Nonstandard finite difference method for the approximate solution of two-point fourth order boundary value problems in ODEs

P. K. Pandey

Abstract. In this article, the boundary value problem involving fourth order differential equations in ODEs in a close open domain is considered and boundary conditions prescribed at the two points. A non conventional finite difference method developed for the approximate numerical solution of the problem. Our proposed technique is simple in application and solves the problem directly. The method uses boundary conditions in a natural way. We have established accuracy in numerical experiments and convergence of the proposed method analytically. The computational experiments on model test problem both nonlinear and linear, verify the competency of the proposed method. A singular problem considered and satisfying numerical results obtained in computational experiments as an extended application of the method.

M.S.C. 2010: 65L10, 65L12.
Key words: Boundary value problem; fourth-order bvp; second kind boundary conditions; second order convergence; unconventional method.

1 Introduction

It is well known that fourth-order differential equations and corresponding boundary value problems appear in the mathematical modeling of the problems and explaining the deformation of structures [20], deformation of elastic membrane [17] and theory and application of elasticity [22, 13, 11]. In this article we consider following differential equation of order four,

$$u^{(4)}(x) = f(x, u, u''), \quad x \in [a, \infty) \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \alpha_0, \quad u(\infty) = \beta_0, \quad u''(a) = \alpha_1 \quad and \qquad u''(\infty) = \beta_1$$

where $\alpha_0, \beta_0, \alpha_1$, and β_1 are finite real constant and $f(x, u)$ is continuous on $[0, \infty) \times D$, $D \subseteq \mathbb{R}$.

Applied Sciences, Vol.23, 2021, pp. 87-98.
© Balkan Society of Geometers, Geometry Balkan Press 2021.

The study and analysis of solution of the differential equation under the prescribed boundary conditions is an area attracted much attention in view of its direct application in different area of social sciences, engineering and sciences. The existence of the solution of the problem and unique or multiple solutions in the domain / region of interest, studied and reported in the literature by many researchers. For the existence and uniqueness of the solution of boundary value problem involving higher order differential equation [1, 2, 3, 24], multiplicity of solution in finite domain [15] and references therein.

It is desired to find a useful general analytical solution of the modelled problem. Sometime it is difficult to find a useful analytical solution of the considered problem (1.1) without specific case. It is also possible that we find a non-useful analytical solution after relaxing some parameter or no analytical solution for the modeled problem. In these cases we rely on some approximate solution. A good number of research articles using different approaches are available in literature and some of literary work can be found in [21, 5, 16, 18] and references therein.

We considered the problem (1.1) in open domain, i.e. a boundary condition prescribed at infinite. The classical above methods for numerical solution of problems (1.1) applicable if we truncate the boundary condition [23, 7]. It means we replaced the original problem by one defined on a finite interval. However, the drawback in truncated boundary method is that the accuracy of finite difference methods compromised. The current trend indicate the interest of researchers in developing technique for accurate numerical solution of the original problem without truncating the boundary condition [14, 8] and references there in.

In this article, we propose a technique for the accurate numerical solution of linear and nonlinear problems involving fourth order differential equations in an open domain without truncating boundary conditions. However, we assumed existence and uniqueness of the solution of the problem (1.1) in semi-open interval/ domain instead of defining and imposing conditions on the forcing function for the existence and uniqueness of the solution of the problem (1.1). To demonstrate the application and computational efficiency, model problems were considered and numerical results compared with analytical solution.

We have presented our work in this article as follows. In the next section 2, we proposed a nonstandard finite difference method. We have outlined the discussion and derivation in Section 3. The convergence and application of the method presented in the Section 4 and Section 5 respectively. A conclusion and brief discussion on the performance of the method are presented in Section 6.

2 The difference method

The boundary condition at infinity, it is challenging task to incorporate. Thus, in developing an efficient and acceptable numerical technique for solution of problem incorporating the boundary condition at infinity we consider following smooth function $x = x(t)$ defined in [4, 6],

$$x(t) = \rho \frac{t}{1-t}, \quad 0 \le t \le 1$$

where $\rho > 0$ is parameter and $0 \leq x(t) \leq \infty$. The function the $x(t)$ is strictly monotonic and semi uniform. Let us define equidistance $N-1$ nodes $t_i = ih$ in $(0,1)$ with uniform step distance $h = \frac{1}{N}$. Also we find that $t_0 = 0$ and $t_N = t_{N-1} + \frac{1}{N} = t_{N-1} + h$. These nodes enable us to generate single parameter family of semi uniform nodes $x_i = x(t_i)$, $i = 0, 1, \cdots, N$ in interval $[0, \infty]$.

The discretization of a problem (1.1) in $[0,1]$, the computational domain with N+1 nodes is the transformation of continuous problem into a discrete problem at these nodes and results in $N-1$ equations in $N-1$ unknown. We compute these unknown at the $N-1$ nodes and call them the numerical solution of the problem. Let us respectively denote u_i and f_i as the numerical approximation of $u(x)$ and $f(x, u(x), u''(x))$ at the node $x = x_i$ for $i = 1, 2, .., N-1$. We will follow the same definition for the other notations used in the present article.

Let us define the following approximations,

$$x_1 = x_i - \frac{3}{5}h, \quad x_2 = x_i - \frac{1}{5}h, \quad x_3 = x_i + \frac{1}{5}h, \quad x_4 = x_i + \frac{3}{5}h \tag{2.1}$$

and

$$A_i = \frac{x_4 - x_3}{x_4 - x_1}, \quad B_i = \frac{x_3 - x_2}{x_4 - x_1}, \quad C_i = \frac{x_2 - x_1}{x_4 - x_1} \tag{2.2}$$

Let we set,

$$\begin{aligned} u_{i+\frac{1}{2}} &= A_i u_{i-1} + B_i u_i + C_i u_{i+1}, \\ u''_{i+\frac{1}{2}} &= A_i u''_{i-1} + B_i u''_i + C_i u''_{i+1} \qquad \text{and} \\ f_{i+\frac{1}{2}} &= f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, u''_{i+\frac{1}{2}}), \quad i = 1, 2, \cdots, N-1 \end{aligned} \tag{2.3}$$

Considered problem (1.1) transformed into following discrete problem at node $x_{i+\frac{1}{2}}$ in computational domain,

$$u^{(4)}_{i+\frac{1}{2}} = f_{i+\frac{1}{2}} \tag{2.4}$$

subject to the boundary conditions

$$u_0 = \alpha_0, \quad u''_0 = \alpha_1, \quad u_N = \beta_0 \quad \text{and} \qquad u''_N = \beta_1$$

Though the idea of discretization in [18, 19] use boundary conditions in a natural way, we discretize the problem (2.4) at nodes $x_{i+\frac{1}{2}}$ in computational domain $(0,1)$,

$$-12(u_{i-1} - 2u_i + u_{i+1}) + 6h^2(u''_{i+1} + u''_{i-1}) = 5h^4 f_{i+\frac{1}{2}} \tag{2.5}$$

$$u''_{i+1} - 2u''_i + u''_{i-1} = h^2 f_{i+\frac{1}{2}}, \quad 1 \leq i \leq N-1. \tag{2.6}$$

The governing equation at each node x_i, $i = 1, 2, \cdots, N-1$ leads to $N-1$ equations. Thus we obtained an algebraic system of equations in unknown u_i, $i = 1, 2, \cdots, N-1$. The algebraic system of equations are linear if the forcing function $f(x, u, u'')$ is linear otherwise nonlinear. We considered the solution of a system of equations is the approximate solution of the considered problem at nodes x_i in the considered computational domain. The proposed method (2.5) and (2.6) uses respectively $u_N = u(\infty)$ and $u''_N = u''(\infty)$ not $u_N = \infty$ and $u''_N = \infty$.

3 Development of the non-standard finite difference method

We outline the systematics development of the proposed non-standard finite difference method discussed in preceding section. Using the definition of nodes i.e. $x_{i+a} = x_i + ah$, we simplify coefficients A_i, B_i, C_i. Thus, we have

$$A_i = \frac{1}{3}, \quad B_i = \frac{1}{3}, \quad \text{and} \quad C_i = \frac{1}{3} \tag{3.1}$$

After substituting A_i, B_i, C_i, $i = 1, 2, \cdots, N-1$ in definition of $u_{i+\frac{1}{2}}$

$$u_{i+\frac{1}{2}} = \frac{1}{3}(u_{i-1} + u_i + u_{i+1}) \tag{3.2}$$

Expand each term on right hands side in (3.2) in Taylor's series and simplify the expression, we observed that $\frac{1}{3}(u_{i-1} + u_i + u_{i+1})$ will provide $O(h)$ approximation for $u_{i+\frac{1}{2}}$. Similarly we can prove that $\frac{1}{3}(u''_{i-1} + u''_i + u''_{i+1})$ will provide $O(h)$ approximation for $u''_{i+\frac{1}{2}}$. Thus,

$$\begin{aligned} f(x_{i+\frac{1}{2}}, \frac{1}{3}(u_{i-1} + u_i + u_{i+1}), \frac{1}{3}(u''_{i-1} + u''_i + u''_{i+1})) \\ = f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}} - \frac{h}{2}u_{i+\frac{1}{2}}, u''_{i+\frac{1}{2}} - \frac{h}{2}u'''_{i+\frac{1}{2}}) \\ = f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, u''_{i+\frac{1}{2}}) + fR_i \end{aligned} \tag{3.3}$$

where fR_i, the remainder terms in Taylor series expansion of forcing function $f(x, u, u'')$ at the node $x_{i+\frac{1}{2}}$,

$$fR_i = -\frac{h}{2}(u'_{i+\frac{1}{2}}(\frac{\partial f}{\partial u})_{i+\frac{1}{2}} + u'''_{i+\frac{1}{2}}(\frac{\partial f}{\partial u''})_{i+\frac{1}{2}}) + \cdots, \quad i = 1, 2 \cdots, N-1. \tag{3.4}$$

Let us consider the following linear combination of solution, second derivative of the solution of the problem (1.1) and forcing function $f(x, u)$ at the discrete points x_{i-1}, x_i and x_{i+1} in a computational domain,

$$h^4 f_{i+\frac{1}{2}} = a_0 u_{i-1} + a_1 u_i + a_2 u_{i+1} + h^2(b_0 u''_{i-1} + b_1 u''_{i+1}) \tag{3.5}$$

In (3.5), a_0, a_1, a_2, b_0, b_1 are constant and to be determined under appropriate conditions. To determine these constants we write each term in (3.5) in a Taylor's series about the nodes $x_{i+\frac{1}{2}}$. Using (2.4)and comparing the coefficients of $h^p, p = 0, 1, .., 4$ in the Taylor series expansion (3.5). Thus, we obtained following system of equations,

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_0 - a_1 - 3a_2 &= 0 \\ a_0 + a_1 + 9a_2 + 8(b_0 + b_1) &= 0 \\ a_0 - a_1 - 27a_2 + 24(b_0 - 3b_1) &= 0 \\ a_0 + a_1 + 81a_2 + 48(b_0 + 9b_1) &= 384 \end{aligned} \tag{3.6}$$

For the discretization of the considered problem, let us assume the system of equations is consistent, i.e. these equations are linearly independent. Thus, the solution of the system of equations (3.6) is,

$$(a_0, a_1, a_2, b_0, b_1) = -\frac{6}{5}(2, -4, 2, -1, -1). \tag{3.7}$$

Using (3.7) in (3.5), we have

$$\frac{5}{6}h^4 f_{i+\frac{1}{2}} = -2(u_{i-1} - 2u_i + u_{i+1}) + h^2(u''_{i-1} + u''_{i+1}) + R_i \tag{3.8}$$

where R_i, the leading term i.e. after fifth term in Taylor series expression of (3.5) and

$$R_i = -\frac{1}{2}h^5 u^{(5)}_{i+\frac{1}{2}}, \quad i = 1, 2, \cdots, N-1. \tag{3.9}$$

Following the similar procedure as above, we obtained,

$$h^2 f_{i+\frac{1}{2}} = u''_{i-1} - 2u''_i + u''_{i+1} + dR_i \quad i = 1, 2, \cdots, N-1 \tag{3.10}$$

where dR_i, the leading term i.e. after third term if we write (3.10) in Taylor series about nodes $x_{i+\frac{1}{2}}$ and

$$dR_i = -\frac{1}{2}h^3 u^{(5)}_{i+\frac{1}{2}}, \quad i = 1, 2, \cdots, N-1. \tag{3.11}$$

After truncating the remainder terms R_i and dR_i in (3.8) and (3.10) respectively, we obtained our proposed method ((2.5)-(2.6)).

Let us define computational truncation error in proposed method ((2.5)-(2.6)). From (3.4), we have obtained $O(h)$ approximations for the forcing function $f_{i+\frac{1}{2}}$ i.e. the truncating terms fR_i are of $O(h)$. From (3.4), (3.8) and (3.9), the leading truncating terms is given by

$$TR_i = \frac{5h^5}{12}(u^{(5)}_{i+\frac{1}{2}} - (u'_{i+\frac{1}{2}}(\frac{\partial f}{\partial u})_{i+\frac{1}{2}} + u'''_{i+\frac{1}{2}}(\frac{\partial f}{\partial u''})_{i+\frac{1}{2}})) \tag{3.12}$$

Also from (3.4) and rationalising (3.11), we have

$$TdR_i = \frac{h^5}{2}(u^{(5)}_{i+\frac{1}{2}} - (u'_{i+\frac{1}{2}}(\frac{\partial f}{\partial u})_{i+\frac{1}{2}} + u'''_{i+\frac{1}{2}}(\frac{\partial f}{\partial u''})_{i+\frac{1}{2}})) \tag{3.13}$$

There are leading term of $O(h^5)$ in computational truncation error generated by the approximations and discretizations in development of proposed method ((2.5)-(2.6)). Thus, the estimated accuracy of the proposed method ((2.5)-(2.6)) is $O(h)$.

4 Convergence analysis

In this section we shall discuss the convergence of the proposed finite difference method (2.4) for the solution of the problem (1.1). Let the application of (2.1), transform problem (1.1) and boundary conditions into following linear test problem,

$$u^{(4)}(x(t)) = f(x, u, u''), \quad 0 < t < 1 \tag{4.1}$$

subject to the boundary conditions

$$u_0 = \alpha_0, \quad u''_0 = \alpha_1, \quad u_N = \beta_0 \quad \text{and} \qquad u''_N = \beta_1$$

Let us define following error equations,

$$\begin{aligned} \boldsymbol{\epsilon} &= \mathbf{u} - \mathbf{U}, \\ \boldsymbol{\delta} &= \mathbf{u}'' - \mathbf{U}'' \end{aligned} \tag{4.2}$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ are respectively error in solution and second derivative of solution of the problem. These $\mathbf{u}$, $\mathbf{u}''$ are approximate solution and second derivative of solution and $\mathbf{U}$, $\mathbf{U}''$ exact solution and second derivative of exact solution of the problem (4.1) and defined as,

$$\begin{aligned} \boldsymbol{\epsilon} &= [\epsilon_0, \epsilon_1, \cdots, \epsilon_N]^T, \quad \boldsymbol{\delta} = [\delta_0, \delta_1, \cdots, \delta_N]^T \\ \mathbf{u} &= [u_0, u_1, \cdots, u_N]^T, \quad \mathbf{u}'' = [u''_0, u''_1, \cdots, u''_N]^T \\ \mathbf{U} &= [U_0, U_1, \cdots, U_N]^T, \quad \mathbf{U}'' = [U''_0, U''_1, \cdots, U''_N]^T. \end{aligned}$$

and set

$$F_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, U_{i+\frac{1}{2}}, U''_{i+\frac{1}{2}}), \quad f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, u''_{i+\frac{1}{2}})$$

Thus, we write proposed finite difference method ((2.5)-(2.6)) in following matrix equation using the approximate and exact solution of the problem (4.1),

$$\mathbf{Ju} + h^2\mathbf{Au}'' = \frac{5}{6}h^4\mathbf{f} + \text{boundary conditions} \tag{4.3}$$

$$\mathbf{JU} + h^2\mathbf{AU}'' = \frac{5}{6}h^4\mathbf{F} + \mathbf{T} + \text{boundary conditions} \tag{4.4}$$

$$h^2\mathbf{Ju}'' = h^4\mathbf{f} + \text{boundary conditions} \tag{4.5}$$

$$h^2\mathbf{JU}'' = h^4\mathbf{F} + \mathbf{T}_1 + \text{boundary conditions} \tag{4.6}$$

where $\mathbf{T}$ and $\mathbf{T}_1$ are respectively truncation errors in proposed method ((2.5)-(2.6)),

$$\begin{aligned} \mathbf{f} &= [f_{\frac{3}{2}}, f_{\frac{5}{2}}, \cdots, f_{N-\frac{1}{2}}]^T, \quad \mathbf{F} = [F_{\frac{3}{2}}, F_{\frac{5}{2}}, \cdots, F_{N-\frac{1}{2}}]^T \\ \mathbf{T} &= [TR_1, TR_2, \cdots, TR_{N-1}]^T, \quad \mathbf{T}_1 = [TdR_1, TdR_2, \cdots, TdR_{N-1}]^T \end{aligned}$$

and the terms TR_i, TdR_i are defined by ((3.12)-(3.13)). Subtract (4.3) from (4.4), (4.5)) from (4.6) and using (4.2), we obtain

$$\mathbf{J}\boldsymbol{\epsilon} + h^2\mathbf{A}\boldsymbol{\delta} = \frac{5}{6}h^4(\mathbf{F} - \mathbf{f}) + \mathbf{T} \tag{4.7}$$

$$h^2\mathbf{J}\boldsymbol{\delta} = h^4(\mathbf{F} - \mathbf{f}) + \mathbf{T}_1 \tag{4.8}$$

Let us linearize forcing function $f(x, U, U'')$, so we have

$$f(x_{i+\frac{1}{2}}, U_{i+\frac{1}{2}}, U''_{i+\frac{1}{2}}) - f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, u''_{i+\frac{1}{2}}) = (U_{i+\frac{1}{2}} - u_{i+\frac{1}{2}})G_{i+\frac{1}{2}} + (U''_{i+\frac{1}{2}} - u''_{i+\frac{1}{2}})H_{i+\frac{1}{2}}, \quad i = 1, 2, \cdots, N-1. \tag{4.9}$$

where $G_{i+\frac{1}{2}} = (\frac{\partial f}{\partial u})_{i+\frac{1}{2}}$ and $H_{i+\frac{1}{2}} = (\frac{\partial f}{\partial u''})_{i+\frac{1}{2}}, \quad i = 1, 2, \cdots, N-1.$

Using simplified form of (2.3) and (4.2) in (4.9), we obtained

$$F_{i+\frac{1}{2}} - f_{i+\frac{1}{2}} = \frac{1}{3}((\epsilon_{i-1} + \epsilon_i + \epsilon_{i+1})G_{i+\frac{1}{2}} + (\delta_{i-1} + \delta_i + \delta_{i+1})H_{i+\frac{1}{2}}), \quad i = 1, 2, \cdots, N-1. \tag{4.10}$$

From (4.10), we obtained

$$\mathbf{F} - \mathbf{f} = \frac{1}{3}(\mathbf{L}_1\boldsymbol{\epsilon} + \mathbf{L}_2\boldsymbol{\delta}) \tag{4.11}$$

Substituting $\mathbf{F} - \mathbf{f}$ from (4.11) in (4.7) and (4.8), we have

$$(\mathbf{J} - \frac{5h^4}{18}\mathbf{L}_1)\boldsymbol{\epsilon} + h^2(\mathbf{A} - \frac{5}{6}h^2\mathbf{L}_2)\boldsymbol{\delta} = \mathbf{T} \tag{4.12}$$

$$-\frac{1}{3}h^4\mathbf{L}_1\boldsymbol{\epsilon} + h^2(\mathbf{J} - \frac{1}{3}h^2\mathbf{L}_2)\boldsymbol{\delta} = \mathbf{T}_1 \tag{4.13}$$

where

$$\mathbf{J} = 2\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}_{N-1\times N-1},$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 0 & & & 1 & 0 \end{pmatrix}_{N-1\times N-1},$$

$$\mathbf{L}_1 = \begin{pmatrix} G_{\frac{3}{2}} & G_{\frac{3}{2}} & & & 0 \\ G_{\frac{5}{2}} & G_{\frac{5}{2}} & G_{\frac{5}{2}} & & \\ & \ddots & \ddots & \ddots & \\ & & G_{N-\frac{3}{2}} & G_{N-\frac{3}{2}} & G_{N-\frac{3}{2}} \\ 0 & & & G_{N-\frac{1}{2}} & G_{N-\frac{1}{2}} \end{pmatrix}_{N-1\times N-1},$$

$$\mathbf{L}_2 = \begin{pmatrix} H_{\frac{3}{2}} & H_{\frac{3}{2}} & & & 0 \\ H_{\frac{5}{2}} & H_{\frac{5}{2}} & H_{\frac{5}{2}} & & \\ & \ddots & \ddots & \ddots & \\ & & H_{N-\frac{3}{2}} & H_{N-\frac{3}{2}} & H_{N-\frac{3}{2}} \\ 0 & & & H_{N-\frac{1}{2}} & H_{N-\frac{1}{2}} \end{pmatrix}_{N-1\times N-1},$$

Thus, equations (4.12) and (4.13) may be written as,

$$\mathbf{ME} = \mathbf{LT}, \tag{4.14}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{J} - \frac{5h^4}{18}\mathbf{L}_1 & h^2(\mathbf{A} - \frac{5}{6}h^2\mathbf{L}_2) \\ -\frac{1}{3}h^4\mathbf{L}_1 & h^2(\mathbf{J} - \frac{1}{3}h^2\mathbf{L}_2) \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\delta} \end{pmatrix} \quad \text{and} \quad \mathbf{LT} = \begin{pmatrix} \mathbf{T} \\ \mathbf{T}_1 \end{pmatrix}.$$

Let us assume,

$$G^* = \max_{x(t)\in[0,1]} \left|\frac{\partial f}{\partial u}\right|, \quad u^{(5)*} = \max_{x(t)\in[0,1]} \left|u^{(5)}(x)\right|, \text{and} \quad H^* = \max_{x(t)\in[0,1]} \left|\frac{\partial f}{\partial u''}\right|$$

Further, let $G^* < \frac{96}{5h^2}$, $H^* < 16$ and $3G^*(4 - 5h^2H^*) < (96 + 5h^2G^*)(16 + H^*)$. Then, the matrix $\mathbf{M}$ defined by (4.14) is invertible and moreover $\|\mathbf{M}^{-1}\|$ is bounded [9, 12, 10]. Let matrix $\|\mathbf{M}^{-1}\|$ is bounded by K. Thus, from (4.14) we obtained,

$$\|\mathbf{E}\| = \|\mathbf{M}^{-1}\mathbf{LT}\| \leq \|\mathbf{M}^{-1}\|\|\mathbf{LT}\| \leq K\|\mathbf{LT}\| \tag{4.15}$$

Hence, from (4.15) we conclude that matrix $\|\mathbf{E}\|$ is bounded. It means error will not become larger as h decrease. The leading truncation errors in proposed method are given by

$$TR_i = \frac{5h^5}{12}u^{(5)*} \quad \text{and} \quad TdR_i = \frac{h^5}{2}u^{(5)*}$$

Thus, as $h \to 0$ and leading terms in matrix $\mathbf{LT}$ defined by ((3.12)-(3.13)) and (4.15) together suggest $\|\mathbf{E}\| \to 0$. Thus we have arrived at the conclusion of the convergence of the proposed method ((2.5)-(2.6)).

5 Numerical experiments

To verify the computational efficiency of the proposed method, we have considered model linear and nonlinear bvps. We have tabulated the numerical results so obtained in numerical experiment. In each tabulated numerical results, we have shown MAU and MAV the maximum absolute error respectively in the approximate solution $u(x)$ and $u''(x)$ of the problems (1.1) for different values of N. Through out our experiment we took uniform step length i.e. $Nh = 1$. We have used the following formula in computation of MAU and MAV,

$$MAU = \max_{1\leq i\leq N} |U(x_i) - u_i|$$

and

$$MAV = \max_{1\leq i\leq N} |U''(x_i) - u''_i|.$$

where $U(x_i)$, u_i are respectively exact and computed solution of the problem (1.1) and similarly we have defined these terms $U''(x_i)$, u''_i.

For the solution of system of equations (2.5) and (2.6), we have used Gauss Seidel and Newton-Raphson method respectively for linear and nonlinear system of equations. All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^4.

Problem 1. The linear model problem is given by

$$u^{(4)}(x) = xu(x) + f(x), \quad x \in [0, \infty)$$

subject to the boundary conditions

$$u(0) = A \quad , \quad u(\infty) = 0, \quad u''(0) = 2A, \quad \text{and} \quad u''(\infty) = 0$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = \frac{A}{1+x}$. The MAU and MAV computed by method (2.5) and (2.6)) for parameter $A = -0.001$ and different values of N are presented in Table 1.

Problem 2. The non-linear model problem is given by,

$$u^{(4)}(x) = u(x)u'(x) + f(x), \quad x \in [0, \infty)$$

subject to the boundary conditions

$$u(0) = A\exp(-1) \quad , \quad u(\infty) = 0, \quad u''(0) = A\exp(-1), \quad \text{and} \quad u''(\infty) = 0$$

The forcing function $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = A\exp(-x-1)$. The MAU and MAV computed by method (2.5) and (2.6) for $A = -1.0 \times 10^{-3}$ and different values of N are presented in Table 2.

Problem 3. The singular model problem is given by,

$$u^{(4)}(x) = \frac{1}{1-x}u(x) + f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 0 \quad , \quad u(1) = 0, \quad u''(0) = 0, \quad \text{and} \quad u''(1) = \sinh(1)$$

The forcing function $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = \sinh(x) - x\sinh(1)$. The MAU and MAV computed by method (2.5) and (2.6) for different values of N are presented in Table 3.

Table 1: The maximum absolute error (Problem 1).

N	Error	
	MAU	MAV
128	.97725987e -4	.48510226e -3
256	.96911099e -4	.48209959e -3
512	.94430955e -4	.47657390e -3
1024	.84356384e -4	.45150509e -3
2048	.36318941e -4	.23392896e -3

Table 2: The maximum absolute error (Problem 2).

N	Error	
	MAU	MAV
128	.60583826e -4	.64081328e -4
256	.60280481e -4	.63530019e -4
512	.59003843e -4	.61715400e -4
1024	.53748018e -4	.55408704e -4
2048	.39975353e -4	.40596024e -4

Table 3: The maximum absolute error (Problem 3).

N	Error	
	MAU	MAV
8	.10299096e -2	.98250317e -2
16	.55052047e -3	.52661569e -2
32	.27984200e -3	.26741952e -2
64	.13961867e -3	.13356761e -2
128	.69285718e -4	.66446215e -3
256	.14615953e -4	.33078732e -3

We have considered two-point fourth order model linear and nonlinear boundary value problem in ODEs to test the computational efficiency as the application of the proposed method for computation of solution of the problem. We observed in numerical experiment for different values of N presented in tables, the maximum absolute error in solution and second derivative the solution decreases with increase in N. So, it is evident from the tabulated results that method (2.5) and (2.6) has slow rate of the convergent.

6 Conclusions

We have presented a unconventional finite difference method for the numerical solution of two-point fourth order boundary value problems in ODEs in a close-open domain. We used boundary conditions at the open end, i.e. at ∞ in natural way. We have not used approximate boundary conditions. The method leads to a system of equations and solution of the problems are obtained by solving a system of equations. The computational results in experiment satisfy the conceptual development of the method presented in the article. A singular problem in closed domain considered in computational experiment as an application of the method. We observed that proposed method is efficient and produced accurate solution of the problem. However in our numerical experiment, we find proposed method approves theoretical development only in closed domain, not in closed-open domain. Proposed method is convergent and improvement is possible either in the order of accuracy or idea in non conventional proposed method. Studies are in advance stage in these directions.

References

[1] R. P. Agarwal, D. O'Regan, *Higher Order Boundary Value Problems. In: Infinite Interval Problems for Differential, Difference and Integral Equations*, Springer, Dordrecht 2001.

[2] R. P. Agarwal, P. S. Kelevedjiev, *On the solvability of fourth-order two-point boundary value problems*, Mathematics 8 (2020), 603; doi:10.3390/math8040603.

[3] J. V. Baxley, *Existence and uniqueness for nonlinear boundary value problems on infinite intervals*, J. Math. Anal. Appl. 147 (1990), 122-133.

[4] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover, New York, USA, 2001.

[5] R. A. Bun, Y. D. Varsil'yev, *A numerical method for solving differential equations with derivatives of any orders*, Comp. Math. Phys. 32(3) (1992), 317-330.

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Jr. Zang, *Spectral Methods - Fundamentals in Single Domains*, Springer-Verlag Berlin Heidelberg, 2006.

[7] R. Fazio, *A survey on free boundary identification of the truncated boundary in numerical BVPs on infinite intervals*, Journal of Computational and Applied Mathematics 140 (1-2) (2002), 331-344.

[8] R. Fazio, A. Jannelli, *Finite difference schemes on quasi-uniform grids for BVPs on infinite intervals*, Journal of Computational and Applied Mathematics 269 (2014), 14-23.

[9] C. E. Froberg, *Introduction to Numerical Analysis*, 2nd ed., Addison-Wesley, New York, 1969.

[10] M. I. Gil, *Invertibility conditions for block matrices and estimates for norms of inverse matrices*, Rocky Mountain Journal of Mathematics 33, 4 (2003), 1323-1335.

[11] L. G. Jaeger, *Elementary Theory of Elastic Plates. Pergamon Press Ltd.*, Headington Hill Hall, Oxford 4 & 5, Fitzroy Square, London W.l, 1969.

[12] M. K. Jain, S. R. K. Iyenger, R. K. Jain, *Numerical Methods for Scientific and Engineering Computation (2/e)*, Willey Eastern Limited, New Delhi, 1987.

[13] H. Kraus, *Thin Elastic Shells*, John Wiley & sons Inc., New York, 1967.

[14] M. Lentini, H. B. Keller, *Boundary value problems on semi-infinite intervals and their numerical solutions*, SIAM J. Numer. Anal. 17 (1980), 577-604.

[15] X. L. Liu, W. T. Li, *Existence and multiplicity of solutions for fourth-order boundary value problems with parameters*, Journal of Mathematical Analysis and Applications 327, 1 (2007), 362-375.

[16] R. K. Mohanty, P. K. Pandey, *Difference methods of order two and four for systems of mildly non-linear biharmonic problems of second kind in two space dimensions*, Numerical Methods for Partial Differential Equations 12, 6 (1996), 707-717.

[17] P. K. Palamides, *Boundary-value problems for shallow elastic membrane caps*, IMA Journal of Applied Mathematics, 67, 3 (2002), 281-299.

[18] P. K. Pandey, *A fourth order accurate finite difference method for the second biharmonic equations*, J. Adv. Res. Sci. Comput. 4 (3)(2012), 35-48.

[19] P. K. Pandey, *A method for an approximate numerical solution of two point boundary value problems: nonstandard finite difference method on semi open interval*, to appear.

[20] W. Soedel, *Vibrations of Shells and Plates*, Marcel Dekker Inc., New York, NY, 2004.

[21] J. W. Stephenson, *Single cell discretization of order two and four for biharmonic problems*, Journal of Computational Physics, 55, 1 (1984), 65-80.

[22] S. P. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[23] S. V. Tsynkov, *Numerical solution of problems on unbounded domains: A review*, Applied Numerical Mathematics, 27, 4 (1998), 465-532.

[24] Ş. Yardamca, E. Uğurlu, *Nonlinear fourth order boundary value problem*, Bound. Value Probl. 2014, 189 (2014). https://doi.org/10.1186/s13661-014-0189-0

Author's address:

Pramod Kumar Pandey
Department of Mathematics,
Dyal Singh College (Univ. of Delhi),
Lodhi Road, New Delhi-110003, India.
E-mail: pramod_10p@hotmail.com