On sharp Chebyshev polynomial bounds for a general subclass of bi-univalent functions

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Abstract. In the present paper, we introduce a subclass $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu, \sigma, \rho)$ of the bi-univalent function class Σ, which is defined in the open unit disk $\mathcal{U}$ using the Chebyshev polynomials along with subordination. Further, we obtain sharp bounds for the initial coefficients a_2, a_3 and the Fekete-Szegö functional $a_3 - \delta a_2^2$ for the functions belong to this subclass.

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1 Introduction

Let $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ be the standard open unit disc in the complex plane. Consider the following well-known function classes:

$$\mathcal{W} = \{f : \mathcal{U} \to \mathcal{C} : f \text{ is analytic in the open unit disk } \mathcal{U}\},$$
$$\mathcal{A} = \{f \in \mathcal{W} : f \text{ is normalized by } f(0) = f^{'}(0) - 1 = 0\},$$

where the class $\mathcal{A}$ consist the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}, a_n \in \mathcal{C}) \tag{1.1}$$

and the class $\mathcal{S}$ is defined as:

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathcal{U}\}.$$

If the function f given by (1.1) and its inverse $L = f^{-1}$ are univalent in $\mathcal{U}$, we say that the function f is bi-univalent in $\mathcal{U}$. Let Σ define the class of all the functions that are bi-univalent in $\mathcal{U}$.

The Koebe one-quarter theorem [8] asserts that the image of $\mathcal{U}$ under each univalent function f in $\mathcal{S}$ contains a disk of radius 1/4. According to this, any function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U}),$$

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and

$$f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4).$$

Indeed, the analytic extension of f^{-1} to $\mathcal{U}$ is

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

For a brief history of the class Σ and various subclasses of it, see the work by Srivastava et al. [22] and Frasin and Aouf [11], see also [1, 2, 3, 10, 14, 18, 19, 20].

Given two functions $f, h \in \mathcal{A}$. The function $f(z)$ is said to be subordinate to $h(z)$ in $\mathcal{U}$, written $f(z) \prec h(z)$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$, with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for all } z \in \mathcal{U},$$

such that

$$f(z) = h(\omega(z)) \text{ for all } z \in \mathcal{U}.$$

Also, if h is univalent in $\mathcal{U}$, then we have the following equivalence (see [16, 23]):

$$f(z) \prec h(z), \ (z \in \mathcal{U}) \quad \Longleftrightarrow \quad f(\mathcal{U}) \subset h(\mathcal{U}) \quad \text{and} \quad f(0) = h(0).$$

The classical Chebyshev polynomials of degree m of the first and second kinds, which are denoted respectively by $\mathcal{T}_m(l)$ and $U_m(l)$, have generated a great deal of interest in recent years. These orthogonal polynomials, in a real variable l and a complex variable z, have played an important role in applied mathematics, approximation theory and numerical analysis. For brief history of the Chebyshev polynomials of first and second kind and their applications, see [4, 5, 6, 13, 21, 12]. The Chebyshev polynomials of the first and the second kinds are orthogonal for $l \in [-1, 1]$ and defined as follows:

Definition 1.1. The Chebyshev polynomials of the first kinds are defined by the following three-terms recurrence relations:

$$\begin{aligned} \mathcal{T}_0(l) &= 1, \\ \mathcal{T}_1(l) &= l, \\ \mathcal{T}_{m+1}(l) &= 2l\mathcal{T}_m(l) - \mathcal{T}_{m-1}(l) \quad (m \in \mathcal{N} := \{1, 2, 3, \cdots\}). \end{aligned}$$

The first few of the Chebyshev polynomials of the first kind are

$$\mathcal{T}_2(l) = 2l^2 - 1, \quad \mathcal{T}_3(l) = 4l^3 - 3l, \quad \mathcal{T}_4(l) = 8t^4 - 8t^2 + 1, \cdots .$$

The generating function for Chebyshev polynomials of the first kind $\mathcal{T}_m(l)$, is given by:

$$F(z, l) = \frac{1 - lz}{1 - 2lz + z^2} = \sum_{m=0}^{\infty} \mathcal{T}_m(l) z^m \quad (z \in \mathcal{U}).$$

Definition 1.2. The Chebyshev polynomials of the second kinds are defined by the following three-terms recurrence relations:

$$\begin{aligned} U_0(l) &= 1, \\ U_1(l) &= 2l, \\ U_{m+1}(l) &= 2lU_m(l) - U_{m-1}(l) \quad (m \in \mathcal{N} := \{1, 2, 3, \cdots\}). \end{aligned}$$

The first few of the Chebyshev polynomials of the second kind are

$$U_2(l) = 4l^2 - 1, \quad U_3(l) = 8l^3 - 4l, \quad U_4(l) = 16t^4 - 12t^2 + 1, \cdots .$$

The generating function for Chebyshev polynomials of the second kind $U_m(l)$, is given by:

$$\mathcal{H}(z,l) := \frac{1}{1-2lz+z^2} = \sum_{m=0}^{\infty} U_m(l) z^m \quad (z \in \mathcal{U}).$$

The Chebyshev polynomials of the first and the second kinds are connected by the following relations:

$$\frac{d\mathcal{T}_m(l)}{dt} = mU_{m-1}(l); \quad \mathcal{T}_m(l) = U_m(l) - lU_{m-1}(l); \quad 2\mathcal{T}_m(l) = U_m(l) - U_{m-2}(l).$$

Definition 1.3. For $0 \le \nu < 1$, $\sigma \ge 1$, $|\rho| \le 1$ but $\rho \ne 1$ and $l \in (1/2, 1]$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu, \sigma, \rho)$ if the following subordination holds for all $z, w \in \mathcal{U}$:

$$\frac{((1-\rho)z)^{1-\nu}(f'(z))^{\sigma}}{(f(z)-f(\rho z))^{1-\nu}} \prec \mathcal{H}(z,l) := \frac{1}{1-2lz+z^2} \tag{1.3}$$

and

$$\frac{((1-\rho)w)^{1-\nu}(L'(w))^{\sigma}}{(L(w)-L(\rho w))^{1-\nu}} \prec \mathcal{H}(w,l) := \frac{1}{1-2lw+w^2}, \tag{1.4}$$

where the function $L(w) = f^{-1}(w)$ is defined by (1.2).

Remark 1.4. For $\nu = 0$ and $\rho = 0$, we have the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(0, \sigma, 0) := \mathfrak{B}_{\Sigma}^{\mathcal{H}}(\sigma)$ of the functions given by (1.1), that satisfies the following subordination conditions for all $z, w \in \mathcal{U}$:

$$\frac{z(f'(z))^{\sigma}}{f(z)} \prec \mathcal{H}(z,l) := \frac{1}{1-2lz+z^2}$$

and

$$\frac{w(L'(w))^{\sigma}}{L(w)} \prec \mathcal{H}(w,l) := \frac{1}{1-2lw+w^2}.$$

Remark 1.5. For $\nu = \rho = 0$ and $\sigma = 1$, we have the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(0, 1, 0) := \mathfrak{B}_{\Sigma}^{\mathcal{H}}$ of the functions given by (1.1), that satisfies the following subordination conditions for all $z, w \in \mathcal{U}$:

$$\frac{zf'(z)}{f(z)} \prec \mathcal{H}(z,l) := \frac{1}{1-2lz+z^2}$$

and

$$\frac{wL'(w)}{L(w)} \prec \mathcal{H}(w,l) := \frac{1}{1-2lw+w^2}.$$

Motivated by the earlier work of Dziok et al. [9], the main focus of this work is to utilize the Chebyshev polynomial expansions to obtain sharp bounds for the initial coefficients a_2 and a_3 and to solve the Feketa-Szegö problem for the subclass $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu, \sigma, \rho)$ of bi-univalent functions (see, for example, [5, 7, 9, 15]).

2 Coefficient bounds for the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu,\sigma,\rho)$

We begin with the following result involving the sharp initial coefficient bounds and the Feketa-Szegö problem for the function class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu,\sigma,\rho)$.

Theorem 2.1. *Let the function $f(z)\in\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu,\sigma,\rho)$ is given by (1.1). Then*

$$(2.1)\qquad |a_2|\le\begin{cases}\sqrt{\frac{4l}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}}, & l\in(\frac{1}{2},\frac{1+\sqrt{5}}{4}]\\ \sqrt{\frac{2(4l^2-1)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}}, & l\in[\frac{1+\sqrt{5}}{4},1],\end{cases}$$

$$(2.2)\qquad |a_3|\le\begin{cases}\frac{4l}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & l\in(\frac{1}{2},\frac{1+\sqrt{5}}{4}]\\ \frac{2\left(4l^2-1\right)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & l\in[\frac{1+\sqrt{5}}{4},1],\end{cases}$$

and for some $\delta\in\mathbb{R}$,

$$(2.3)\qquad |a_3-\delta a_2^2|\le\begin{cases}\frac{4l|1-\delta|}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & l\in(\frac{1}{2},\frac{1+\sqrt{5}}{4}]\\ \frac{2\left(4l^2-1\right)|1-\delta|}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & l\in[\frac{1+\sqrt{5}}{4},1].\end{cases}$$

All the three inequalities are sharp.

Proof. Let $f\in\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\nu,\sigma,\rho)$. Thus, from (1.3) and (1.4), we have

$$(2.4)\qquad \frac{((1-\rho)z)^{1-\nu}(f'(z))^{\sigma}}{(f(z)-f(\rho z))^{1-\nu}}=1+U_1(l)s(z)+U_2(l)s^2(z)+\cdots$$

and

$$(2.5)\qquad \frac{((1-\rho)w)^{1-\nu}(L'(w))^{\sigma}}{(L(w)-L(\rho w))^{1-\nu}}=1+U_1(l)t(w)+U_2(l)t^2(w)+\cdots,$$

for some analytic functions:

$$s(z)=u_1z+u_2z^2+u_3z^3+\cdots\qquad(z\in\mathcal{U})$$

and

$$t(w)=v_1w+v_2w^2+v_3w^3+\cdots\qquad(w\in\mathcal{U}),$$

such that $s(0)=t(0)=0$, $|s(z)|<1$ $(z\in\mathcal{U})$ and $|t(w)|<1$ $(w\in\mathcal{U})$. It is well known that for such functions $s(z)$ and $t(w)$, (see Nehari [17]), we have:

$$(2.6)\qquad |u_1|\le1,\quad |v_1|\le1,\quad |u_2|\le1-|u_1|^2,\quad |v_2|\le1-|v_1|^2.$$

It follows from (2.4) and (2.5) that

$$\frac{((1-\rho)z)^{1-\nu}(f'(z))^{\sigma}}{(f(z)-f(\rho z))^{1-\nu}}=1+U_1(l)u_1z+[U_1(l)u_2+U_2(l)u_1^2]z^2+\cdots$$

and

$$\frac{((1-\rho)w)^{1-\nu}(L'(w))^{\sigma}}{(L(w)-L(\rho w))^{1-\nu}} = 1 + U_1(l)v_1 w + [U_1(l)v_2 + U_2(l)v_1^2]w^2 + \cdots .$$

On short calculations and then equating the coefficients, we obtain

$$[2\sigma - (1-\nu)(1+\rho)]\, a_2 = U_1(l)u_1, \tag{2.7}$$

$$\begin{aligned}\left[3\sigma - (1-\nu)(1+\rho+\rho^2)\right] a_3 + \Big[2\sigma(\sigma - 1 - (1-\nu)(1+\rho)) \\ + \frac{\nu(1-\nu)(1+\rho)^2}{2} + (1+\rho)^2(1-\nu)^2\Big] a_2^2 = U_1(l)u_2 + U_2(l)u_1^2,\end{aligned} \tag{2.8}$$

$$-\,[2\sigma - (1-\nu)(1+\rho)]\, a_2 = U_1(l)v_1, \tag{2.9}$$

$$\begin{aligned}\Big[2\sigma^2 + 4\sigma + \frac{\nu(1-\nu)(1+\rho)^2}{2} + (1+\rho)^2(1-\nu)^2 - 2(1-\nu)(1+\rho+\rho^2) \\ - 2\sigma(1-\nu)(1+\rho)\Big] a_2^2 + \left[(1-\nu)(1+\rho+\rho^2) - 3\sigma\right] a_3 = U_1(l)v_2 + U_2(l)v_1^2.\end{aligned} \tag{2.10}$$

From (2.7) and (2.9), we have

$$u_1 = -v_1. \tag{2.11}$$

By summing (2.8) and (2.10), we get

$$\begin{aligned}\left[4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]\right] a_2^2 \\ = U_1(l)(u_2+v_2) + U_2(l)(u_1^2+v_1^2).\end{aligned} \tag{2.12}$$

Or equivalently,

$$a_2^2 = \frac{U_1(l)(u_2+v_2) + U_2(l)(u_1^2+v_1^2)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]}. \tag{2.13}$$

By using (2.6) and (2.11), we have

$$\begin{aligned}|a_2|^2 &\le \frac{2\left[\left(1-|u_1|^2\right)U_1(l) + |u_1|^2 U_2(l)\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|} \\ &= \frac{2\left[U_1(l) + (U_2(l) - U_1(l))\,|u_1|^2\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|},\end{aligned}$$

which, according to the cases when the coefficient of $|u_1|^2$ is positive or negative along with $|u_1| \le 1$ gives us:

$$|a_2|^2 \le \begin{cases} \frac{2U_1(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_1(l) \ge U_2(l) \\ \frac{2\,U_2(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_2(l) \ge U_1(l). \end{cases}$$

This, for $l \in (\frac{1}{2}, 1]$ and the values of $U_1(l)$ and $U_2(l)$ proves the inequality (2.1). Next, By subtracting (2.10) from (2.8), we get

$$a_3 = a_2^2 + \frac{U_1(l)\,(u_2 - v_2)}{2\,[3\sigma - (1-\nu)\,(1+\rho+\rho^2)]} \tag{2.14}$$
$$= \frac{U_1(l)(u_2+v_2) + U_2(l)(u_1^2+v_1^2)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]}$$
$$+ \frac{U_1(l)\,(u_2 - v_2)}{2\,[3\sigma - (1-\nu)\,(1+\rho+\rho^2)]},$$

which on cross multiplying and compiling the coefficients of u_2 and v_2 gives

$$a_3 = A_1/B_1,$$

where

$$\begin{aligned} A_1 = {} & U_1(l)u_2\big[4\sigma^2 + 8\sigma + (1-\nu)\,[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 4(1+\rho+\rho^2)]\big] \\ & + U_1(l)v_2\big[-4\sigma^2 + 4\sigma - (1-\nu)\,[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho)]\big] \\ & + 2U_2(l)\left(u_1^2+v_1^2\right)\left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right] \end{aligned}$$

and

$$\begin{aligned} B_1 = {} & 2\big[4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]\big] \\ & \left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right]. \end{aligned}$$

This equation of a_3, on some simple computations using (2.6) and (2.11) yields

$$|a_3| \leq \frac{2\left[\left(1-|u_1|^2\right)U_1(l) + |u_1|^2 U_2(l)\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|}$$
$$= \frac{2\left[U_1(l) + (U_2(l) - U_1(l))\,|u_1|^2\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|},$$

which is same as in the case of $|a_2|^2$ and leads to the result:

$$|a_3| \leq \begin{cases} \frac{2U_1(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_1(l) \geq U_2(l) \\ \frac{2\,U_2(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_2(l) \geq U_1(l). \end{cases}$$

This, for $l \in (\frac{1}{2}, 1]$ and the values of $U_1(l)$ and $U_2(l)$ proves the inequality (2.2). Finally, for the Feketa-Szegö problem, from (2.14) we have

$$a_3 - \delta a_2^2 = (1-\delta)\,a_2^2 + \frac{U_1(l)\,(u_2 - v_2)}{2\,[3\sigma - (1-\nu)\,(1+\rho+\rho^2)]}$$
$$= \frac{(1-\delta)\left[U_1(l)(u_2+v_2) + U_2(l)(u_1^2+v_1^2)\right]}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]}$$
$$+ \frac{U_1(l)\,(u_2 - v_2)}{2\,[3\sigma - (1-\nu)\,(1+\rho+\rho^2)]},$$

which on cross multiplying and compiling the coefficients of u_2 and v_2 gives

$$a_3 - \delta a_2^2 = A_2/B_2,$$

where

$$\begin{aligned} A_2 = U_1(l)u_2\big[&4\sigma^2 + 8\sigma + (1-\nu)\left[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 4(1+\rho+\rho^2)\right] \\ &- 2\delta\left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right]\big] + \\ &U_1(l)v_2\big[-4\sigma^2 + 4\sigma - (1-\nu)\left[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho)\right] \\ &- 2\delta\left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right]\big] + \\ &2\left(1-\delta\right)U_2(l)\left(u_1^2 + v_1^2\right)\left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right] \end{aligned}$$

and

$$\begin{aligned} B_2 = 2&\left[4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]\right] \\ &\left[3\sigma - (1-\nu)\left(1+\rho+\rho^2\right)\right]. \end{aligned}$$

In light of (2.6) and (2.11), the above equation of $a_3 - \delta a_2^2$ yields

$$\begin{aligned} |a_3 - \delta a_2^2| &\le \frac{2|1-\delta|\left[\left(1-|u_1|^2\right)U_1(l) + |u_1|^2 U_2(l)\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|} \\ &= \frac{2|1-\delta|\left[U_1(l) + (U_2(l) - U_1(l))\,|u_1|^2\right]}{|4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]|}. \end{aligned}$$

This leads to the result:

$$|a_3 - \delta a_2^2| \le \begin{cases} \frac{2|1-\delta|U_1(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_1(l) \ge U_2(l) \\ \frac{2|1-\delta|\,U_2(l)}{|4\sigma^2+2\sigma+(1-\nu)[(1+\rho)^2(2-\nu)-4\sigma(1+\rho)-2(1+\rho+\rho^2)]|}, & U_2(l) \ge U_1(l), \end{cases}$$

which, for the values of $U_1(l)$, $U_2(l)$ and $l \in (\frac{1}{2}, 1]$ proves the inequality (2.3). Also for $\delta = 0$, we get the bound on a_3 and for $\delta = 1$, we get the result $|a_3| = |a_2|^2$.

If we set $s(z) = z^2$ and $t(w) = w^2$ in (2.4) and (2.5) respectively, we get the function $\phi(z) \in \mathfrak{B}_\Sigma^{\mathcal{H}}(\nu, \sigma, \rho)$ given by

$$\begin{aligned} \phi = z &+ \left[\frac{2U_1(l)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]}\right]^{\frac{1}{2}} z^2 \\ &+ \frac{2U_1(l)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]} z^3 + \cdots, \end{aligned}$$

which proves the sharpness for the first part of all the three inequalities. Whereas, if we set $s(z) = z$ and $t(w) = -w$ in (2.4) and (2.5) respectively, we get the function $\psi(z) \in \mathfrak{B}_\Sigma^{\mathcal{H}}(\nu, \sigma, \rho)$ given by

$$\begin{aligned} \psi = z &+ \left[\frac{2U_2(l)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]}\right]^{\frac{1}{2}} z^2 \\ &+ \frac{2U_2(l)}{4\sigma^2 + 2\sigma + (1-\nu)[(1+\rho)^2(2-\nu) - 4\sigma(1+\rho) - 2(1+\rho+\rho^2)]} z^3 + \cdots, \end{aligned}$$

which proves the sharpness for the second part of all the three inequalities. □

Taking $\nu = 0$ and $\rho = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let the function f given by (1.1) be in the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}(\sigma)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2l}{2\sigma^2-\sigma}}, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ \sqrt{\frac{4l^2-1}{2\sigma^2-\sigma}}, & l \in [\frac{1+\sqrt{5}}{4}, 1], \end{cases}$$

$$|a_3| \leq \begin{cases} \frac{2l}{2\sigma^2-\sigma}, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ \frac{4l^2-1}{2\sigma^2-\sigma}, & l \in [\frac{1+\sqrt{5}}{4}, 1], \end{cases}$$

and for some $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{2l|1-\delta|}{2\sigma^2-\sigma}, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ \frac{\left(4l^2-1\right)|1-\delta|}{2\sigma^2-\sigma}, & l \in [\frac{1+\sqrt{5}}{4}, 1]. \end{cases}$$

All the three inequalities are sharp.

Taking $\nu = \rho = 0$ and $\sigma = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let the function f given by (1.1) be in the class $\mathfrak{B}_{\Sigma}^{\mathcal{H}}$. Then*

$$|a_2| \leq \begin{cases} \sqrt{2l}, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ \sqrt{4l^2-1}, & l \in [\frac{1+\sqrt{5}}{4}, 1], \end{cases}$$

$$|a_3| \leq \begin{cases} 2l, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ 4l^2-1, & l \in [\frac{1+\sqrt{5}}{4}, 1], \end{cases}$$

and for some $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \begin{cases} 2l|1-\delta|, & l \in (\frac{1}{2}, \frac{1+\sqrt{5}}{4}] \\ \left(4l^2-1\right)|1-\delta|, & l \in [\frac{1+\sqrt{5}}{4}, 1]. \end{cases}$$

All the three inequalities are sharp.

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