & if \ \sup\limits_{\gamma_n\in\Lambda^c}\dfrac{n}{2n+1} < \dfrac{1}{2},\\ \dfrac{1}{2}, & otherwise,\end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} ni, & if\ \gamma = \gamma_n, \\ 0, & otherwise, \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$, where i is the imaginary unit. Then, the sequence (ζ_n) statistically converges almost surely to ζ. However, for a given $\varepsilon > 0$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leqslant n : \mathcal{M}(\|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \frac{1}{2} \right\} \right| \\ &= \lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leqslant n : \mathcal{M}(\gamma : \|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \frac{1}{2} \right\} \right| \\ &= \lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leqslant n : \mathcal{M}\{\gamma_n\} \geqslant \frac{1}{2} \right\} \right| \\ &= 0. \end{aligned}$$

Therefore, the sequence (ζ_n) is not statistically convergent completely to ζ.

Theorem 3.4. *If the complex uncertain sequence (ζ_n) statistically converges completely to ζ, then (ζ_n) statistically converges uniformly almost surely to ζ.*

Proof. If the uncertain sequence (ζ_n) statistically converges completely to ζ then

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leqslant n : \sum_{k=n}^{\infty} \mathcal{M}(\|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \delta \right\} \right| = 0.$$

It follows from Axiom 3 that

$$\delta \left(\mathcal{M} \left(\bigcup_{k=n}^{\infty} \{\|\zeta_k - \zeta\| \geq \varepsilon\} \right) \right) \leq \delta \left(\sum_{k=n}^{\infty} \mathcal{M}\{\|\zeta_k - \zeta\| \geq \varepsilon\} \right).$$

Taking the limit, as $n \longrightarrow \infty$ on both side of the above inequality, we obtain

$$\delta \left(\mathcal{M} \left(\bigcup_{k=n}^{\infty} \{\|\zeta_k - \zeta\| \geq \varepsilon\} \right) \right) = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leqslant n : \mathcal{M}(\bigcup_{k=n}^{\infty} \|\zeta_k - \zeta\| \geqslant \varepsilon) \geqslant \delta \right\} \right| = 0.$$

Hence, (ζ_n) statistically converges uniformly almost surely to ζ. □

Theorem 3.5. *If the complex uncertain sequence (ζ_n) statistically converges completely to ζ, then (ζ_n) statistically converges in measure to ζ.*

Proof. Since statistically convergence completely implies statistically convergence uniformly almost surely, then it follows from Lemma 2.4 that (ζ_n) is statistically convergent in measure to ζ. □

In view of Theorem 3.5 and Lemma 2.2, we state the following result without proof.

Theorem 3.6. *Let the complex uncertain sequence* (ζ_n) *have the real and imaginary parts* (ξ_n) *and* (η_n) *respectively, for* $n = 1, 2, \ldots$. *If the uncertain sequences* (ξ_n) *and* (η_n) *statistically converge completely to* ξ *and* η *respectively, then the complex uncertain sequence* (ζ_n) *statistically converges in distribution to* $\zeta = \xi + i\eta$.

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