Bifurcation analysis of a transportation network for energy with distributed delayed carrying capacity

B.A. Pansera, M. Gangemi, L. Guerrini, M. Ferrara

Abstract. Recently, [10] have developed a mathematical model of an economy viewed as a transport network for energy. Based on [24], [5] have adapted their ideas and proposed a generalization by introducing a logistic-type equation for population with delayed carrying capacity. This study examines the consequences of replacing time delays with distributed time delays in their model. The local asymptotic stability of the equilibrium point is studied by analyzing the corresponding characteristic equation. It is found that the destructive impact of the agents on the carrying capacity leads the system dynamic behavior to exhibit stability switches and Hopf bifurcations to occur.

M.S.C. 2010: 34K18, 34K20, 91B62.
Key words: Bifurcation; energy; network; delayed carrying capacity.

1 Introduction

The Solow model ([19]) has been a point of reference of economic growth since the 1950s. It attempted to explain how increased capital stock generates greater per capita production and revealed the role of savings, population growth and technological change in the long-term determination of labor productivity. In this regard, various augmented versions of this model were built over the decades, including among others the contribution of [17], employing a physical and an intangible capital (human capital), and the Solow-[**?**] residual model, addressing issues concerning intangibles, such as the contribution of intangible capital to output growth and how does the inclusion of intangibles affects the allocation of output growth between capital formation and multifactor productivity growth (see e.g. [7]).

[10] shed some light on the relationship between energy distribution and economic growth, viewing the economy as a transportation network for electricity and getting a model whose dynamics are formally the same as in the Solow model. Specifically, energy originates from a power plant and is diffused across the economy to the sites at which it is used via a power grid. Energy is needed to run, maintain and create capital. Without an electricity supply, any investment in machinery at a particular place or time will not lead to economic growth. They set a supply relationship between

Applied Sciences, Vol.23, 2021, pp. 99-108.

electricity consumption per capita and the size of the economy measured by capital per capita. Techniques taken from biological sciences were used in their modeling: a power law association between consumption per capita of electricity and capital per capita with an exponent assigned to capital bounded between 1/2 and 3/4 with the final exponent dependent on the efficiency of the network (see [1]). Notice that the ability to harness energy for the betterment of society and how efficiently it can be deployed was first mentioned in 1862 by [20]. To prevent limits to growth as a result of energy shortages will require technological change even if the supply of energy increases. The distribution of energy through a network is therefore crucial.

It is worth stressing that in Dalgaard and Strulik's model population increases at a positive constant rate ([16] growth), which is a totally unrealistic assumption because it doesn't consider the environmental limits that have consequences on the population. Exponential growth may happen for a while in environments where there are few individuals and plentiful resources. But when the number of individuals gets large enough, resources will be depleted, slowing the growth rate. Eventually, the growth rate will level off, making an S-shaped curve. This population size at which it levels off represents the maximum population size a particular environment can support and is called the carrying capacity. Therefore, to model more realistic population growth, scientists developed the logistic growth model (see [21]). [13] pointed out that the process of reproduction is not instantaneous, there is a lag in some of the processes involved, and so the logistic model is inappropriate for the description of population growth. For this reason, he proposed to introduce a time delay in the logistic model. The resulting model is known as the delayed logistic equation or Hutchinson's equation or, under a suitable change of variables, Wright's equation ([23]).

Many authors (e.g. [11]) have stressed that, due to the influence on the carrying capacity resulting from the existing populations, its original definition implying a constant value has lost its meaning. The society activity does influence its own carrying capacity that can be either enhanced by producing new goods, materials, knowledge, and so on, or can be destroyed by unreasonable exploitation of resources, e.g., by deforestation, polluting water, and spoiling climate. The idea that the carrying capacity may be not a constant but a function of population fractions has repeatedly appeared in the literature in the form of general discussions. The article by [24] suggests an interesting approach to the transformation of logistic equations with certain time lags and variable carrying capacity of the biosphere to the shape that allows to model punctuated staircase-like growth (or decline), which corresponds better to real processes in natural and social macrosystems. Taking into account the fact that the carrying capacity is not a simple constant describing the available resources, but that these resources are subjected to the change due to the activity of the system individuals, who can either increase the carrying capacity by creative work or decrease it by destructive actions, [5] have improved the Dalgaard-Strulik model ([10]). However, they used a discrete delay, which, sometimes, might be seen as a rough approximation in modeling the delay distribution over a large size population since it is implicitly assumed that each individual within the population is subject to the same maturation delay. According to this fact, [22] introduced distributed delay into models, taking the length of the delay from a probability distribution (see e.g. [8],[9],[15]). Since the delay might not be exactly the same for every member, this will provide a more appropriate description. There is also some experimental evidence which indicates that such continuously distributed delays are more accurate than those with instantaneous time lags (see [6]).

In this paper, we propose and study an extension of the model introduced by [5], whose principal feature is the functional dependence of the population carrying capacities on the population species and delay maturation delay distributed by the continuous Gamma distributions [18]. The stability and Hopf bifurcation analysis of the equilibrium under different conditions are carried out and the influence on the system behaviour of the destroyed or created carrying capacity is pointed out. The organization of this paper is as follows. In Section 2, the model is presented. In Sections 3 and 4, local stability of the positive equilibrium and existence of the local Hopf bifurcation are discussed in case of weak or strong kernel, respectively. Some main conclusions are drawn in Section 5.

2 The model

We begin by considering the model developed by [5], who introduced a logistic-type equation with delayed carrying capacity in the transport network model for energy of [10] and obtained the following system of delay differential equations

$$\text{(2.1)} \qquad \begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}\,[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + BL(t-\tau)}\right] k(t), \\[2ex] \dot{L}(t) = \gamma L(t) - \dfrac{C[L(t)]^2}{A + BL(t-\tau)}, \end{cases}$$

where k is capital per capita, L denotes labor/population, μ and v represent the energy required to operate and maintain the generic capital good and the energy costs to create a new capital good, respectively, $a \in (0,1)$ is a real constant proportional to the dimension and efficiency of the network, $\mu > 0$ is a real constant. C defines the balance between competition and cooperation. $A > 0$ is a natural carrying capacity, provided by Nature, i.e. it is the pre-existing carrying capacity. The parameter B is the created or destroyed capacity, depending on whether the society activity is constructive or destructive. It is a production factor if it is positive ($B > 0$) and a destruction factor in case it is negative ($B < 0$). Since it is well known that dynamical systems with distributed delays are more general than those with discrete delay, we now assume that the time delay is not the same for all members of the population, but rather is distributed according to the gamma distribution function [18]

$$\text{(2.2)} \qquad g(u,T,n) = \left(\frac{n}{T}\right)^n \frac{u^{n-1} e^{-\frac{n}{T}u}}{(n-1)!},$$

with n a positive integer that determines the shape of the weighting function and $T \geq 0$ a parameter associated with the mean time delay of the distribution. Accordingly, we derive that the dynamics of our model are then governed by the following integrodifferential equations system

$$
(2.3)\qquad \begin{cases} \dot{k}(t) & = & \dfrac{\varepsilon}{v}\,[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + BF(t,T,n)}\right] k(t), \\[2ex] \dot{L}(t) & = & \gamma L(t) - \dfrac{C[L(t)]^2}{A + BF(t,T,n)}, \end{cases}
$$

where

$$
F(t,T,n) = \int_{-\infty}^{t} g(t-r,T,n)L(r)dr.
$$

Notice that as $T \to 0$ the distribution function approaches the Dirac distribution, and thus system (2.3) is reduced to system (2.1), i.e. one recovers the discrete delay case. In order to analyse the local behaviour of system (2.3) one should consider the characteristic equation of the linearized system at the equilibrium point for which it is known to be hard to derive general stability conditions. For this reason, we follow the standard procedure that consists in drawing attention to some special cases and examine stability of the equilibrium analytically by applying the Routh-Hurwitz theorem, which provides conditions that are both necessary and sufficient for this polynomial to have roots with negative real parts. Henceforth, we will concentrate on the two particular cases: $n = 1$ and $n = 2$. The first case corresponds to a weak delay kernel in the sense that the maximum weighted response of the growth rate is to current population density while past densities have exponentially decreasing influence; the second case instead represents a strong delay kernel in the sense that the maximum influence on growth rate response at any time t is due to population density at the previous time $t - T$.

3 Stability analysis and Hopf bifurcation (case weak kernel)

Let $g(\cdot)$ in (2.2) be a weak kernel, i.e.

$$
g(t-r,T,1) = \frac{1}{T} e^{-\frac{1}{T}(t-r)}.
$$

For convenience, a new variable is introduced and defined as follows

$$
x(t) = \int_{-\infty}^{t} \left(\frac{1}{T}\right) e^{-\frac{1}{T}(t-r)} L(r)dr.
$$

Applying the linear chain trick technique (see [15]), system (2.3) is then transformed into the following system of three dimensional ODEs

$$\text{(3.1)} \qquad \begin{cases} \dot{k}(t) &= \dfrac{\varepsilon}{v}\,[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A+Bx(t)}\right] k(t), \\[2ex] \dot{L}(t) &= \gamma L(t) - \dfrac{C[L(t)]^2}{A+Bx(t)}, \\[2ex] \dot{x}(t) &= \dfrac{1}{T}\,[L(t)-x(t)]\,, \end{cases}$$

The equilibrium points of (2.3) are the same as those for (2.1). Hence, it follows from [5] the existence of a unique non-trivial equilibrium $(k_{*,}L_*, x_*)$, where

$$\text{(3.2)} \qquad \varepsilon k_*^{a-1} = \mu, \quad x_* = L_* = \frac{\gamma A}{C-\gamma B},$$

when

$$\text{(3.3)} \qquad C - \gamma B \neq 0 \text{ and } sign\,(\gamma) = sign\,(C-\gamma B)\,.$$

It is well known that the local stability of $(k_{*,}L_*, x_*)$ is governed by the roots of the associated characteristic equation for (3.1). By linearising (3.1) at the equilibrium point , we derive that the characteristic equation of system (3.1) is given by

$$\det \begin{bmatrix} -(1-\alpha)\dfrac{\mu}{v} - \lambda & \dfrac{\gamma k_*}{L_*} & -\dfrac{BCk_*L_*}{(A+BL_*)^2} \\[2ex] 0 & -\gamma - \lambda & \dfrac{\gamma^2 B}{C} \\[2ex] 0 & \dfrac{1}{T} & -\dfrac{1}{T} - \lambda \end{bmatrix} = 0.$$

By simple calculation, one obtains

$$\text{(3.4)} \qquad \left[-(1-\alpha)\frac{\mu}{v} - \lambda\right]\left[\lambda^2 + a_1(T)\lambda + a_2(T)\right] = 0.$$

where the coefficients are

$$a_1(T) = \gamma + \frac{1}{T}, \quad a_2(T) = \frac{\gamma}{T}\left(1 - \frac{\gamma B}{C}\right).$$

The Routh-Hurwitz criterion implies that the equilibrium is locally asymptotically stable, i.e. all roots of the polynomial (3.4) are negative or have negative real parts, if and only if the following conditions hold

$$\text{(3.5)} \qquad a_1(T) > 0, \quad a_2(T) > 0.$$

By (3.3), conditions (3.5) leads to

$$\text{(3.6)} \qquad \gamma + \frac{1}{T} > 0, \quad C > 0.$$

We have two cases: $\gamma > 0$, in which case (3.6) reduces to $C > 0$ with $C > \gamma B$, and so we get $C > \max\{0, \gamma B\}$; $\gamma < 0$, then (3.6) gives $T < -1/\gamma$, $0 < C < \gamma B$ with $B < 0$. In this latter case, the curve $T = -1/\gamma$ divides the parameter space into stable and unstable parts. In order to check the possibility of the emergency of a limit cycle at $T = -1/\gamma$, we use the Hopf bifurcation theorem and check if the characteristic equation (3.4) possesses a pair of purely imaginary roots and the real parts of these roots change signs with a bifurcation parameter. Supposing that $\lambda = i\omega$ $(\omega > 0)$ is a root of (3.4), then, at the critical value $T = -1/\gamma$, one in particular find

$$-\omega^2 + a_1(-1/\gamma)i\omega + a_2(-1/\gamma) = 0. \tag{3.7}$$

Separating the real and imaginary parts of (3.7) yields the contradiction $0 = \gamma + 1/(-1/\gamma) > 0$. Thus, there is no imaginary root for (3.4). Summarizing the analysis above we arrive at the following stability results.

Proposition 3.1.

1) *The equilibrium point* (k_*, L_*, x_*) *of* (3.1) *is locally asymptotically stable for all* $T > 0$ *if* $\gamma > 0$ *and* $C > \max\{0, \gamma B\}$.

2) *The equilibrium point* (k_*, L_*, x_*) *of* (3.1) *is locally asymptotically stable for* $T < -1/\gamma$ *and unstable for* $T > -1/\gamma$ *if* $\gamma < 0$ *and* $0 < C < \gamma B$, $B < 0$. *There can be no Hopf bifurcation at the equilibrium.*

4 Stability analysis and Hopf bifurcation (case strong kernel)

Let $g(\cdot)$ in (2.2) be a strong kernel, i.e.

$$g(t-r, T, 2) = \left(\frac{2}{T}\right)^2 (t-r)e^{-\frac{2}{T}(t-r)}.$$

Introducing the new variables

$$z(t) = \int_{-\infty}^{t} \left(\frac{2}{T}\right)^2 e^{-\frac{2}{T}(t-r)} u(r)dr, \quad u(t) = \int_{-\infty}^{t} \left(\frac{2}{T}\right) e^{-\frac{2}{T}(t-r)} L(r)dr,$$

again an application of the linear chain trick technique leads system (2.3) to be rewritten as a four dimensional system of ODEs

$$\begin{cases} \dot{k}(t) &= \dfrac{\varepsilon}{v}[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + Bz(t)}\right] k(t), \\ \dot{L}(t) &= \gamma L(t) - \dfrac{C[L(t)]^2}{A + Bz(t)}, \\ \dot{z}(t) &= \dfrac{2}{T}[u(t) - z(t)], \\ \dot{u}(t) &= \dfrac{2}{T}[L(t) - u(t)]. \end{cases} \tag{4.1}$$

The characteristic equation of the linearized system of (4.1) at the equilibrium (k_*, L_*, z_*, u_*), where $z_* = u_* = L_*$, and k_*, L_* are defined as in (3.2), is provided by

$$\det \begin{bmatrix} -(1-\alpha)\dfrac{\mu}{v} - \lambda & \dfrac{\gamma k_*}{L_*} & -\dfrac{BCk_*L_*}{(A+BL_*)^2} & 0 \\ 0 & -\gamma - \lambda & \dfrac{\gamma^2 B}{C} & 0 \\ 0 & 0 & -\dfrac{2}{T} - \lambda & \dfrac{2}{T} \\ 0 & \dfrac{2}{T} & 0 & -\dfrac{2}{T} - \lambda \end{bmatrix} = 0. \tag{4.2}$$

By expanding (4.2), the characteristic equation becomes

$$\left[-(1-\alpha)\frac{\mu}{v} - \lambda\right]\left[\lambda^3 + b_1(T)\lambda^2 + b_2(T)\lambda + b_3(T)\right] = 0, \tag{4.3}$$

where

$$b_1(T) = \gamma + \frac{4}{T}, \quad b_2(T) = \frac{4}{T}\left(\gamma + \frac{1}{T}\right), \quad b_3(T) = \frac{4\gamma}{T^2}\left(1 - \frac{\gamma B}{C}\right).$$

Due to the Routh-Hurwitz criterion, we have local asymptotic stability for the equilibrium of system (4.1) if and only if the following conditions

$$b_1(T) > 0, \quad b_3(T) > 0, \quad b_1(T)b_2(T) - b_3(T) > 0 \tag{4.4}$$

hold true. The first two conditions are equivalent to

$$\gamma + \frac{4}{T} > 0, \quad C > 0. \tag{4.5}$$

A direct calculation changes the latter condition in (4.4) to

$$(\gamma T + 2)^2 + \frac{\gamma^2 BT}{C} > 0. \tag{4.6}$$

Proposition 4.1.

1) *The equilibrium point* (k_*, L_*, z_*, u_*) *of* (4.1) *is locally asymptotically stable for all* $T > 0$ *if* $B > 0$, $\gamma > 0$, $C > \gamma B$, *or if* $B < 0$, $\gamma > 0$, $C + \gamma B \geq 0$.

2) *The equilibrium point* (k_*, L_*, z_*, u_*) *of* (4.1) *is locally asymptotically stable for* $T < T_*$ *and unstable for* $T > T_*$ *if* $B < 0$, $\gamma > 0$, $C + \gamma B < 0$, *or if* $B < 0$, $\gamma < 0$, $C + \gamma B > 0$, *where*

$$T_* = -\frac{2C}{\gamma(C + \gamma B)}.$$

Proof. Let $B > 0$. Then (4.6) is always verified so that the conditions for stability reduce to (4.5). In addition, we must have $\gamma > 0$ since $C > \gamma B$. Next, let $B < 0$. From (4.6) we have

$$|\gamma T + 2| > -\frac{\gamma^2 BT}{C},$$

and so

$$\frac{\gamma^2 BT}{C} < \gamma T + 2, \quad \gamma T + 2 > -\frac{\gamma^2 BT}{C}.$$

The first inequality gives $\gamma(C - \gamma B) T + 2C > 0$, which is obviously true. The second one instead yields $\gamma(C + \gamma B) T + 2C > 0$. The conclusions now hold considering the cases $\gamma > 0$ and $\gamma < 0$. Notice that when $\gamma < 0$ it is $C + \gamma B > 0$ and $T_* < -4/\gamma$. □

We now return to the characteristic equation (4.3) and show the possibility of the birth of a limit cycle at $T = T_*$ by applying the Hopf bifurcation theorem. We start noticing that at the critical value T_*, one has $b_1(T_*)b_2(T_*) = b_3(T_*)$, so that the characteristic equation (4.3) factors as

$$\left[-(1-\alpha)\frac{\mu}{v} - \lambda\right][\lambda + b_1(T_*)]\left[\lambda^2 + b_2(T_*)\right] = 0.$$

This shows that two roots are real and negative

$$\lambda_1 = -(1-\alpha)\frac{\mu}{v} < 0, \quad \lambda_2 = -b_1(T_*) < 0,$$

and two are purely imaginary

$$\lambda_{3,4} = \pm i\omega_*, \quad \text{with } \omega_* = \sqrt{b_2(T_*)}.$$

Next, we need to check whether the real part of the conjugate complex root changes its sign as the bifurcation parameter T passes through its critical value T_*. Assuming $\lambda = \lambda(T)$, differentiating the characteristic equation (4.3) with respect to T, and arranging terms, we get

$$\frac{d\lambda}{dT} = \frac{-b_1'(T)\lambda^2 - b_2'(T)\lambda}{3\lambda^2 + 2b_1(T)\lambda + b_2(T)},$$

where

$$b_1'(T) = -\frac{4}{T^2}, \quad b_2'(T) = -\frac{4}{T^2}\left(\gamma + \frac{2}{T}\right), \quad b_3'(T) = -\frac{8\gamma}{T^3}\left(1 - \frac{\gamma B}{C}\right).$$

From being $\omega_*^2 = b_2(T_*)$ and $b_1(T_*)b_2(T_*) = b_3(T_*)$, after some calculations, we have

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} = -\frac{b_1'(T_*)b_2(T_*) + b_1(T_*)b_2'(T_*)}{2\left[b_2(T_*) + b_1^2(T_*)\right]}. \tag{4.7}$$

Conditions (4.4) in particular yield $b_2(T_*) > 0$, i.e. $\gamma + 1/T_* > 0$. Thus,

$$-b_1'(T_*)b_2(T_*) + b_1(T_*)b_2'(T_*) = \frac{4}{T_*^2}\left[\frac{4}{T_*}\left(\gamma + \frac{1}{T_*}\right) + \left(\gamma + \frac{4}{T_*}\right)\left(\gamma + \frac{2}{T_*}\right)\right] > 0,$$

Since the sign of (4.7) is positive, only crossing the imaginary axis from left to right is possible as T increases.

In conclusion we have the following theorem.

Theorem 4.2. *The equilibrium point* (k_*, L_*, z_*, u_*) *of* (4.1) *loses stability at* $T = T_*$ *and bifurcates to chaos as* T *increases.*

5 Conclusions

An extension of the model proposed by [5] is examined in which distributed time delays are assumed into a transportation network for energy model. Since the delay might not be exactly the same for every member of the population, but rather might vary according to some distribution, it is more appropriate to use a distributed delay or integrodifferential equation in their model. Conditions are given to the local asymptotical stability of the equilibrium and the appearance of Hopf bifurcations is investigated. It is shown that the creative processes impacting the carrying capacity do not affect stability, while the destructive processes may have a stabilizing effect as well as a destabilizing effect. In the latter case, chaotic behaviour emerges at the stability switch. In future research we will apply this methodology taking into account the possibility of including the conservation of global resources through the use of the thermostatted kinetic theory for active particles proposed in [2] and generalized in [3], [4].

Acknowledgements. The authors would like to express their gratitude to the anonymous referees for several helpful comments and suggestions.

References

[1] J.R. Banavar, A. Maritan, A. Rinaldo, *Size and form in efficient transportation networks*, Nature 399 (1999), 130-132.

[2] C. Bianca, *Kinetic theory for active particles modelling coupled to Gaussian thermostats*, Appl. Math. Sci. 6, (2012), 651-660.

[3] C. Bianca, *Onset of nonlinearity in thermostatted active particles models for complex systems*, Nonlinear Anal. Real World Appl. 3 (2012), 2593-2608.

[4] C. Bianca, *Modeling complex systems by functional subsystems representation and thermostatted-KTAP methods*, Appl. Math. Inf. Sci. 6, (2012), 495-499.

[5] C. Bianca, L. Guerrini, *On the Dalgaard-Strulik model with logistic population growth rate and delayed-carrying capacity*, Acta Appl. Math. 28 (1) (2013), 39-48.

[6] J. Caperon, *Time lag in population growth response of Isochrysis Galbana to a variable nitrate environment*, Ecology 50 (2) (1969), 188-192.

[7] C. Corrado, C. Hulten, D. Sichel, *Intangible capital and US economic growth* Rev. Income Wealth 55 (3) (2009), 661-685.

[8] J.M. Cushing, *Stable positive periodic solutions of the time-dependent logistic equation under possible hereditary influences*, J. Math. Anal. Appl. 60 (3), (1977), 747-754.

[9] J.M. Cushing, *Time delays in single species growth models*, J. Math. Biol. 4 (3) (1977), 257-264.

[10] C.L. Dalgaard, H. Strulik, *Energy distribution and economic growth*, Resour. Energy Econ. 3 (2011), 782-797.

[11] P. Del Monte-Luna, B.W. Brook, M.J. Zetina-Rejon, V.H. Cruz-Escalona, *The carrying capacity of ecosystems*, Global. Ecol. Biogeogr. 3(6) (2004), 485-495.

[12] M. Ferrara, M. Gangemi, L. Guerrini, B.A. Pansera, *Stability and Hopf bifurcation analysis of a distributed time delay energy model for sustainable economic*

growth, Atti della accademia peloritana dei pericolanti, classe di scienze fisiche matematiche e naturali 98(1), 2020.

[13] G.E. Hutchinson, *Circular causal systems in ecology*, Ann. N.Y. Acad. Sci. 50 (1948), 221-246.

[14] D.W. Jorgenson, Z. Griliches, *The explanation of productivity change*, Rev. Econ. Studies 34 (3) (1967), 249-283.

[15] N. MacDonald, *Biological delay systems: linear stability theory*, Cambridge University Press, Cambridge 2008.

[16] T.R. Malthus, *An Essay on the Principle of Population*, Oxford University Press, Oxford 1999.

[17] N.G. Mankiw, D. Romer, D.N. Weil, *A contribution to the empirics of economic growth*, Q. J. Econ. 107 (2) (1992), 407-437.

[18] S. Ruan, *Delay differential equations in single species dynamics*. In: O. Arino, M. Hbid, E. Ait Dads (Eds.), *Delay Differential Equations with Applications*, NATO Science Series II, 477-517, Springer, Berlin 2006.

[19] R.M. Solow, *A contribution to the theory of economic growth*, Q. J. Econ. 70 (1956), 65-94.

[20] H. Spencer, *First Principles*, Cambridge University Press, 2009.

[21] P.F. Verhulst, *Notice sur la loi que la population suit dans son accroissement*, Corresp. Math. Phys. 10 (1838), 113-121.

[22] V. Volterra, *Theory of functionals and of integral and integro-differential equations*, Dover, Mineola, 1959.

[23] E.M. Wright, *A non-linear difference-differential equation*. Jour. Reine Angewandte Math. 194 (1955), 66-87.

[24] V.I. Yukalov, E.P. Yukalova, D. Sornette, *Punctuated evolution due to delayed carrying capacity*, Physica D. 238 (2009), 1752-1767.

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