Bipolar (λ, δ)-fuzzy ideals in LA-semigroups

T. Gaketem

Abstract. In this paper, the notation of bipolar (λ, δ)-fuzzy LA-subsemigroups is introduced and also some properties of bipolar-(λ, δ)-fuzzy LA-subsemigroup, left (right generalized bi, bi-) ideals of LA-semigroups have been discussed.

M.S.C. 2010: 06F35, 03E72, 20N10, 20M12.
Key words: LA-semigroup; bipolar (λ, δ)-fuzzy ideals; bipolar (λ, δ)-fuzzy bi-ideals.

1 Introduction

The concept of a fuzzy set was introduced by Zadeh in 1965. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. There has been a rapid growth in the interest of fuzzy set theory and its applications from the past several years. Many researchers published high-quality research articles on fuzzy sets in a variety of international journals. Lee [2] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree indicate that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ assign that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ assign that elements somewhat satisfy the implicit counter-property. They introduced the concept of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCH-algebra. Lee applied the notion of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. Also some results on bipolar-valued fuzzy BCK/BCI-algebras are introduced by Saeid.

B. Yao [5], introduced (λ, δ)-fuzzy normal subfields. Khan et al. [5] characterized ordered semigroups in terms of (λ, δ)-fuzzy bi-ideals.

In 1972, M. A. Kazim and M. Naseerudin [7], have introduced a *pseudo associate law* or *left invertive law* in a groupoid $\mathfrak{S}$ by $(ab)c = (cb)a$ for all $a, b, c \in \mathfrak{S}$, and called the groupoid a *left almost semigroup* (abbreviated as a LA-semigroup). An LA-semigroup satisfies the medial law $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in \mathfrak{S}$. It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup. An LA-semigroup $\mathfrak{S}$ with left identity the satisfies paramedial law, that is, $(ab)(cd) = (db)(ca)$ and $\mathfrak{S}$ satisfies the following law $a(bc) = b(ac)$ for all $a, b, c, d \in \mathfrak{S}$. A non-empty subset $\mathfrak{A}$ of an LA-semigroup S is called a left (right) ideal of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{A} \subseteq \mathfrak{A}(\mathfrak{A}\mathfrak{S} \subseteq \mathfrak{A})$ and it is called a two-sided ideal if it is both left and a right ideal of $\mathfrak{S}$.

Applied Sciences, Vol.23, 2021, pp. 49-55.

In 2011, N. Yaqoob [4] have introduced a bipolar valued fuzzy LA-subsemigroups and bipolar-valued fuzzy left (right, bi-, interior) ideals in LA-semigroups.

In this paper we defined the notation bipolar (λ, δ)-fuzzy ideal in LA-semigroups and some properties of these ideals are studied.

2 Preliminaries

Definition 2.1. [7] A groupoid $\mathfrak{S}$ is called a left almost semigroup (abbreviated as an LA-semigroup) if it satisfy the left invertive law :

$$(ab)c = (cb)a \text{ for all } a, b, c \in \mathfrak{S}.$$

Lemma 2.1. *[7] If $\mathfrak{S}$ is a unitary LA-semigroup, then the* medial law *holds, that is,*

$$(ab)(cd) = (ac)(bd), \quad \textit{for all} \quad a, b, c, d \in \mathfrak{S}.$$

Definition 2.2. If there is an element 0 of an LA -semigroup $\mathfrak{S}$ such that $x0 = 0x = x$ for all $x \in \mathfrak{S}$, we call 0 a *zero element*of $\mathfrak{S}$.

Proposition 2.2. *[8] If $\mathfrak{S}$ is an LA-semigroup with left identity, then*

$$a(bc) = b(ac), \quad \textit{for all} \quad a, b, c \in \mathfrak{S}.$$

Definition 2.3. [8] An LA-semigroup $\mathfrak{S}$ is called a *paramedial* if its satisfies

$$(ab)(cd) = (dc)(ba), \quad \text{for all} \quad a, b, c, d \in \mathfrak{S}.$$

Definition 2.4. [4] An LA-semigroup $\mathfrak{S}$ is called a *locally associative* LA-semigroup if it satisfies

$$(aa)a = a(aa), \quad \text{for all} \quad a \in \mathfrak{S}.$$

Definition 2.5. Let $\mathfrak{S}$ be an LA-semigroup. A non-empty subset $\mathfrak{A}$ of $\mathfrak{S}$ is called an *LA-subsemigroup* of $\mathfrak{S}$ if $\mathfrak{A}^2 \subseteq \mathfrak{A}$.

Definition 2.6. [9] A non-empty subset $\mathfrak{A}$ of an LA-semigroup $\mathfrak{S}$ is called a *left (right) ideal* of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{A} \subseteq \mathfrak{A}(\mathfrak{A}\mathfrak{S} \subseteq \mathfrak{A})$, and is called an *ideal* of $\mathfrak{S}$ if it is both left and right ideal of $\mathfrak{S}$.

Now we will recall the concept of bipolar-valued fuzzy sets.

Definition 2.7. [4] Let $\mathfrak{X}$ be a nonempty set. A bipolar-valued fuzzy subset (BVF-subset, in short) $\mathfrak{B}$ of $\mathfrak{X}$ is an object having the form

$$\mathfrak{B} = \{\langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x)\rangle : x \in \mathfrak{X}\}, \tag{2.1}$$

where $\mu^+_{\mathfrak{B}}(x) : \mathfrak{X} \to [0, 1]$ and $\mu^-_{\mathfrak{B}}(x) : \mathfrak{X} \to [-1, 0]$

The positive membership degree $\mu^+_{\mathfrak{B}}(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mathfrak{B}$ od the form (2.1) and the negative membership degree $\mu^-_{\mathfrak{B}}(x)$ denotes the satisfaction degree of x to some implicit counter property of $\mathfrak{B} = \{\langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x)\rangle : x \in \mathfrak{X}\}$. For the sake of simplicity, we shall use the symbol $\mathfrak{B} = \{\langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x)\rangle\}$ for the bipolar-valued fuzzy set $\mathfrak{B} = \{\langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x)\rangle : x \in \mathfrak{X}\}$.

Definition 2.8. [4] Let $\mathfrak{B}_1 = \langle \mu^+_{\mathfrak{B}_1}, \mu^-_{\mathfrak{B}_1} \rangle$ and $\mathfrak{B}_2 = \langle \mu^+_{\mathfrak{B}_2}, \mu^-_{\mathfrak{B}_2} \rangle$ be two BVF-subsets of a nonempty set $\mathfrak{X}$. Then the product of two BVF-subsets is denoted by $\mathfrak{B}_1 \circ \mathfrak{B}_2$ and defined as:

$$\mu^+_{\mathfrak{B}_1} \circ \mu^+_{\mathfrak{B}_2}(x) = \begin{cases} \bigvee_{x=yz} \{\mu^+_{\mathfrak{B}_1}(y) \vee \mu^+_{\mathfrak{B}_2}(z)\}, & \text{if} \quad x = yz \quad \text{for some} \quad y, z \in \mathfrak{S} \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu^-_{\mathfrak{B}_1} \circ \mu^-_{\mathfrak{B}_2}(x) = \begin{cases} \bigwedge_{x=yz} \{\mu^-_{\mathfrak{B}_1}(y) \wedge \mu^-_{\mathfrak{B}_2}(z)\}, & \text{if} \quad x = yz \quad \text{for some} \quad y, z \in \mathfrak{S} \\ 0, & \text{otherwise.} \end{cases}$$

Note that an LA-semigroup $\mathfrak{S}$ can be considered as a BVF-subset of itself and let

$$\begin{aligned} \Gamma &= \langle \mathcal{S}^+_\Gamma(x), \mathcal{S}^-_\Gamma(x) \rangle \\ &= \langle \mathcal{S}^+_\Gamma(x), \mathcal{S}^-_\Gamma(x) \rangle : \mathcal{S}^+_{\Gamma(x)} = 1 \quad \text{and} \quad \mathcal{S}^+_\Gamma(x) = -1, \quad \text{for all} \quad x \in \mathfrak{S} \end{aligned}$$

be a BVF-subset and $\Gamma = \langle \mathcal{S}^+_\Gamma(x), \mathcal{S}^-_\Gamma(x) \rangle$ will be carried out in operations with a BVF-subset $\mathfrak{B} = \{\langle \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x) \rangle$ such that $\mathcal{S}^+$ and $\mathcal{S}^-$ will be used in collaboration with $\mu^+_{\mathfrak{B}}$ and $\mu^-_{\mathfrak{B}}$ respectively. Let BVF($\mathfrak{S}$) denote the set of all BVF-subsets of an LA-semigroup $\mathfrak{S}$.

Proposition 2.3. *[4] Let $\mathfrak{S}$ be an LA-semigroup, then the set $(BVF(\mathfrak{S}), \circ)$ is an LA-semigroup.*

Corollary 2.4. *[4] If $\mathfrak{S}$ is an LA-semigroup, then the medial law holds in $BVF(\mathfrak{S})$.*

Definition 2.9. [5] Let $\mathfrak{B} = \langle x, \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ be a bipolar fuzzy set and $(s, t) \in [-1, 0] \times [0, 1]$. Define:

(1) the set $\mathfrak{B}^+_t = \{x \in \mathfrak{X} \mid \mu^+_{\mathfrak{B}}(x) \geq t\}$ and $\mathfrak{B}^-_s = \{x \in X \mid \mu^-_{\mathfrak{B}}(x) \leq s\}$, which are called positive t-cut of $B = \langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x) \rangle$ and the negative s-cut of $\mathfrak{B} = \langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x) \rangle$, respective,

(2) the set ${}^>\mathfrak{B}^+_t = \{x \in \mathfrak{X} \mid \mu^+_{\mathfrak{B}}(x) > t\}$ and ${}^<\mathfrak{B}^-_s = \{x \in \mathfrak{X} \mid \mu^-_{\mathfrak{B}}(x) < s\}$, which are called strong positive t-cut of $\mathfrak{B} = \langle x, \mu^+_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(x) \rangle$ and the strong negative s-cut of $\mathfrak{B} = \langle x, \mu^+_B(x), \mu^-_B(x) \rangle$, respective,

(3) the set $X^{(t,s)}_{\mathfrak{B}} = \{x \in \mathfrak{X} \mid \mu^+_{\mathfrak{B}}(x) \geq t, \mu^+_{\mathfrak{B}}(x) \leq s\}$ is called an (s, t)-level subset of $\mathfrak{B}$,

(4) the set ${}^S X^{(t,s)}_{\mathfrak{B}} = \{x \in \mathfrak{X} \mid \mu^+_{\mathfrak{B}}(x) > t, \mu^+_{\mathfrak{B}}(x) < s\}$ is called a strong (s, t)-level subset of $\mathfrak{B}$

for all $x \in \mathfrak{S}$.

3 Bipolar (λ, δ) Fuzzy Ideals in LA-semigroups

In this section, we will define the notion of bipolar (λ, δ)-fuzzy ideals in ternary semigroups and discuss some properties of these ideals. In what follows, let $\lambda, \delta \in [0, 1]$ be such that $0 \leq \lambda < \delta \leq 1$. Both λ and δ are arbitrary but fixed.

Definition 3.1. A bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of an LA-semigroup $\mathfrak{S}$ is called a bipolar (λ, δ)-fuzzy LA-subsemigruop of $\mathfrak{S}$ if

(1) $\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \geq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\}$,

(2) $\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \leq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\}$,

for all $x, y \in \mathfrak{B}$

Definition 3.2. A bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of an LA-semigroup $\mathfrak{S}$ is called a bipolar (λ, δ)-fuzzy left (right) ideal of $\mathfrak{S}$ if

(1) $\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \geq \min\ \{\mu^+_{\mathfrak{B}}(y), \delta\}(\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \leq \min\ \{\mu^-_{\mathfrak{B}}(x), \delta\})$,

(2) $\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \leq \max\ \{\mu^-_{\mathfrak{B}}(x), -\delta\}(\max\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \geq \min\ \{\mu^-_{\mathfrak{B}}(x), -\delta\})$,

for all $x, y \in \mathfrak{S}$. A bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of a LA-semigroup $\mathfrak{S}$ is called a bipolar (λ, δ)-fuzzy ideal of $\mathfrak{S}$ if it is a bipolar (λ, δ)-fuzzy left ideal and bipolar (λ, δ)-fuzzy right ideal of $\mathfrak{S}$.

Definition 3.3. A bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of an LA-semigroup $\mathfrak{S}$ is called a bipolar (λ, δ)-fuzzy generalized bi-ideal of $\mathfrak{S}$ if

(1) $\max\ \{\mu^+_{\mathfrak{B}}((xy)z), \lambda\} \geq \min\ \{\mu^+_{\mathfrak{B}}(x) \wedge \mu^+_{\mathfrak{B}}(z), \delta\}$

(2) $\min\ \{\mu^-_{\mathfrak{B}}((xy)z), -\lambda\} \leq \max\ \{\mu^-_{\mathfrak{B}}(x) \vee \mu^-_{\mathfrak{B}}(z), -\delta\}$

for all $x, y, z \in \mathfrak{S}$.

Definition 3.4. A bipolar (λ, δ)-fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of an LA-semigroup $\mathfrak{S}$ is called a bipolar (λ, δ)-fuzzy bi-ideal of $\mathfrak{S}$ if

(1) $\max\ \{\mu^+_{\mathfrak{B}}((xy)z), \lambda\} \geq \min\ \{\mu^+_{\mathfrak{B}}(x) \wedge \mu^+_{\mathfrak{B}}(z), \delta\}$

(2) $\min\ \{\mu^-_{\mathfrak{B}}((xy)z), -\lambda\} \leq \max\ \{\mu^-_{\mathfrak{B}}(x) \vee \mu^-_{\mathfrak{B}}(z), -\delta\}$

for all $x, y \in \mathfrak{S}$.

Theorem 3.1. *A bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ of an LA-semigroup $\mathfrak{S}$ is a bipolar (λ, δ)-fuzzy LA-subsemigroup, left (right generalized bi, bi-)ideal of $\mathfrak{S}$ if and only if $\varnothing \neq \mathfrak{S}^{(t,s)}_{\mathfrak{B}}$ is an LA-subsemigroup, left (right generalized bi, bi-)ideal of $\mathfrak{S}$ for all $(s, t) \in [-\delta, -\lambda) \times (\lambda, \delta]$.*

Proof. Suppose that $B = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ be a bipolar (λ, δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$. Let $x, y \in \mathfrak{S}, (s, t) \in [-\delta, -\lambda) \times (\lambda, \delta]$ and $x, y \in S^{(t,s)}_{\mathfrak{B}}$. Then $\mu^+_{\mathfrak{B}}(x) \geq t, \mu^+_{\mathfrak{B}}(y) \geq t$ also $\mu^-(x) \leq s$ and $\mu^-(y) \leq s$. Since $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ, δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$. Then

$$\begin{aligned} \max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} &\geq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\} \\ &\geq \min\ \{t, t, \delta\} = t. \end{aligned}$$

and

$$\begin{aligned} \min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} &\leq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\} \\ &\geq \min\ \{s, s, -\delta\} = s. \end{aligned}$$

This implies that $\mu^+_{\mathfrak{B}}(xy) \geq t$ and $\mu^-_{\mathfrak{B}}(xy) \leq s$. Thus $xy \in \mathfrak{S}^{(s,t)}_{\mathfrak{B}}$.
Hence $\mathfrak{S}^{(s,t)}_{\mathfrak{B}}$ is an LA-subsemigroup of $\mathfrak{S}$.

Conversely, suppose that $\mathfrak{S}^{(s,t)}_{\mathfrak{B}}$ is an LA-subsemigroup of $\mathfrak{S}$. Let $x, y \in \mathfrak{S}$ such that

$$\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda)\} < \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\}$$

and

$$\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} > \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\}.$$

Then there exists $(s,t) \in [-\delta, \lambda) \times (\lambda, \delta]$ such that

$$\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} < t \leq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+\mathfrak{B}(y), \delta\}$$

and

$$\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} > s \geq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\}.$$

This shows that $\mu^+_{\mathfrak{B}}(x) \geq t, \mu^+_{\mathfrak{B}}(y) \geq t$ and $\mu^+_{\mathfrak{B}}(xy) < t$, also $\mu^-_{\mathfrak{B}}(x) \leq s$, $\mu^-_{\mathfrak{B}}(y) \leq s$ and $\mu^+_{\mathfrak{B}}(xy) > s$. Thus $x, y \in \mathfrak{S}^{(t,s)_{\mathfrak{B}}}$, since $\mathfrak{S}^{(s,t)}_{\mathfrak{B}}$ is an LA-subsemigroup of $\mathfrak{S}$. Therefore $xy \in \mathfrak{S}^{(s,t)}_{\mathfrak{B}}$, but this is a contradiction to $\mu^+_{\mathfrak{B}}(xy) < t$ and $\mu^+_{\mathfrak{B}}(xy) > s$. Thus

$$\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \geq \min\ \{\mu^+\mathfrak{B}(x), u^+(y), \delta\}$$

and

$$\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \leq \max\ \{\mu^-_B(x), \mu^-(y), -\delta\}.$$

Hence $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ,δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$.
The other cases can be seen in a similar way. □

Corollary 3.2. *Every bipolar fuzzy LA-subsemigroup, left (right generalized bi, bi-)ideal $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ,δ)-fuzzy LA-subsemigroup, left (right generalized bi, bi-)ideal of $\mathfrak{S}$ with $\lambda = 0$ and $\delta = 1$.*

Theorem 3.3. *If a bipolar fuzzy subset $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ,δ)-fuzzy LA-subsemigroup, left (right generalized bi, bi-)ideal of S, then the set $\mathfrak{B}_\lambda = \langle {}^>\mathfrak{B}^+_\lambda, {}^<\mathfrak{B}^-_{-\lambda} \rangle$ is an LA-subsemigroup, left (right generalized bi, bi-)ideal of $\mathfrak{S}$, where ${}^>B^+_\lambda = \{x \in \mathfrak{S} \mid \mu^+_{\mathfrak{B}}(x) > \lambda\}$ and ${}^<\mathfrak{B}^-_{-\lambda} = \{x \in \mathfrak{S} \mid \mu^+_{\mathfrak{B}}(x) < -\lambda\}$.*

Proof. Suppose that $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ,δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$ and $x, y \in \mathfrak{S}$ such that $x, y, \in \mathfrak{B}\lambda$. Then $\mu^+_{\mathfrak{B}}(x) > \lambda, \mu^+_{\mathfrak{B}}(y) > \lambda$ and $\mu^-_{\mathfrak{B}}(x) > -\lambda, \mu^-_{\mathfrak{B}}(y) > -\lambda$. Since $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ,δ)-fuzzy LA-subsemigroup we have

$$\begin{aligned} \max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} &\geq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\} \\ &\geq \min\ \{\lambda, \lambda, \delta\} = \lambda. \end{aligned}$$

and

$$\begin{aligned} \min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} &\leq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\} \\ &\geq \min\ \{-\lambda, -\lambda, -\delta\} = -\lambda. \end{aligned}$$

Hence $\mu^+_{\mathfrak{B}}(xy) > \lambda$ and $\mu^-_{\mathfrak{B}}(xy) < -\lambda$ so $xy \in \mathfrak{B}_\lambda$.
Therefore $\mathfrak{B}_\lambda$ is an LA-subsemigroup of $\mathfrak{S}$.
The other cases can be seen in a similar way. □

Theorem 3.4. *A non-empty subset* $\mathfrak{A}$ *of an LA-semigroup* $\mathfrak{S}$ *is an LA-subsemigroup, left (right, generalized bi-, bi-) ideal of* $\mathfrak{S}$ *if and only if the bipolar fuzzy subset* $\mathfrak{B}_=(\mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}})$ *of* $\mathfrak{S}$ *defined as follows:*

$$\mu^+_{\mathfrak{B}}(x) = \begin{cases} \geq \delta & if \quad x \in \mathfrak{A} \\ \lambda, & if \quad x \notin \mathfrak{A} \end{cases} \qquad and \quad \mu^-_B(x) = \begin{cases} \leq -\delta & if \quad x \in \mathfrak{A} \\ -\lambda, & if \quad x \notin \mathfrak{A} \end{cases}$$

is a bipolar (λ, δ)*-fuzzy LA-subsemigroup, left (right, generalized bi-, bi-) ideal of* $\mathfrak{S}$*.*

Proof. Suppose that A is an LA-subsemigroup of $\mathfrak{S}$ and $x, y \in \mathfrak{S}$.

Case 1. If $x, y \in \mathfrak{A}$ then $xy \in \mathfrak{A}$. Thus $\mu^+_{\mathfrak{B}}(xy) \geq \delta$ and $\mu^-_{\mathfrak{B}}(xy) \leq -\delta$. Hence

$$\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \geq \delta = \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\}$$

and

$$\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \leq -\delta = \max\ \{\mu^-_B(x), \mu^-_{\mathfrak{B}}(y), -\delta\}.$$

Case 2. If $x \notin \mathfrak{A}$ or $y \notin \mathfrak{A}$ then $\min\ \{\mu^+_B(x), \mu^+_{\mathfrak{B}}(y), \delta\} = \lambda$ and $\max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\} = -\lambda$. Thus

$$\max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} \geq \lambda = \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\}$$

and

$$\min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} \leq -\lambda = \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\}.$$

Consequently $\mathfrak{B} = (\mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}})$ is a bipolar (λ, δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$.

Conversely, let $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_{\mathfrak{B}} \rangle$ is a bipolar (λ, δ)-Fuzzy LA-subsemigroup of $\mathfrak{S}$ and $x, y \in \mathfrak{A}$. Then $\mu^+_{\mathfrak{B}}(x) \geq \delta, \mu^-_{\mathfrak{B}}(y) \geq \delta$ and $\mu^-_{\mathfrak{B}}(x) \leq -\delta . \mu^-_{\mathfrak{B}}(y) \leq -\delta$.
Since $\mathfrak{B} = \langle \mu^+_{\mathfrak{B}}, \mu^-_B \rangle$ is a bipolar (λ, δ)-Fuzzy LA-subsemigroup of $\mathfrak{S}$, we have

$$\begin{aligned} \max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} &\geq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\} \\ &\geq \min\ \{\delta, \delta, \delta\} = \delta. \end{aligned}$$

and

$$\begin{aligned} \min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} &\leq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-_{\mathfrak{B}}(y), -\delta\} \\ &\geq \min\ \{-\delta, -\delta, -\delta\} = -\delta. \end{aligned}$$

This implies that $xy \in \mathfrak{A}$. Hence $\mathfrak{A}$ is an LA-subsemigroup of $\mathfrak{S}$.
The other cases can be seen in a similar way. □

Theorem 3.5. *A non-empty subset* $\mathfrak{A}$ *of an LA-semigroup* $\mathfrak{S}$ *is an LA-subsemigroup, left (right, generalized bi-, bi-) ideal of* $\mathfrak{B}$ *if and only if* $\mathfrak{B}_{\mathfrak{A}} = \langle \mu^+_{\mathfrak{B}_{\mathfrak{A}}}, \mu^-_{\mathfrak{B}_{\mathfrak{A}}} \rangle$ *is a bipolar* (λ, δ)*-fuzzy LA-subsemigroup, left (right, generalized bi-, bi-) ideal of* $\mathfrak{S}$*.*

Proof. Let $\mathfrak{A}$ be an LA-subsemigroup of $\mathfrak{S}$. Then $B_A = \langle \mu^+_{\mathfrak{B}_{\mathfrak{A}}} \mu^-_{\mathfrak{B}_{\mathfrak{A}}} \rangle$ is a bipolar fuzzy LA-subsemigroup of $\mathfrak{S}$ and by Corollary 3.2, $\mathfrak{B}_{\mathfrak{A}}$ is a bipolar (λ, δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$.

Conversely, suppose that $\mathfrak{B}_{\mathfrak{A}}$ is a bipolar (λ, δ)-Fuzzy LA-subsemigroup of $\mathfrak{S}$ and $x, y \in \mathfrak{A}$. Let $x, y \in \mathfrak{A}$. Then $\mu^+_{\mathfrak{B}_{\mathfrak{A}}}(x) = \mu^+_{\mathfrak{B}_{\mathfrak{A}}}(y) = 1$ and $\mu^-_{\mathfrak{B}_{\mathfrak{A}}}(x) = \mu^-_{\mathfrak{B}_{\mathfrak{A}}}(y) = -1$. Since $\mathfrak{B}_{\mathfrak{A}}$ is bipolar (λ, δ)-fuzzy LA-subsemigroup of $\mathfrak{S}$ we have

$$\begin{aligned} \max\ \{\mu^+_{\mathfrak{B}}(xy), \lambda\} &\geq \min\ \{\mu^+_{\mathfrak{B}}(x), \mu^+_{\mathfrak{B}}(y), \delta\} \\ &\geq \min\ \{1, 1, \delta\} = \lambda. \end{aligned}$$

and

$$\begin{aligned} \min\ \{\mu^-_{\mathfrak{B}}(xy), -\lambda\} &\leq \max\ \{\mu^-_{\mathfrak{B}}(x), \mu^-\mathfrak{B}(y), -\delta\} \\ &\geq \min\ \{-1, -1, -\delta\} = -\delta. \end{aligned}$$

It implies that $\mu^+_{\mathfrak{B}_{\mathfrak{A}}}(xy) \geq \delta$ and $\mu^-_{\mathfrak{B}_{\mathfrak{A}}}(xy) \leq -\delta$. Thus $xy \in \mathfrak{A}$.
Hence $\mathfrak{B}_{\mathfrak{A}} = \langle \mu^+_{\mathfrak{B}_{\mathfrak{A}}} \mu^-_{\mathfrak{B}_{\mathfrak{A}}} \rangle$ is a bipolar fuzzy LA-subsemigroup of $\mathfrak{S}$.
The other cases can be seen in a similar way. □

Acknowledgements. The author is grateful to School of Science, University of Phayao for a grant support.

References

[1] Faisal, N. Yaqoob and A.B. Saeid, *Some results in bipolar-valued fuzzy ordered AG-groupoids*, Disc. Math. Gen. Algebra Appl. 32 (2012), 55-76.

[2] K.M. Lee, *Bi-polar-valued fuzzy sets and their operations*, Proc. Int. Conf. Intel. Tech. Bangkok Thailand, 2000; 307-312.

[3] K.M. Lee, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets*, J. Korean Inst. Intel. Syst. 14, 2 (2004), 125-129.

[4] N. Yaqoob, *Bipolar-Valued Fuzzy Ideals in LA-semigroups*, Journal of Advanced Studies in Topology 3, 1 (2012), 60-71.

[5] N. Yaqoob and M. A. Ansari, *Bipolar (λ, δ)-Fuzzy Ideals in in Ternary Semigroups*, Internaltion Journal of Math. Analysis 7, 36 (2013), 1775-1782.

[6] M. Aslam, S. Abdullah and M. Maqsood, *Bipolar fuzzy ideals in LA-semigroups*, World. Appl. Sci. J. 17, 12 (2012), 1769-1782.

[7] M. A. Kazim, M. Naseeruddin, *On almost semigroups*, The Alig. Bull. Math. 2, (1972), 41-47

[8] M. Khan, Faisal, and V. Amjid, *On some classes of Abel-Grassmann's groupoids*, arXiv:1010.5965v2 [math.GR]. 2 (2010), 1-6.

[9] M. Khan,V. Amjid and Faisal. *Ideals in intra-regular left almost semigroups*, arXiv:1012.5598v1 [math.GR]. (2010), 1-10.

[10] M. Naseeruddin. *Some studies in almost semigroups and flocks*, PhD Thesis, Aligarh Muslim University, Aligarh, India, 1970.

[11] M. Sarwar (Kamran), *Conditions for LA-semigroup to resemble associative structures*, PhD Thesis, Quaid-i-Azam University, 1993.

[12] Q. Mushtaq, S.M. Yousuf, *On LA-semigroups*, The Alig. Bull. Math. 8 (1978), 65-70.

[13] R. Bala, B. Ram, *Trigonometric series with semi-convex coefficients*, Tamang J. Math. 18, 1 (1987), 75-84.

Author's address:

Thiti Gaketem
Department of Mathematics, School of Sciences,
University of Phayao, Phayao, 56000, Thailand.
E-mail: newtonisaac41@yahoo.com