On subgroups of the projective special linear group containing the projective special unitary group

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Abstract. Let m be a field of $char \neq 2$, and let M be an algebraic extension of m. To determine maximal subgroups of the projective special linear group $PSL_2(M)$ one of the essential assignments of our research. Therefore, it is necessary to portray intermediate subgroups of $PSL_2(M)$ over a field M generated by projective transvections, and containing the projective special unitary group $PSU_2(M)$ over a field M, specifically of degree 2. But the substantial modernity in the current investigation maybe a description of the intermediate subgroup H, which are generated by projective transvections. Considering H as a normal subgroup lying between $PSL_2(M)$ over M and $PSU_2(m)$ over m, such that $PSU_2(m) \leq H \leq PSL_2(M)$.

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1 Introduction

The point of the present paper is to acquire more conception of classical groups, which are known as projective special linear groups PSL_2 and projective special unitary groups PSU_2 of degree 2 over various fields. They can be defined as images of the corresponding classical group under the canonical homomorphism from $SL_2(M)$ contains $SU_2(M)$ to quotient $SL_2(M)/Z$, where Z is the center of $SL_2(M)$ consists of all scalar matrices whose determinants 1. With the consideration, there is an intersection of PSL_2 with PSU_2 as in [8, 11, 12], concerning some results of maximal subgroups of $PSL_2(M)$, and structure subgroups of $PSL_2(M)$, which contains $PSU_2(M)$. Bashkirov [1] depicted subgroups of SL_2 for arbitrary (infinite) fields of degree 2, and then in [3, 4] described subgroups of the general linear group of degree 4, and degree 7 over various fields, respectively. Sabbar [13] added some results to Bashkirov's results, when characterized subgroups of $PSL_2(K)$ over the field K of degree 2, definitely under his supervision. Also, in [15] we described subgroups of $PSL_2(K)$ that contain a projective root subgroup (see theorem 1.2).

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The generating set for the groups $SL_2(M)$ and $SU_2(M)$, which are known as transvections and unitary transvections, or more accurately elementary transvections and elementary unitary transvections, respectively, is formed by matrices. Shangzhi [17] proved over groups of $SU_n(K)$ in $GL_n(K)$, so he proved in [18] analogous results of unitary group in $GL_2(K)$. In [2] described subgroups of $GL_n(K)$ containing $SU_n(K)$. A significant destination within the current study is to add some considerable consequences of the previous studies, which are including $PSL_2(M)$ and $PSU_2(m)$.

Let G be a group and H is a normal subgroup of G, then we shall symbolize by the quotient $G/H = \{aH \mid a \in G\}$, which is the collection of the distinct left coset of H in G. That is, also the collection of all right cosets too. Since $aH = Ha$ guarantees that the definition of a normal subgroup for every $a \in G$. A foundation of multiplication may be defined on G/H by the formalization $(aH) \cdot (bH) = abH$. Imperatively, this is often valid only when H may be a normal subgroup. The significance of normal subgroups is also asserted by the following results.

Theorem 1.1. *([7]) Let H be a normal subgroup of a group G, then the operation of multiplication $(aH)(bH) = abH$ is the quotient group of G by H forms the set G/H.*

$G = \langle X \rangle$ denote the set X is a generating set for G. Thus the set $X \subseteq G$ generates G for all $g \in G$, every element of G can be expressed as a product $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} ......x_n^{\varepsilon_n}$, for all $x_i \in X$, and $\varepsilon_i = \pm 1$.

Lemma 1.2. *([13]) Let G be a group, H is a normal subgroup of G. Then $G = \langle X \rangle$, which implies $G/H = \langle xH \mid x \in X \rangle$.*

Indeed, if H is an arbitrary (not normal) subgroup of G this is not true. This elucidates, the significant role which normal subgroups plays within the present investigation of groups. For that reason, it is necessary to clarify the relevance between normal subgroups of the linear group and the unitary group by subsequent lemma.

Lemma 1.3. *Let $\Phi : SU_2(M) \to PSL_2(M)$ be a group homomorphism. Let L be a normal subgroup of $PSL_2(M)$, and define $U = \{u \in SU_2(M) : \Phi(u) \in L\}$, then U is a normal subgroup of $SU_2(M)$.*

Proof. Since Φ is a group homomorphism from $SU_2(M)$ to $PSL_2(M)$, so U is nonempty. If $u \in U$, then $\Phi(u) = uZ \in L$, when Z is the center consists of all scalar matrices whose determinants 1, since L is a group, then the inverse of $(\Phi(u))^{-1} = (uZ)^{-1} = u^{-1}Z = \Phi(u^{-1}) \in L$, and thus $u^{-1} \in U$ the construction of U. If $u, y \in U$, then $\Phi(u),\ \Phi(y) \in L$. Also, by Theorem 1.1,

$$\Phi(u)\Phi(y) = (uZ)(yZ) = (uy)Z = \Phi(uy) \in L.$$

Now necessary to show the normality of U, i.e., if $a \in SU_2(M)$, and $u \in U$, then $aua^{-1} \in U,\ \forall a \in SU_2(M)$, applying homomorphism Φ. Thus

$$\Phi(aua^{-1}) = (aua^{-1})Z = aZuZa^{-1}Z = \Phi(a)\Phi(u)\Phi(a^{-1}),$$

suppose that $\Phi(a) = s$, and $\Phi(a^{-1}) = s^{-1}$ for some $s \in L$, since L is a normal subgroup of $PSL_2(M)$. Thus

$$\Phi(aua^{-1}) = s\Phi(u)s^{-1} \in L.$$

Which indicate $\Phi(aua^{-1}) \in L$, and then $aua^{-1} \in U$ by the construction of U. Therefore, U is a normal subgroup of $SU_2(M)$. The lemma is proved entirely. □

Let M^n the set of columns of length n established by elements in M, and indicate by ${}^nM$ the set of rows of length n established by elements in M. Let $\psi \in M^n$ as a column, and $s \in {}^nM$ as a row. Then are defined both product $s\psi$, ψs. Assume $\psi s = 0$, whereas $s\psi$ is an $n \times n$ matrix with entries M. Then $g = I_n + s\psi$ ($n \times n$ matrix) is called a transvection. More accurately, g is a transvection corresponding to the pair $(s, \psi) \in M^n \times {}^nM$. The transvection $I_n + \alpha e_{ij}$ is called an elementary transvection, where e_{ij} is a standard matrix in its (i, j) location unit that has 1 and zeros elsewhere denoted by $t_{ij}(\alpha)$. Thus

$$t_{ij}(\alpha) = 1_n + \alpha e_{ij} \quad (i \neq j,\ \alpha \in M).$$

Each elementary transvection has the determinant is 1, and so all elementary transvections are in the $SL_n(M)$. Thus if $B_{ij}(\lambda)$ is an elementary transvection, then $B_{ij}(\lambda) \in SL(n, M)$.

Note that also the inverse of an elementary transvection is another elementary transvection. In particular, $B_{ij}(\lambda)^{-1} = B_{ij}(-\lambda)$:

$$B_{ij}(\lambda)B_{ij}(-\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

The subgroup of $SL_n(M)$ generated by all elementary transvections denoted by $E_n(M)$ is called the elementary subgroup. Thus by definition

$$E_n(M) = \langle t_{ij}(\alpha) \mid \alpha \in M,\ 1 \leqslant i \neq j \leqslant n \rangle .$$

The linear transvection in $SL(n, M)$ is a map of the form : $v \mapsto v + \theta(v).u$, when u is a non-zero vector in V, and θ is a linear form on V with $\theta(u) = 0$. For any pair dimension 1 and $n - 1$, the commutative subgroup of $SL(n, M)$ is generated by all transvections. The above linear transvection it lies in $SU(n, M)$ if and only if u is isotropic, and $\theta(v) = \lambda(u, v)$ for some $\lambda \in M^*$ such that $\lambda = -\overline{\lambda}$. The unitary transvection exists if Witt index ν great than zero, or $\nu \geq 1$, and then the form : $v \mapsto v + a\beta(v, u)u$, where $a \in M$ is an arbitrary symmetric element that satisfies $a + \overline{a} = 0$, and u is an arbitrary isotropic vector. Conversely, every transvection of this form is in the unitary group, and the hyperplane of points of vector space V invariant by the transvection is the hyperplane orthogonal to u. In particular, V contains isotropic vectors if and only if $SU(n, M)$ contains transvections. In [9] proved the special unitary group generated by transvection . Sabbar [16] depicted subgroups of the special unitary group contians all elementary unitary transvections (see theorem 11.15). In [14] we described a significant element of the current study called a projective transvection.

Definition 1.1. An element $y \in PSL_2(m)$ as a coset of Z in $SL_2(m)$ has a representative which is a transvection of $SL_2(m)$, which is a coset $hZ = \{h, -h\}$ is called a projective transvection, whereas h is some element in $SL_2(m)$ has a representative that is a transvection of $SL_2(m)$. Thus an element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z \in PSL_2(m),$$

is a projective transvection.

A matrix g belongs to the center of $SL_n(m)$ or $SU_n(m)$ if and only if g as the form αI_n such that α is an element of m, and $\alpha^n = 1$. The center of $SL_2(m)$ or $SU_2(m)$ is the subgroup of all matrices αI_2, and $\alpha^2 = 1$, where I_2 is the identity matrix. Presume that the characteristic of m is different from 2, then the equation $\alpha^2 = 1$ has precisely two roots ± 1, and subsequently, the center of $SL_2(m)$ and $SU_2(m)$ is the subgroup

$$Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \{\pm I_2\}\,.$$

2 Maximal subgroup of $PSL_2(M)$

A subgroup $H \leq G$ is a maximal normal subgroup of G if there is no a normal subgroup N of G with $H < N < G$. On the other hand, H is a maximal normal subgroup of G if and only if G/H has no normal subgroups (other than itself and 1). We may restate H is a maximal normal subgroup of G if and only if G/H is simple. If H is maximal in G, then every conjugate gHg^{-1} of H is maximal in G. Any a maximal subgroup must be a proper normal subgroup.

King [10] described maximal subgroups of the classical groups, so in [11] proved a significant proposition by using an intermediate subgroup X of $SL_2(K)$ contains $SU_2(K_0)$, when X as a normal subgroup contains every transvection, and K is an extension field of the field K_0 of degree 2, and concluded $SU_2(K_0)$ is maximal of $SL_2(K)$. Can be written this proposition as follows.

Proposition 2.1. *Let K be an extension field of a field K_0. If $K \neq GF(4)$, $GF(9)$ or $GF(25)$ such that $SU_2(K_0) \leq X \leq SL_2(K)$, when X as a normal subgroup of $SL_2(K)$. If X is generated by transvections, then $X = SL_2(K)$.*

So he considered the groups $GU_n(K)$, $PGU_n(K)$, and $PSU_n(K)$, with specified some conditions for them to be maximal in $GL_n(K)$, $PGL_n(K)$, and $PSL_n(K)$, respectively, when K is finite. In this section, if $M \neq GF(4), GF(9),$ *or* $GF(25)$, then according to the next theorem, we can introducing a substantial study of $PSU_2(M)$ to be maximal in $PSL_2(M)$ over the finite field M.

Theorem 2.2. *If M is a field of char $\neq 2$, then $PSU_2(M)$ the quotient group of $SU_2(M)$ by the center Z its subgroup of the scalar matrix, it is a maximal subgroup of $PSL_2(M)$ the quotient group of $SL_2(M)$ by the center Z its subgroup of the scalar matrix.*

Proof. Let X be a normal subgroup generated by projective transvections in the quotient group of $SL_2(M)$ by its subgroup of the scalar matrix, that contains the quotient group of $SU_2(M)$ by its subgroup of the scalar matrix, such that

$$PSU_2(M) \leq X \leq PSL_2(M),$$

therefore, if have been proved $X = PSL_2(M)$, then $PSU_2(M)$ is a maximal subgroup of $PSL_2(M)$. Let Z be the center of $SL_2(M)$ and $SU_2(M)$ consists of all scalar matrices whose determinants 1. Hence $PSL_2(M) = SL_2(M)/Z$, and $PSU_2(M) = SU_2(M)/Z$. Thus

$$SU_2(M)/Z \leq X \leq SL_2(M)/Z.$$

Indicate by the canonical homomorphism from $SL_2(M)$ onto $SL_2(M)/Z$. In other word, $\Omega : SL_2(M) \to SL_2(M)/Z$ is the homomorphism, and $\Omega(u) = uZ$ for every element $u \in SL_2(M)$, that is, the coset $uZ \in SL_2(M)/Z = PSL_2(M)$. Let N be a full preimage of X under Ω as a normal subgroup of $SL_2(M)$ contains $SU_2(M)$. Therefore,

$$N = \Omega^{-1}(X) = \{n \in SL_2(M) : \Omega(n) \in X\}.$$

Since $PSU_2(M) \leq X$, and N the full preimage of X contains $SU_2(M)$ the full preimage of $PSU_2(M)$. Thus

$$SU_2(M) \leq N \leq SL_2(M).$$

Since X is a normal subgroup of $SL_2(M)/Z$ generated by projective transvections contains $SU_2(M)/Z$, and N is a full preimage of X under Ω. Thus we deduce N is a normal subgroup of $SL_2(M)$ generated by transvections contains $SU_2(M)$. Therefore, by Proposition 2.1, $SU_2(M)$ is properly contained in the proper subgroup of $SL_2(M)$, and $N = SL_2(M)$. Consequently, $SU_2(M)$ is a maximal in $SL_2(M)$. Applying the canonical homomorphism under Ω: $\Omega(N) = \Omega(SL_2(M))$, since N the full preimage of X. Therefore, $\Omega(N) = X$, and $\Omega(SL_2(M)) = SL_2(M)/Z = PSL_2(M)$, thus $X = PSL_2(M)$. Which is the proof of the theorem is completed. □

3 Intermediate subgroups

The generations of $SL_2(M)$ over a field M is the central problem, which we are interested. The proof of this result appears in [1] described the subgroups of $SL_2(K)$ generated by transvections. This consequence is established as follows.

Theorem 3.1. *If K is an algebraic extension of k, such that k is a field of char $\neq 2$, and $SL_2(k) \leq G \leq SL_2(K)$, then G contains the group $SL_2(L)$ as a normal subgroup such that L is a subfield of K contains k, $(k \leq L \leq K)$. Then $G = SL_2(L)$, if G is generated by transvections.*

In [5] acquired similar results of the Theorem 3.1, when described subgroups of $GL_2(K)$ contains $SL_2(k)$ over commutative rings (e.g., see Theorem 1), so in [6] acquired analogous results (see Theorems 2.3, 4.1, and 5.1). In [13] we obtained some additional results of Bashkirov's Theorems, with regard to the case of $PSL_2(K)$ over a field K, when we described intermediate subgroups lying between $PSL_2(K)$ and $PSL_2(k)$ over various fields, when K is an extension field of the field k.

The purpose of the current study is to acquire some supplementary results of Bashkirove's Theorem and Sabbar's Theorem, with consideration to the status $PSL_2(M)$ and $PSU_2(m)$.

Let m be a field of *char* $\neq 2$, and M is an algebraic extension of m. A describe of intermediate subgroups lying between $PSL_2(M)$ and $PSU_2(m)$ over various fields is the our main goal in the paper with some exceptions, when $M \neq GF(4)$, $GF(9)$, *or* $GF(25)$ is a finite field. The eventual consequence is to prove the subsequent theorem.

Theorem 3.2. *Let m be a field of char $\neq 2$, and let M be an algebraic extension of m. If $PSU_2(m) \leq H \leq PSL_2(M)$, when H contains as a normal subgroup the group $PSL_2(D)$, where D is a subfield of M contains m such that $(m \leq D \leq M)$. If H is generated by projective transvections, then $H = PSL_2(D)$.*

Proof. Let Z be the center as a normal subgroup of the scalar matrix of $SL_2(M)$ and $SU_2(m)$, then $PSL_2(M) = SL_2(M)/Z$, and $PSU_2(m) = SU_2(m)/Z$. Thus we can reformulate our assumption

$$SU_2(m)/Z \leq H \leq SL_2(M)/Z.$$

Denote by Ψ the canonical homomorphism from $SU_2(M)$ onto $SL_2(M)/Z$. In other words, it is the homomorphism

$$\Psi : SU_2(M) \rightarrow SL_2(M)/Z,$$

since H is a normal subgroup of $PSL_2(M)$, then by Lemma 1.3, there exist a subgroup F as a normal subgroup of $SU_2(M)$ such that $F = \{f \in SU_2(M) : \Psi(f) \in H\}$, that is, $\Psi(f) = fZ$ for every $f \in SU_2(M)$, and fZ is the coset belongs to H. Now let G be the full preimage of H contains the full preimage of $PSL_2(m)$ under Ψ. Since $PSU_2(m) \leq H$. Thus

$$SU_2(m) \leq G \leq SL_2(M).$$

Using Theorem 3.1, there exists a subfield D of the field M such that $m \subseteq D$, and G contains $SL_2(D)$ as a normal subgroup of G. The form $\Psi(g)$ for some $g \in G$ is any element $a \in H$, while could be written $\Psi(h)$ as an element $b \in PSL_2(D)$, where $h \in SL_2(D)$. Since $SL_2(D)$ is a normal subgroup of G. Therefore, $g^{-1}hg \in SL_2(D)$, and by applying Ψ gives

$$\Psi(g^{-1}hg) \in \Psi(SL_2(D)) = PSL_2(D).$$

But Ψ is a homomorphism. Therefore,

$$\Psi(g^{-1}hg) = \Psi(g^{-1})\Psi(h)\Psi(g) = a^{-1}ba.$$

In this case, we conclude that $a^{-1}ba \in PSL_2(D)$, for any $a \in H$ and $b \in PSL_2(D)$. Thus by Sabbar's Theorem, H is generated by projective transvections containing the group $PSL_2(D) = SL_2(D)/Z$ as the normal subgroup of H. If we proving G is generated by transvections, then by Theorem 3.1, $G = SL_2(D)$. Applying Ψ to this relation gives $\Psi(G) = \Psi(SL_2(D))$. For achieving that, let us take an arbitrary element $y \in G$ such that $\Psi(y) = yZ$, and according to our assumption yZ is a product of projective transvections, $x_1Zx_2Z....x_SZ$, we say that $yZ = (x_1Z)(x_2Z)....(x_SZ)$. Using Theorem 1.1 of the coset multiplication, $yZ = (x_1x_2....x_S)Z$, and recalling that $Z = \{\pm I_2\}$, we conclude

$$y \in (x_1x_2....x_s)\{I_2, \ -I_2\} = \{I_2x_1x_2....x_s, \ -I_2 \ x_1x_2....x_s\},$$

so either $y = I_2x_1x_2....x_s$, or else $y = -I_2x_1x_2....x_s$. In the first case, y is visibly a product of transvections because of all $x_1x_2....x_s$ are transvections. In the second case, $y = -I_2 x_1 x_2 .... x_s$, we knew that $-I_2$ is generated by transvections that belong to the group $SU_2(m)$. Thence, $-I_2 x_1 x_2 .... x_s$ is a product of transvections too, thus y is the product of transvections. But y of G is an arbitrary element, subsequently, we infer that the group G is generated by the transvections. As stated by Theorem 3.1, $G = SL_2(D)$. Applying the homomorphism Ψ, yields $\Psi(G) = \Psi(SL_2(D))$, since G is the full preimage of H generated by transvection, that implies $\Psi(G) = H$, and $\Psi(SL_2(D)) = PSL_2(D)$. Therefore, $H = PSL_2(D)$. The theorem is proved completely. □

Conclusions

The subgroups of $SU_2(M)$, are described by canonical homomorphism from $SU_2(M)$ onto $PSL_2(M)$. The maximal subgroups that are generated by projective transvections of $PSL_2(M)$ contains $PSU_2(M)$, are described. If a field m of $char \neq 2$ and M be an algebraic extension of a field m, the intermediate subgroups that are contained between $PSU_2(m)$ and $PSL_2(M)$, are described.

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