Dynamics of a delayed mathematical model for one predator sharing teams of two preys

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Abstract. A delayed mathematical model consisting of teams of two prey and one predator is studied. Existence of Hopf bifurcation and local stability are obtained by linearizing the model at the positive equilibrium. By using the normal form method and center manifold theorem, Hopf bifurcation properties are studied. Some numerical simulations and a qualitative analysis are presented.

M.S.C. 2010: 34Axx, 34F10, 37K50.
Key words: delay, Hopf bifurcation, predator-prey system, stability, team of prey.

1 Introduction

In a biological point of view, prey-predator dynamics are one of the most basic phenomena in the World for getting the equilibria of the species and a lot of applied mathematicians, economists and ecologists have in time paid much attention to this issue. In [22], Murray presented the first fundamental prey-predator system as Lotka-Volterra system which was proposed to study the dynamics oscillatory of certain fish species in the Adriatic sea. Many kinds of predator-prey systems have been investigated by scholars since the classical works by Lotka [16] and Volterra [28], especially the predator-prey systems describing the interaction among multiple species in nature due to the more complex relationships among species. In [19]-[17], Ment et al. formulated different two predator-one prey competition systems. In [4]-[8], Djomegni et al. studied two prey-one predator competition systems. In [18]- [14], Mbava et al. investigated food chain systems with different functional responses. In [6, 24]. authors introduced two advantages for some animals to form technically a team. The first is that it is more efficient to look for food in a team than doing that alone. The second is that it can reduce predation risk to live in a team. The pursing behavior and the prey reaction could be interesting in modeling observed dynamics to investigate prey-predator system with the assumption that during predation the members of preys help each other. In [6], Elettreby proposed a two-prey one-predator system with linear functional response in which the prey teams help each other. Elettreby studied the local and global stability of the system [6]. In [24], Tripathi et al. formulated a two-prey one-predator system with linear functional response in which teams of preys

Applied Sciences, Vol.23, 2021, pp. 39-48.
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group help each other in the presence of predator while compete each other in the absence of predator and they studied local and global stability of the system. In [25], Tripathi et al. studied two preys one-predator competitive system with Beddington-DeAngelis functional response in which the two teams of prey help each other against the predator and the two teams of prey interact each other competitively in the absence of predator, and they investigated the existing of Hopf bifurcation of the system. Kundu and Maitra formulated a three species predator-prey system with cooperation among the prey in which delays are taken just in the growth components for each of the species and the preys are picked such that there is no competition among them at any condition [15]. They studied the effects of the delays on the proposed system and derived the sufficient conditions for the existence of Hopf bifurcation by choosing the delays as the bifurcation parameter.

In [20], Mishra and Raw obtained a prey- predator system by considering teams of one predator and two preys with Monod-Haldane and Holling type II functional response based on the system proposed by Tripathi et al. [24] and the assumption that the first prey is dangerous and the second prey is harmless for predator:

$$\text{(1.1)} \qquad \begin{cases} \frac{du(t)}{dt} &= a_1u(t) - b_1u^2(t) - \frac{mu(t)p(t)}{iu^2(t)+c_1} + \sigma_1 u(t)v(t)p(t), \\ \frac{dv(t)}{dt} &= a_2v(t) - b_2v^2(t) - \frac{nv(t)p(t)}{v(t)+c_2} + \sigma_2 u(t)v(t)p(t), \\ \frac{dp(t)}{dt} &= \frac{m_1u(t)p(t)}{iu^2(t)+c_1} + \frac{n_1v(t)p(t)}{v(t)+c_2} - \delta_1 p(t) - \delta_2 p^2(t), \end{cases}$$

where $u(t)$ and $v(t)$ are the densities of the two preys at time t, respectively. $p(t)$ is the density of the predator at time t. a_1 and a_2 are the growth rates of the dangerous prey and the innocent prey, respectively; b_1 and b_2 are the intra-specific components of the dangerous prey and the innocent prey, respectively; c_1 and c_2 are the intra-specific components of the dangerous prey and the innocent prey, respectively; σ_1 and σ_2 are the coefficients of the help between teams of two prey; $\frac{m_1}{m}$ and $\frac{n_1}{n}$ are the conversion efficiencies from the dangerous prey and the innocent prey to the predator, respectively; i represents the inverse measure of inhibitory effect. in [20] authors studied the stability and Hopf bifurcation of system (1.1).

Time delays have been incorporated into predator-prey systems in many related works [30]-[5]. Specially, time delay due to the gestation is a common example in predator-prey systems, because that the increment in the predator population due to predator does not appear immediately after consuming the prey and the predator needs a certain period to reproduce their progeny. Therefore, the present birth rate of the predator depends upon the number of individuals present at time $t - \tau$, where τ should be regarded as the gestation period [30]. Based on the above discussions, we incorporate time delay due to gestation of the predator into system (1.1) and investigate the following delayed predator-prey system:

$$\text{(1.2)} \qquad \begin{cases} \frac{du(t)}{dt} &= a_1u(t) - b_1u^2(t) - \frac{mu(t)p(t)}{iu^2(t)+c_1} + \sigma_1 u(t)v(t)p(t), \\ \frac{dv(t)}{dt} &= a_2v(t) - b_2v^2(t) - \frac{nv(t)p(t)}{v(t)+c_2} + \sigma_2 u(t)v(t)p(t), \\ \frac{dp(t)}{dt} &= \frac{m_1u(t-\tau)p(t-\tau)}{iu^2(t-\tau)+c_1} + \frac{n_1v(t-\tau)p(t-\tau)}{v(t-\tau)+c_2} - \delta_1 p(t) - \delta_2 p^2(t), \end{cases}$$

where τ is the time delay due to gestation of the predator. The initial conditions for system (1.2) take the form

$$u(\theta) = \varphi_1(\theta) \geq 0,$$
$$v(\theta) = \varphi_2(\theta) \geq 0,$$
$$p(\theta) = \varphi_3(\theta) \geq 0, \tag{1.3}$$

where $\theta \in [-\tau, 0)$, $(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta)) \in C\{[-\tau, 0], R^3_+\}$, $R^3_+ = \{(u, v, p) : u \geq 0, v \geq 0, p \geq 0\}$.

The rest of this paper is structured as follows. In the next section, local stability and existence of Hopf bifurcation are analyzed. In Section 3, direction and stability of the Hopf bifurcation are determined. In Section 4, simulations are presented in order to verify the correctness of the obtained results. Finally, our paper is end with conclusion in Section 5.

2 Hopf bifurcation and local stability analysis

The characteristic polynomial concerning the positive equilibrium E_* is the following:

$$\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0 + (L_2\lambda^2 + L_1\lambda + L_0)e^{-\lambda\tau} = 0, \tag{2.1}$$

where

$$\begin{aligned}
K_0 &= k_{33}(k_{12}k_{21} - k_{11}k_{22}),\\
K_1 &= k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}k_{21},\\
K_2 &= -(k_{11} + k_{22} + k_{33}),\\
L_0 &= (k_{12}l_{33} - k_{13}l_{32})k_{21} + (k_{13}k_{22} - k_{12}k_{23})l_{31}\\
&\quad +(k_{23}l_{32} - k_{22}l_{33})k_{11},\\
L_1 &= l_{33}(k_{11} + k_{22}) - k_{13}l_{31} - k_{23}l_{32}L_2 = -l_{33},
\end{aligned}$$

and

$$\begin{aligned}
k_{11} &= \frac{2imu_*^2p_*}{(iu_*^2 + c_1)^2} - b_1u_*, k_{12} = \sigma_1u_*p_*,\\
k_{13} &= \sigma_1u_*v_* - \frac{mu_*}{iu_*^2 + c_1},\\
k_{21} &= \sigma_2v_*p_*, a_{22} = -b_2v_* + \frac{nv_*p_*}{(v_* + c_2)^2},\\
k_{23} &= \sigma_2u_*v_* - \frac{mu_*}{iu_*^2 + c_1}, k_{33} = -\delta_1 - 2\delta_2p_*,\\
l_{31} &= \frac{m_1(c_1 - iu_*^2)p_*}{(iu_*^2 + c_1)^2}, l_{32} = \frac{n_1c_2p_*}{(v_* + c_2)^2},\\
l_{33} &= \frac{m_1u_*}{iu_*^2 + c_1} + \frac{n_1v_*}{v_* + c_2}.
\end{aligned}$$

Lemma 2.1. *[20] If the condition (H_1) holds, that is, $2imu_*p_* < b_1(iu_*^2 + c_1)^2$, $np_* < b_2(v_*+c_2)^2$, $v_*(iu_*^2+c_1) < \frac{m}{\sigma_1}$, $u_*(v_*+c_2) < \frac{n}{\sigma_2}$ and $u_*^2 < \frac{c_1}{i}$, then $E_*(u_*, v_*, p_*)$ is locally asymptotically stable (l.a.s) if $\tau = 0$.*

When $\tau > 0$, substituting $\lambda = i\omega(\omega > 0)$ into Eq.(2.1), we obtain

$$-\omega^3 i - K_2\omega^2 + K_1\omega i + (-L_2\omega^2 + L_1\omega i + L_0)e^{-i\omega\tau} = 0. \tag{2.2}$$

Then from (2.2) separating real and imaginary part we get

$$\begin{cases} L_1\omega\sin\omega\tau + (L_0 - L_2\omega^2)\cos\omega\tau = K_2\omega^2 - K_0, \\ L_1\omega\cos\omega\tau - (L_0 - L_2\omega^2)\sin\omega\tau = \omega^3 - K_1\omega. \end{cases} \tag{2.3}$$

Then, we get

$$\omega^6 + (K_2^2 - 2K_1 - L_2^2)\omega^4 + (K_1^2 - 2K_1K_2 - L_1^2)\omega^2 + K_0^2 - L_0^2 = 0. \tag{2.4}$$

We assume that (H_2) from (2.4) has at lease one positive root ω_0. Next, on substituting $K_2^2 - 2K_1 - L_2^2 = K_{02}$, $K_1^2 - 2K_1K_2 - L_1^2 = K_{01}$, $K_{00} = K_0^2 - L_0^2$ and $\omega^2 = z$, then Eq.(2.4) becomes

$$z^3 + K_{02}z^2 + K_{01}z + K_{00} = 0. \tag{2.5}$$

Eliminating $\sin(\omega\tau)$ from (2.3) and substituting ω_0, where ω_0 is a positive root of (2.4), we get

$$\tau_0 = \frac{1}{\omega_0} \times \arccos\left\{\frac{(K_1 - K_2L_2)\omega_0^4 + (K_0L_2 - K_1L_1 + K_2L_1)\omega_0^2 - K_0L_0}{L_2^2\omega_0^4 + (L_1^2 - 2L_0L_2)\omega_0^2 + L_0^2}\right\}.$$

Now, differentiating Eq.(2.1) with respect to τ and then substituting $\lambda = i\omega_0$, we have

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{f'(z_0)}{L_2^2\omega_0^4 + (L_1^2 - 2L_0L_2)\omega_0^2 + L_0^2},$$

where $f(z) = z^3 + K_{02}z^2 + K_{01}z + K_{00}$ and $z_0 = \omega_0^2$. Thus, if $f'(z_0) \neq 0$, then $\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} \neq 0$. Summarizing the above analysis, we can obtain the following results.

Theorem 2.2. *For system (1.2), if the conditions (H_1)-(H_3) hold, then the positive equilibrium $E_*(u_*, v_*, p_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$; system (1.2) undergoes a Hopf bifurcation at $E_*(u_*, v_*, p_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from $E_*(u_*, v_*, p_*)$.*

3 Properties of the Hopf bifurcation

According to [35] we can define the following statements. Let $\mu = \tau - \tau_0$ with $\mu \in R$, then $\mu = 0$ is the Hopf bifurcation value of system (1.2). Let $u_1(t) = u(t) - u_*$, $u_2(t) = v(t) - v_*$, $u_3(t) = p(t) - p_*$, and normalize τ by $t \to (t/\tau)$. Then, system (1.2) can be transformed into the equation in the phase space $C = C([-1, 0], R^3)$ as follows

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1, u_2, u_3)^T \in R^3$, L_μ: $C \to R^3$, F: $R \times C \to R^3$, with

$$L_\mu \phi = (\tau_0 + \mu)(K\phi(0) + L\phi(-1)).$$

where

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

and

$$F(\mu, \phi) = (\tau_0 + \mu)(F_1, F_2, F_3)^T,$$

with

$$\begin{aligned} F_1 &= k_{14}\phi_1(0)\phi_2(0) + k_{15}\phi_1(0)\phi_3(0) + k_{16}\phi_2(0)\phi_3(0) + k_{17}\phi_1^2(0) \\ &\quad + k_{18}\phi_1^2(0)\phi_3(0) + k_{19}\phi_1^3(0) + k_{110}\phi_1(0)\phi_2(0)\phi_3(0) + \cdots, \\ F_2 &= k_{24}\phi_1(0)\phi_2(0) + k_{25}\phi_1(0)\phi_3(0) + k_{26}\phi_2(0)\phi_3(0) + k_{27}\phi_2^2(0) \\ &\quad + k_{28}\phi_2^2(0)\phi_3(0) + k_{29}\phi_2^3(0) + k_{210}\phi_1(0)\phi_2(0)\phi_3(0) + \cdots, \\ F_3 &= k_{34}\phi_3^2(0) + l_{34}\phi_1(-1)\phi_3(-1) + l_{35}\phi_2(-1)\phi_3(-1) + l_{36}\phi_1^2(-1) \\ &\quad + l_{37}\phi_2^2(-1) + l_{38}\phi_1^2(-1)\phi_3(-1) + l_{39}\phi_2^2(-1)\phi_3(-1) \\ &\quad + l_{310}\phi_1^3(-1) + l_{311}\phi_2^3(-1) + \cdots, \end{aligned}$$

and

$$\begin{aligned} l_{14} &= \sigma_1 p_*, l_{15} = \frac{m(iu_*^2 - c_1)}{(iu_*^2 + c_1)^2} + \sigma_1 v_*, l_{16} = \sigma_1 u_*, \\ l_{17} &= \frac{imu_* p_*(3c_1 - iu_*^2)}{(iu_*^2 + c_1)^3} - b_1, l_{18} = \frac{imu_*(3c_1 - iu_*^2)}{(iu_*^2 + c_1)^3}, \\ l_{19} &= \frac{imp_*}{(iu_*^2 + c_1)^2} - \frac{8i^2 mc_1 u_*^2 p_*}{(iu_*^2 + c_1)^4}, l_{110} = \sigma_1, \\ k_{24} &= \sigma_2 p_*, k_{25} = \sigma_2 v_*, k_{26} = \sigma_2 u_* - \frac{nc_2}{(v_* + c_2)^2}, \\ k_{27} &= \frac{nc_2 p_*}{(v_* + c_2)^3} - b_2, k_{28} = \frac{nc_2}{(v_* + c_2)^3}, \\ k_{29} &= \frac{nc_2 p_*}{(v_* + c_2)^4}, k_{210} = \sigma_2, \\ k_{34} &= -\delta_2, l_{34} = \frac{m_1(c_1 - iu_*^2)}{(iu_*^2 + c_1)^2}, l_{35} = \frac{n_1 c_2}{(v_* + c_2)^2}, \\ l_{36} &= \frac{im_1 u_* p_*(iu_*^2 - 3c_1)}{(iu_*^2 + c_1)^3}, l_{37} = -\frac{n_1 c_2 p_*}{(v_* + c_2)^3}, \\ l_{38} &= \frac{im_1 u_*(iu_*^2 - 3c_1)}{(iu_*^2 + c_1)^3}, l_{39} = -\frac{n_1 c_2}{(v_* + c_2)^3}, \\ l_{310} &= \frac{8i^2 m_1 c_1 u_*^2 p_*}{(iu_*^2 + c_1)^4} - \frac{im_1 p_*}{(iu_*^2 + c_1)^2}, l_{311} = \frac{n_1 c_2 p_*}{(v_* + c_2)^4}. \end{aligned}$$

By using Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \phi \in C. \tag{3.2}$$

Choosing

$$\eta(\theta,\mu) = (\tau_0+\mu)(K\delta(\theta) + L\delta(\theta+1)),$$

and $\delta(\theta)$ is the Dirac delta function. For $\phi \in C([-1,0],R^3)$, define the operator $A(\mu)$

$$A(\mu_0)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\mu_0)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu_0,\phi), & \theta = 0. \end{cases}$$

Then, system (1.2) becomes

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{3.3}$$

For $\varphi \in C^1([0,1],(R^3)^*)$, define A^*

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

and a bilinear product

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.4}$$

with $\eta(\theta) = \eta(\theta,0)$.

Then, we can conclude that $i\omega_0$ are eigenvalues of $A(0)$ and $A^*(0)$. Suppose that $\rho(\theta) = (1,\rho_2,\rho_3)^T e^{i\omega_0\theta}$ and $\rho^*(s) = V(1,\rho_2^*,\rho_3^*)e^{i\omega_0 s}$ are the corresponding eigenfunctions. By direct calculation we obtain

$$\begin{aligned} q_2 &= \frac{i\omega_0 - a_{33} - b_{33}e^{-i\omega_0}}{a_{32}} q_3, \\ q_3 &= \frac{a_{32}(i\omega_0 - a_{11})}{a_{12}(i\omega_0 - a_{33} - b_{33}e^{-i\omega_0})}, \\ q_2^* &= -\frac{i\omega_0 + a_{11}}{a_{21}}, \\ q_3^* &= \frac{a_{32}(i\omega_0 + a_{11}) - a_{21}b_{13}e^{i\omega_0}}{a_{21}(i\omega_0 + a_{33} + b_{33}e^{i\omega_0})}. \end{aligned}$$

From $\langle \rho^*, \rho \rangle = 1$, we have

$$\bar{V} = [1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \tau_0 e^{-i\tau_0\omega_0}(b_{13} + b_{33}\bar{\rho}_3^*)]^{-1}.$$

Using the algorithm given in [32] and the similar computation process to that in

[28], we can obtain the expressions of g_{20}, g_{11}, g_{02} and g_{21} as follows:

$$
\begin{aligned}
g_{20} &= 2\bar{V}\tau_0[a_{13}\rho_2 + a_{14}\rho_2^2 + \bar{\rho}_2^*(a_{24}\rho_2 + a_{25}\rho_2^2)],\\
g_{11} &= \bar{V}\tau_0[a_{13}(\rho_2 + \bar{\rho}_2) + 2a_{14}\rho_2\bar{\rho}_2 + \bar{\rho}_2^*(a_{24}(\rho_2 + \bar{\rho}_2) + 2a_{25}\rho_2\bar{\rho}_2)],\\
g_{02} &= 2\bar{V}\tau_0[a_{13}\bar{\rho}_2 + a_{14}\bar{\rho}_2^2 + \bar{\rho}_2^*(a_{24}\bar{\rho}_2 + a_{25}\bar{\rho}_2^2)],\\
g_{21} &= 2\bar{V}\tau_0\Bigg[a_{13}\left(W_{11}^{(1)}(0)\rho_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\\
&\quad + a_{14}(2W_{11}^{(2)}(0)\rho_2 + W_{20}^{(2)}(0)\bar{\rho}_2)\\
&\quad + \bar{\rho}_2^*\left(a_{24}\left(W_{11}^{(1)}(0)\rho_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\right.\\
&\quad \left. + a_{25}(2W_{11}^{(2)}(0)\rho_2 + W_{20}^{(2)}(0)\bar{\rho}_2)\right)\Bigg],
\end{aligned}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2.
\end{aligned}
$$

E_1 and E_2 can be solved by

$$
E_1 = 2\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & -b_{13}e^{-2i\tau_0\omega_0} \\ -a_{21} & 2i\omega_0 - a_{22} & -a_{23} \\ 0 & -a_{32} & 2i\omega_0 - a_{33} - b_{33}e^{-2i\tau_0\omega_0} \end{pmatrix}^{-1} \times \begin{pmatrix} a_{13}\rho_2 + a_{14}\rho_2^2 \\ a_{24}\rho_2 + a_{25}\rho_2^2 \\ 0 \end{pmatrix},
$$

$$
E_2 = 2\begin{pmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} + b_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} a_{13}(\rho_2 + \bar{\rho}_2) + 2a_{14}\rho_2\bar{\rho}_2 \\ a_{24}(\rho_2 + \bar{\rho}_2) + 2a_{25}\rho_2\bar{\rho}_2 \\ 0 \end{pmatrix},
$$

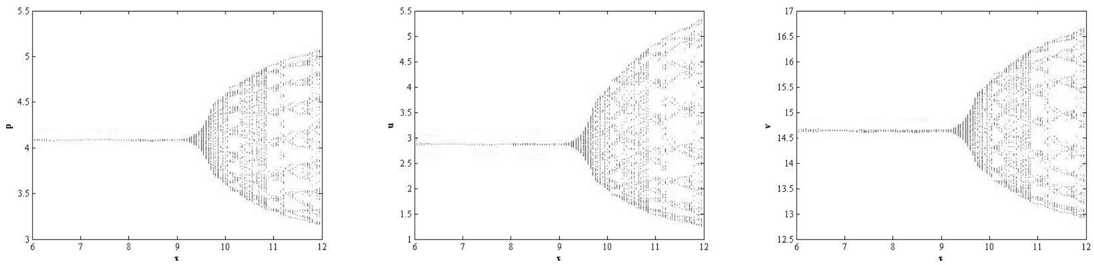
Thus, we have

$$
\begin{aligned}
&C_1(0) = \tfrac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \tfrac{|g_{02}|^2}{3}\right) + \tfrac{g_{21}}{2}\\
&\mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},\\
&\beta_2 = 2\mathrm{Re}\{C_1(0)\},\\
&T_2 = -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0},
\end{aligned}
\tag{3.5}
$$

where μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcating periodic solutions and T_2 determines the period of the bifurcating periodic solutions. In conclusion, we can obtain the following results based on the fundamental results about Hopf bifurcation in the literature [32].

According to [35], we have the following result:

Theorem 3.1. *For system (1.2), the Hopf bifurcation is supercritical (or subcritical) if $\mu_2 > 0$ (or $\mu_2 < 0$); the bifurcating periodic solutions are stable (or unstable) if $\beta_2 < 0$ (or $\beta_2 > 0$); the period of the bifurcating periodic solutions increase (or decrease) if $T_2 > 0$ (or $T_2 < 0$).*



(a) (b) (c)

Bifurcation diagram with respect to time delay.

4 Numerical simulation

Choosing $a_1 = 0.25$, $b_1 = 0.02$, $m = 0.45$, $i = 0.035$, $c_1 = 7$, $\sigma_1 = 0.001$, $a_2 = 0.3$, $b_2 = 0.02$, $n = 0.3$, $c_2 = 8$, $\sigma_2 = 0.004$, $m_1 = 0.1$, $n_1 = 0.1$, $\delta_1 = 0.1$, $\delta_2 = 0.001$. Then, system (1.2) becomes

$$(4.1)\quad \begin{cases} \frac{du(t)}{dt} &= 0.25u(t) - 0.02u^2(t) - \frac{0.45u(t)p(t)}{0.035u^2(t)+7} + 0.001u(t)v(t)p(t), \\ \frac{dv(t)}{dt} &= 0.3v(t) - 0.02v^2(t) - \frac{0.3v(t)p(t)}{v(t)+8} + 0.004u(t)v(t)p(t), \\ \frac{dp(t)}{dt} &= \frac{0.1u(t-\tau)p(t-\tau)}{0.035u^2(t-\tau)+7} + \frac{0.1v(t-\tau)p(t-\tau)}{v(t-\tau)+8} - 0.101p(t)p^2(t), \end{cases}$$

5 Conclusions

In this paper, we proposed a delayed mathematical model for one predator sharing teams of two preys by considering time delay due to the dynamics of the predator into the model formulated in [20]. The aim of this research is to investigate the effect of the time delay on stability of the proposed model. It is found that the time delay due to behavior of the predator is responsible for the stability of the model. When the time delay is suitable small, then the model is locally asymptotically stable. Once the delay passes through a certain critical value, the model will lose its stability and a Hopf bifurcation occurs, which indicates that the species in the model will coexist periodic oscillation mode under some certain conditions. The obtained results in the present paper are supplements to [20].

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