A distinguished geometry perspective on multi-time affine quadratic Lagrangians

M. Neagu

Abstract. For a space endowed with a general quadratic multi-time Lagrangian and an associated non-linear connection, the paper constructs the main Riemann-Lagrange distinguished geometric objects (linear connection, torsion and curvature). Some Einstein-like equations for a canonical geometrical abstract multi-time gravitational potential, together with a trivial geometrical abstract electromagnetic-like theory, are derived from the given quadratic affine multi-time Lagrangian and its associated non-linear connection.

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1 Introduction

It is notable fact that quadratic multi-time Lagrangians are present in most physical domains. Illustrative examples are present in the theory of elasticity [7], the dynamics of ideal fluids, the magnetohydrodynamics [3], [4] and in the theory of bosonic strings [2]. This fact encourages the natural attempt of geometrization for quadratic multi-time Lagrangians. This framework implies, as can be seen below, the introduction of a corresponding Riemann-Lagrange geometry on 1-jet spaces.

2 The generalized multi-time Lagrange space of a quadratic Lagrangian

Let $U_{(i)}^{(\alpha)}(t^\gamma, x^k)$ be a d-tensor (*distinguished* tensor, in brief) on the 1-jet space $J^1(T, M)$, and let $F : T \times M \to \mathbb{R}$ be a smooth function. We further consider the quadratic multi-time Lagrangians $L : J^1(T, M) \to \mathbb{R}$, of the form

$$L = G_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t^\gamma, x^k)x_\alpha^i + F(t^\gamma, x^k), \tag{2.1}$$

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whose fundamental vertical metrical d-tensor

$$G_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k) = \frac{1}{2}\frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j}$$

is symmetric, of rank $n = \dim M$ and has a constant signature with respect to the indices i and j. By using a semi-Riemannian metric $h = (h_{\alpha\beta}(t^\gamma))_{\alpha,\beta=\overline{1,p}}$ on T and by considering the canonical Kronecker h-regular vertical metrical d-tensor attached to the Lagrangian function (2.1), given by

$$\mathcal{G}_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k) = \frac{1}{p} h^{\alpha\beta}(t^\gamma) h_{\mu\nu}(t^\gamma) G_{(i)(j)}^{(\mu)(\nu)}(t^\gamma, x^k),$$

where $p = \dim T$, we then consider the pair

$$GL(J) = \left(J^1(T, M), \mathcal{G}_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k),\right)$$

which is a generalized multi-time Lagrange space, whose spatial metrical d-tensor is given by the formula

$$g_{ij}(t^\gamma, x^k) = \frac{1}{p} h_{\mu\nu}(t^\gamma) G_{(i)(j)}^{(\mu)(\nu)}(t^\gamma, x^k). \tag{2.2}$$

Definition 2.1. We call the space $GL(J)$ *the canonical generalized multi-time Lagrange space associated with the quadratic Lagrangian function* given by (2.1).

In order to construct the main Riemann-Lagrange geometric objects of the space $GL(J)$, i.e., its d-linear connection, torsions and curvatures, one needs a nonlinear connection $\Gamma = \left(M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)}\right)$ on $J^1(T, M)$. The fundamental vertical metrical d-tensor $\mathcal{G}_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k)$, where g_{ij} is given by (2.2), produces the following natural nonlinear connection [6, p. 88]:

$$M_{(\alpha)\beta}^{(i)} = -H_{\alpha\beta}^\gamma x_\gamma^i, \quad N_{(\alpha)j}^{(i)} = \Gamma_{jm}^i x_\alpha^m + \frac{g^{im}}{2}\frac{\partial g_{jm}}{\partial t^\alpha},$$

where $H_{\alpha\beta}^\gamma$ are the Christoffel symbols of the temporal semi-Riemannian metric $h_{\alpha\beta}$,

$$\Gamma_{jk}^i(t^\mu, x^m) = \frac{g^{ir}}{2}\left(\frac{\partial g_{jr}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r}\right)$$

are the generalized Christoffel symbols of the spatial metric $g_{ij}(t^\gamma, x^k)$. Let

$$\left\{\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i}\right\} \subset \mathcal{X}\left(J^1(T, M)\right) \text{ and } \left\{dt^\alpha, dx^i, \delta x_\alpha^i\right\} \subset \mathcal{X}^*\left(J^1(T, M)\right)$$

be the adapted bases of the nonlinear connection Γ, where [1]

$$\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - M_{(\beta)\alpha}^{(j)}\frac{\partial}{\partial x_\beta^j}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(\beta)i}^{(j)}\frac{\partial}{\partial x_\beta^j},$$

$$\delta x_\alpha^i = dx_\alpha^i + M_{(\alpha)\beta}^{(i)} dt^\beta + N_{(\alpha)j}^{(i)} dx^j.$$

[1] Throughout the rest of the paper, the constructed geometrical objects will be expressed in local adapted components with respect to previous adapted bases.

3 The Riemann-Lagrange geometry of the space $GL(J)$

We further adopt the formalism introduced and developed in core seminal research studies ([1],[6],[5]), and accordingly provide the main results of the Riemann-Lagrange geometry of the generalized multi-time Lagrange space $GL(J)$.

Theorem 3.1 (the Cartan linear connection)**.** *The canonical Cartan linear connection of the space $GL(J)$ is given by its adapted components*

$$C\Gamma = \left(H^{\gamma}_{\alpha\beta}, G^{k}_{j\gamma}, L^{i}_{jk} = \Gamma^{i}_{jk}, C^{i(\gamma)}_{j(k)} = 0\right),$$

where $G^{k}_{j\gamma} = \dfrac{g^{km}}{2}\dfrac{\partial g_{mj}}{\partial t^{\gamma}}$.

Proof. The formulas which describe the adapted coefficients of the Cartan canonical connection are given by

$$G^{k}_{j\gamma} = \frac{g^{km}}{2}\frac{\delta g_{mj}}{\delta t^{\gamma}} = \frac{g^{km}}{2}\frac{\partial g_{mj}}{\partial t^{\gamma}}, \quad L^{i}_{jk} = \frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^{k}} + \frac{\delta g_{km}}{\delta x^{j}} - \frac{\delta g_{jk}}{\delta x^{m}}\right) = \Gamma^{i}_{jk},$$

$$C^{i(\gamma)}_{j(k)} = \frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial x^{k}_{\gamma}} + \frac{\partial g_{km}}{\partial x^{j}_{\gamma}} - \frac{\partial g_{jk}}{\partial x^{m}_{\gamma}}\right) = 0.$$

□

Remark 3.1. The generalized Cartan canonical connection $C\Gamma$ of the space $GL(J)$ satisfies the metricity conditions

$$h_{\alpha\beta/\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta}|^{(\gamma)}_{(k)} = 0, \qquad g_{ij/\gamma} = g_{ij|k} = g_{ij}|^{(\gamma)}_{(k)} = 0,$$

where " $_{/\gamma}$", " $_{|k}$" and " $|^{(\gamma)}_{(k)}$" are the local covariant derivatives produced by the Cartan connection $C\Gamma$.

Theorem 3.2. *The generalized multi-time Lagrange space $GL(J)$ is characterized by the following adapted torsion d-tensors:*

$$T^{m}_{\alpha j} = -G^{m}_{j\alpha}, \quad P^{m(\beta)}_{i(j)} = C^{m(\beta)}_{i(j)} = 0, \quad P^{(m)\ (\beta)}_{(\mu)i(j)} = \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^{j}_{\beta}} - \delta^{\beta}_{\mu}L^{m}_{ij} = 0,$$

$$P^{(m)\ (\beta)}_{(\mu)\alpha(j)} = \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^{j}_{\beta}} - \delta^{\beta}_{\mu}G^{m}_{j\alpha} + \delta^{m}_{j}H^{\beta}_{\mu\alpha} = -\delta^{\beta}_{\mu}G^{m}_{j\alpha},$$

$$R^{(m)}_{(\mu)\alpha\beta} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta t^{\beta}} - \frac{\delta M^{(m)}_{(\mu)\beta}}{\delta t^{\alpha}}, \quad R^{(m)}_{(\mu)\alpha j} = \frac{\delta M^{(m)}_{(\mu)\alpha}}{\delta x^{j}} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta t^{\alpha}},$$

$$R^{(m)}_{(\mu)ij} = \frac{\delta N^{(m)}_{(\mu)i}}{\delta x^{j}} - \frac{\delta N^{(m)}_{(\mu)j}}{\delta x^{i}}, \quad S^{(m)(\alpha)(\beta)}_{(\mu)(i)(j)} = \delta^{\alpha}_{\mu}C^{m(\beta)}_{i(j)} - \delta^{\beta}_{\mu}C^{m(\alpha)}_{j(i)} = 0.$$

Using the general formulas from [6], which provide the curvature d-tensors of a generalized multi-time Lagrange space, we obtain the following result:

Theorem 3.3. *The generalized multi-time Lagrange space $GL(J)$ is characterized by the following adapted curvature d-tensors:*

$$H^{\alpha}_{\eta\beta\gamma} = \frac{\partial H^{\alpha}_{\eta\beta}}{\partial t^{\gamma}} - \frac{\partial H^{\alpha}_{\eta\gamma}}{\partial t^{\beta}} + H^{\mu}_{\eta\beta}H^{\alpha}_{\mu\gamma} - H^{\mu}_{\eta\gamma}H^{\alpha}_{\mu\beta},$$

$$R^{l}_{i\beta\gamma} = \frac{\partial G^{l}_{i\beta}}{\partial t^{\gamma}} - \frac{\partial G^{l}_{i\gamma}}{\partial t^{\beta}} + G^{m}_{i\beta}G^{l}_{m\gamma} - G^{m}_{i\gamma}G^{l}_{m\beta},$$

$$R^{l}_{i\beta k} = \frac{\partial G^{l}_{i\beta}}{\partial x^{k}} - \frac{\partial \Gamma^{l}_{ik}}{\partial t^{\beta}} + G^{m}_{i\beta}\Gamma^{l}_{mk} - \Gamma^{m}_{ik}G^{l}_{m\beta},$$

$$R^{l}_{ijk} = \frac{\partial \Gamma^{l}_{ij}}{\partial x^{k}} - \frac{\partial \Gamma^{l}_{ik}}{\partial x^{j}} + \Gamma^{m}_{ij}\Gamma^{l}_{mk} - \Gamma^{m}_{ik}\Gamma^{l}_{mj},$$

$$P^{l\ \ (\gamma)}_{i\beta(k)} = 0, \quad P^{l\ (\gamma)}_{ij(k)} = 0, \quad S^{l(\beta)(\gamma)}_{i(j)(k)} = 0.$$

4 Generalized multi-time field theories on the space $GL(J)$

4.1 Multi-time gravitational field

The fundamental vertical metrical d-tensor of the space $GL(J)$ naturally induces the multi-time gravitational h-potential G, defined by

$$G = h_{\alpha\beta}dt^{\alpha}\otimes dt^{\beta} + g_{ij}dx^{i}\otimes dx^{j} + h^{\alpha\beta}g_{ij}\delta x^{i}_{\alpha}\otimes \delta x^{j}_{\beta}.$$

We postulate that the generalized Einstein-like equations corresponding to the multi-time gravitational h-potential of the space $GL(J)$, are of the form

$$Ric(C\Gamma) - \frac{Sc(C\Gamma)}{2}G = \mathcal{K}\mathcal{T}, \tag{4.1}$$

where $Ric(C\Gamma)$ represents the Ricci d-tensor associated with the generalized Cartan connection, $Sc(C\Gamma)$ is the scalar curvature, $\mathcal{K}$ is the Einstein curvature scalar and $\mathcal{T}$ is the stress-energy d-tensor of matter.

Using now the general formulas from [6], we infer the following:

Proposition 4.1. *The Ricci d-tensor $Ric(C\Gamma)$ of the space $GL(J)$ has the following adapted components:*

$$R_{\alpha\beta} := H_{\alpha\beta} = H^{\mu}_{\alpha\beta\mu}, \quad R^{\ (\alpha)}_{i(j)} := P^{\ (\alpha)}_{i(j)} = -P^{m\ (\alpha)}_{im(j)} = 0,$$

$$R^{(\alpha)}_{(i)j} := P^{(\alpha)}_{(i)j} = P^{m\ (\alpha)}_{ij(m)} = 0, \quad R^{(\alpha)}_{(i)\beta} := P^{(\alpha)}_{(i)\beta} = P^{m\ (\alpha)}_{i\beta(m)} = 0,$$

$$R^{(\alpha)(\beta)}_{(i)(j)} := S^{(\alpha)(\beta)}_{(i)(j)} = S^{m(\beta)(\alpha)}_{i(j)(m)} = 0, \quad R_{i\alpha} = R^{m}_{i\alpha m}, \quad R_{ij} = R^{m}_{ijm}.$$

Corollary 4.2. *The scalar curvature $Sc(C\Gamma)$ of the space $GL(J)$ is given by*

$$Sc(C\Gamma) = H + R,$$

where $H = h^{\alpha\beta}H_{\alpha\beta}$ and $R = g^{ij}R_{ij}$.

Then we can state the main result of the generalized Riemann-Lagrange geometry of the multi-time gravitational field:

Theorem 4.3. *The global generalized Einstein-like equations (4.1) of the space $GL(J)$ have the local form*

$$\begin{cases} H_{\alpha\beta} - \dfrac{H+R}{2} h_{\alpha\beta} = \mathcal{K}\mathcal{T}_{\alpha\beta} \\ R_{ij} - \dfrac{H+R}{2} g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ -\dfrac{H+R}{2} h^{\alpha\beta} g_{ij} = \mathcal{K}\mathcal{T}^{(\alpha)(\beta)}_{(i)(j)} \\ 0 = \mathcal{T}_{\alpha i}, \qquad R_{i\alpha} = \mathcal{K}\mathcal{T}_{i\alpha}, \quad 0 = \mathcal{T}^{(\alpha)}_{(i)\beta} \\ 0 = \mathcal{T}_{\alpha(i)}^{\ \ (\beta)}, \quad 0 = \mathcal{T}_{i(j)}^{\ \ (\alpha)}, \qquad 0 = \mathcal{T}^{(\alpha)}_{(i)j}, \end{cases}$$

where $\mathcal{T}_{AB}$, $A, B \in \left\{\alpha,\ i,\ {}^{(\alpha)}_{(i)}\right\}$ are the adapted components of the stress-energy d-tensor $\mathcal{T}$.

4.2 The multi-time electromagnetism

The multi-time electromagnetic theory of the space $GL(J)$ relies on the *metrical deflection d-tensors*

$$D^{(\alpha)}_{(i)j} = \left[\mathcal{G}^{(\alpha)(\mu)}_{(i)(m)} x^m_\mu\right]_{|j} = -\frac{h^{\alpha\mu}}{2}\frac{\partial g_{ij}}{\partial t^\mu},$$
$$d^{(\alpha)(\beta)}_{(i)(j)} = \left[\mathcal{G}^{(\alpha)(\mu)}_{(i)(m)} x^m_\mu\right]|^{(\beta)}_{(j)} = h^{\alpha\beta} g_{ij}.$$

This yields the *electromagnetic-like* 2*-form* of the space $GL(J)$, via:

$$F = F^{(\alpha)}_{(i)j}\delta x^i_\alpha \wedge dx^j + f^{(\alpha)(\beta)}_{(i)(j)}\delta x^i_\alpha \wedge \delta x^j_\beta,$$

where

$$F^{(\alpha)}_{(i)j} = \frac{1}{2}\left[D^{(\alpha)}_{(i)j} - D^{(\alpha)}_{(j)i}\right] = 0, \qquad f^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\left[d^{(\alpha)(\beta)}_{(i)(j)} - d^{(\alpha)(\beta)}_{(j)(i)}\right] = 0.$$

Since $F = 0$, we infer that the multi-time electromagnetic theory of the space $GL(J)$ is formally trivial.

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Mircea Neagu
Transilvania University of Braşov,
Department of Mathematics and Informatics,
Blvd. Iuliu Maniu 50, 500091 Braşov, Romania.
E-mail: mircea.neagu@unitbv.ro