Statistical convergence of multisequences on $\mathbb{R}$

S. Debnath and A. Debnath

Abstract. The main aim of this paper is to introduce the statistical convergence of multisequences on $\mathbb{R}$ and study some basic algebraic and topological properties of a multisequence.

M.S.C. 2010: 46B15, 40C05, 11B39.
Key words: Multiset; multisequence; statistical convergent; statistical limsup.

1 Introduction

The ordinary convergence of a real sequence was generalized to statistical convergence by Fast [6] as follows:

If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K; k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} n^{-1} |K_n|$. A real number sequence x is *statistically convergent* to l provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ has natural density zero; in this case we write $st - lim\ x = l$. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [7, 8, 9].

A multiset (mset) is an unordered collection of objects, unlike a standard Cantor set, elements are allowed to repeat. It is observed from the survey of available literature on multiset and its application that the idea of multiset was hinted by R. Dedekind in 1988. The multiset theory which contains set theory as a special case was introduced by Cerf et al [5]. The term multiset, as Knuth [12] notes was first suggested by N.G. de Bruijn in a private communication to him. Further study was carried out by Peterson [14], Yager [19]. Blizard [3, 4] gave a new dimension to the multiset theory. From a practical point of view multiset are very useful structures arising in many areas of mathematics and computer science. The prime factorization of an integer $n > 0$ is an example of a multiset. The terminal string of a non-circular context-free grammar form a multiset which is a set if and only if the grammar is unambiguous.

Research on the multiset theory has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of multisets. It is possible to extend some of the main notion and results of

Applied Sciences, Vol.23, 2021, pp. 29-38.
© Balkan Society of Geometers, Geometry Balkan Press 2021.

sets to the setting of multisets. In 2009, Girish et al. [6] introduced the concepts of relation, function, composition and equivalence in multiset context.

Tella and Daniel [11] have considered set of mappings between multisets and studied about symmetric groups from multiset perspective. Nazmul et. al. [9] have considered the initial universe set to be a group. Then they have defined a group on the multiset derived from the initial universal set. Recently, these concepts have been extended by many authors [1, 2, 7, 12]. Roy et al. [15] introduced the concepts of M-metric space and studied some of its properties. Recently, we have introduced the convergence concepts of multisequences on $\mathbb{R}$. Therefore the study of statistical convergence concepts in the context of multisequences is very natural. In this paper we have introduced the notion of statistical convergence of multisequence on $\mathbb{R}$ and study some basic algebraic and topological properties of multisequence.

2 Definitions and preliminaries

We present first several definitions which are available in the existing literatures, and which will be further used throughout the paper.

Definition 2.1. [6] A collection of elements which are allowed to repeat is called a multiset. Formally if X is a set of elements, a multiset A drawn from the set X is represented by a function C_A defined by $C_A : X \to \mathbb{N}_0$, where $\mathbb{N}_0$ represents the set of non-negative integers.

For each $x \in X$, $C_A(x)$ is the characteristic value of x in A and indicates the number of occurrences of the element x in A. A multiset A is a set if $C_A(x) = 0$ or 1 for all $x \in X$. The word multiset, often written as mset.

Remark 2.2. Let A be a mset from X with x appearing n times in A. It is denoted by $x \in^n A$.

Definition 2.3. [6] Let A_1 and A_2 be two msets drawn from a set X. Then the Cartesian product of A_1 and A_2 is defined by $A_1 \times A_2 = \{mn/(x,y) : x \in^m A_1, y \in^n A_2\}$.

Definition 2.4. [9] A set of real numbers where repetition of real numbers is allowed, is called multiset of real numbers, denoted by $m\mathbb{R}$. Thus $m\mathbb{R}=\{x_i/c_i : x_i \in \mathbb{R}$ and $c_i \in \mathbb{N}\}$.

Definition 2.5. [7] If $x = (x_k)$ is a real sequence we write $\{x_k : k \in \mathbb{N}\}$ to denote the range of x. If $\{x_{k(j)}\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$. In case $\delta(K) = 0$, $\{x\}_K$ is called a *subsequence of density zero* or a *thin subsequence*. On the other hand, $\{x\}_k$ is a *nonthin subsequence* of x if K does not have density zero. It should be noted that $\{x\}_K$ is a nonthin subsequence of x if either $\delta(K)$ is a positive number or K fails to have natural density.

Definition 2.6. [8] The number λ is a *statistical limit point* of the number sequence $x = (x_i)$ provided that there is a nonthin subsequence of x that converges to λ.

Definition 2.7. [8] The number γ is a *statistical cluster point* of the number sequence $x = (x_i)$ provided that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ does not have density zero.

Definition 2.8. [8] A real sequence $x = (x_i)$ is said to be *statistically bounded* if there is a real number M such that $\delta\,(\{i : |x_i| > M\})$=0.

Definition 2.9. [8] For any real sequence $x = (x_i)$, let B_x denotes the set

$$B_x = \{b \in \mathbb{R} : \delta\,(\{i : x_i > b\}) \neq 0\}$$

and A_x denotes the set

$$A_x = \{a \in \mathbb{R} : \delta\,(\{i : x_i < a\}) \neq 0\}.$$

Then the *statistical limit superior* of x is given by

$$st - lim\ sup\ x = \begin{cases} sup\ B_x, & B_x \neq \phi \\ -\infty, & B_x = \phi \end{cases}$$

and *statistical limit inferior* of $x = (x_i)$ is given by

$$st - lim\ inf\ x = \begin{cases} inf\ A_x, & A_x \neq \phi \\ +\infty, & A_x = \phi \end{cases}.$$

Definition 2.10. A function whose domain is the set $\mathbb{N}$ of natural numbers and range, a set of $m\mathbb{R}$ is called a *multisequence*. Thus a multisequence is denoted symbolically as $mx : N \longrightarrow m\mathbb{R}$.

Let $x = (x_i)$ be a sequence. Then a multisequence, denoted by mx, is defined by $mx = \{x_i/c_i : x_i \in \mathbb{R}, c_i \in \mathbb{N}_0\}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}_0$ is the set of non-negative integers.

3 Main results

Definition 3.1. A multisequence $mx = (x_i/c_i)$ of $m\mathbb{R}$ is statistically convergent to l/c of $m\mathbb{R}$ if given for any $\varepsilon > 0$, the set $I(\varepsilon) = \left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\}$ has natural density zero,

i.e., for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} \dfrac{1}{n}\ |i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon| = 0.$

Example 3.2. Consider a multisequence $mx = (x_i/c_i)$, given by

$$x_i = \begin{cases} i, & i = n^2;\ n = 1, 2, 3, ... \\ 1, & \text{otherwise.} \end{cases} \quad \text{and } c_i = \begin{cases} i, & i = n^3;\ n = 1, 2, 3, ... \\ 5, & \text{otherwise.} \end{cases}$$

Then for any $\varepsilon > 0$, $\quad \lim\limits_{n\to\infty} \dfrac{1}{n}\ |i \leq n : \sqrt{(x_i - 1)^2 + (c_i - 5)^2} \geq \varepsilon|$

= Density of perfect squared or perfect cubic positive integers or both

$\leq \lim_{n\to\infty} \frac{1}{n}\left(n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}}\right) = 0.$

Therefore, the multisequence mx statistically converges to $1/5$.

Example 3.3. Consider the multisequence $mx = (x_i/c_i)$, where

$$x_i = \begin{cases} i,\ i = n^p;\ n = 1, 2, 3, ...and\ p \geq 2,\ a\ fixed\ integer \\ 0,\ \text{otherwise.} \end{cases} \quad \text{and } c_i = i.$$

Then it can be easily shown that the multisequence mx is not statistically convergent.

Example 3.4. Consider the multisequence $mx = (x_i/c_i)$, where

$$x_i = 1/i,\ i \in \mathbb{N} \text{ and } c_i = \begin{cases} 5,\ when\ i\ is\ odd \\ 10,\ when\ i\ is\ even \end{cases}$$

It is obvious that the multisequence mx is not statistically convergent.

Remark 3.5. In example (3.3) and (3.4), if we take $c_i = 1\ \forall\ i$, it reduces to usual sequence in either case and hence both are statistical convergence to $0/1$.

Thus, the statistical convergence of multisequences is a natural generalization of the statistical convergence of usual sequences.

Proposition 3.1. *The convergent multisequences are also statistically convergent.*

Proof. Let $mx = (x_i/c_i)$ be a convergent multisequence, which converges to l/c of $m\mathbb{R}$ then (x_i) converges to l and (c_i) converges to c. Now, since (x_i) converges to l and (c_i) converges to c , then for each $\varepsilon > 0$, each of the sets of $I = \{i \in \mathbb{N} : |x_i - l| \geq \varepsilon\}$ and $I^{'} = \{i \in \mathbb{N} : |c_i - l| \geq \varepsilon\}$ have a finite number of elements and all finite subsets of natural numbers have density zero.

Hence, $\lim_{n\to\infty} \frac{1}{n}\ |i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon| = 0$, i.e., mx converges statistically to l/c of $m\mathbb{R}$. □

Definition 3.6. A multisequence $mx = (x_i/c_i)$ is said to be *statistically bounded* if there exists non-negative real number M such that

$$\delta\left(\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > M\right\}\right) = 0.$$

Example 3.7. The multisequence mentioned in the example (3.2) is statistically bounded, for, there exists real numbers M $(\geq \sqrt{17})$ for which

$$\delta\left(\left\{i \leq n : \sqrt{x_i^2 + (c_i - 1)^2} > M\right\}\right) = \lim_{n\to\infty} \frac{1}{n}\left(n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}}\right) = 0.$$

Example 3.8. The multisequence mentioned in the example (3.3), is not statistically bounded, for, there exists no real number M for which

$$\delta\left(\left\{i \leq n : \sqrt{x_i^2 + (c_i - 1)^2} > M\right\}\right) = 0.$$

Proposition 3.2. *A statistically convergent multisequence is statistically bounded.*

Proof. Let $mx = (x_i/c_i)$ converges statistically to l/c. Then for any $\varepsilon > 0$,

$\delta\left(i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right) = 0,$

which is equivalent to $\delta\left(i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon\right) = 1.$

i.e $\delta\left(i \in \mathbb{N} : |x_i - l| < \varepsilon,\ |c_i - c| < \varepsilon\right) = 1.$

which implies, $\delta\left(i \in \mathbb{N} : l - \varepsilon < x_i < l + \varepsilon,\ c - \varepsilon < c_i < c + \varepsilon\right) = 1.$

i.e., $\delta\left(i \in \mathbb{N} : x_i < M', c_i < M'\right) = 1$, where, $M' = max\,(l + \varepsilon, c + \varepsilon)$

i.e., $\delta\left(i \in \mathbb{N} : x_i^2 + c_i^2 < 2M'^2\right) = 1$

i.e., $\delta\left(i \in \mathbb{N} : x_i^2 + (c_i - 1)^2 < 2M'^2\right) = 1$

i.e., $\delta\left(i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} < M\right) = 1$, where $M^2 = 2M'^2$.

which implies, $\delta\left(i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} > N\right) = 0$, where N is any number greater than M. Hence, mx is statistically bounded. □

Corollary 3.3. *The converse of the above proposition may not be true.*

Proof. We have seen that the multisequence given in example (3.4), is not statistically convergent. But it is statistically bounded, for, there exists M (say, 10) for which $\delta\left(\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > 10\right\}\right) = 0.$ □

Definition 3.9. The number λ/c is a *statistical limit point* of the multisequence $mx = (x_i/c_i)$ provided that there is a nonthin multisubsequence of (x_i/c_i) that converges to λ/c.

Example 3.10. Consider the multisequence $mx = (x_i/c_i)$ where

$$x_i = \begin{cases} 1, & \textit{when } n \textit{ is odd} \\ 0, & \textit{when } n \textit{ is even} \end{cases} \quad \text{and } c_i = \begin{cases} 12, & \textit{when } n \textit{ is odd} \\ 13, & \textit{when } n \textit{ is even} \end{cases}$$

Then it can be shown that $1/12$ and $0/13$ are statistical limit points of the given multisequence.

Definition 3.11. The number γ/c of $m\mathbb{R}$ is said to be a *statistical cluster point* of the multisequence $mx = (x_i/c_i)$ provided that for every $\varepsilon > 0$,
the set $\left\{i \in \mathbb{N} : \sqrt{(x_i - \gamma)^2 + (c_i - c)^2} < \varepsilon\right\}$ does not have density zero.

For any multisequence mx, let Λ_{mx} denote the set of statistical limit points and Γ_{mx} denote the set of statistical cluster points.

Example 3.12. Consider the multisequence $mx = (x_i/c_i)$ of $m\mathbb{R}$, each element having same multiplicity, say c, where $(x_i) = \left(1, \frac{1}{2}, 1, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{1}{5}, ...\right)$. Then there is no multisubsequence converging to a finite element of mx,which is non-thin. So, no element of $m\mathbb{R}$ will be a statistical limit point of mx. Hence, $\Lambda_{mx} = \phi$.

But for any $\varepsilon > 0$, there is only one element $0/c$ of $m\mathbb{R}$, for which $\{i \in \mathbb{N} : \sqrt{(x_i - \gamma)^2 + (c_i - c)^2} < \varepsilon\}$ has density non-zero. Hence, $\Gamma_{mx} = \{0/c\}$.

Definition 3.13. For any multisequence $mx = (x_i/c_i)$, let B_{mx} denotes the set

$$B_{mx} = \left\{ b/c \in m\mathbb{R} : \delta \left(i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{b^2 + (c-1)^2} \right) \neq 0 \right\}$$

and A_{mx} denotes the set

$$A_{mx} = \left\{ a/c \in m\mathbb{R} : \delta \left(i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{a^2 + (c-1)^2} \right) \neq 0 \right\}.$$

l/c is called the *supremum* of B_{mx}, denoted by *sup* B_{mx}, if c is the greatest multiplicity in B_{mx} under the condition $c \leq max\,(c_i)$ in mx and l is the supremum among the different sets of real numbers bearing the multiplicity c in B_{mx}, whenever it exists.

l/c is called the $infimum$ of A_{mx}, denoted by inf A_{mx}, if c is the lowest multiplicity in A_{mx} under the condition $c \geq max\,(c_i)$ in mx and l is the infimum among the different sets of real numbers bearing the multiplicity c in A_{mx} , whenever it exists.

Definition 3.14. If $mx = (x_i/c_i)$ is a multisequence of real numbers, then the *statistical limit superior* of mx is given by

$$st - lim\ sup\ mx = \begin{cases} sup\ B_{mx}, & B_{mx} \neq \phi \\ -\infty, & B_{mx} = \phi \end{cases}$$

and the *statistical limit inferior* of $mx = (x_i/c_i)$ is given by

$$st - lim\ inf\ mx = \begin{cases} inf\ A_{mx}, & A_{mx} \neq \phi \\ +\infty, & A_{mx} = \phi \end{cases}$$

For any multiset of real numbers $m\mathbb{R}$, $(a, b)/c$ means each number in the open interval (a, b) of real numbers has multiplicity c. In the same way in $(-\infty, \infty)/c$ or $(-\infty, a)/c$ or $(a, \infty)/c$, each real number in every interval of real numbers has multiplicity c.

Example 3.15. Consider the multisequence $mx = (x_i/c_i)$ of $m\mathbb{R}$, given by

$$(x_i/c_i) = \begin{cases} 3/4, & if\ k\ is\ an\ odd\ square \\ 2/4, & if\ k\ is\ an\ even\ square \\ 1/6, & if\ k\ is\ an\ odd\ non-square \\ 0/6, & if\ k\ is\ an\ even\ non-square \end{cases}$$

then it can be shown that,

$B_{mx} = \{(-1, 1)/6, \left(-\sqrt{11}, \sqrt{11}\right)/5, \left(-\sqrt{20}, \sqrt{20}\right)/4, \left(-\sqrt{27}, \sqrt{27}\right)/3, \left(-\sqrt{32}, \sqrt{32}\right)/2, \left(-\sqrt{35}, \sqrt{35}\right)/1\}$. Therefore, $st - lim\ sup\ mx = 1/6$.

Again,
$A_{mx} = \{\left(-\infty, -\sqrt{35}\right)/1, \left(\sqrt{35}, \infty\right)/1, \left(-\infty, -\sqrt{32}\right)/2, \left(\sqrt{32}, \infty\right)/2, \left(-\infty, -\sqrt{27}\right)/3, \left(\sqrt{27}, \infty\right)/3, \left(-\infty, -\sqrt{20}\right)/4, \left(\sqrt{20}, \infty\right)/4, \left(-\infty, -\sqrt{11}\right)/5, \left(\sqrt{11}, \infty\right)/5, (-\infty, 0)/6, (0, \infty)/6, (-\infty, \infty)/7, (-\infty, \infty)/8, (-\infty, \infty)/9......\}$. Therefore, $st - lim\ inf\ mx = 0/6$.

Proposition 3.4. *Let* $mx = (x_i/c_i)$ *be a multisequence of* $m\mathbb{R}$*. If* b/c = *st-lim sup* mx *is finite, then for every positive number* ε,

$$\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} > \sqrt{(b-\varepsilon)^2 + (c-1)^2}\right\} \neq 0$$

and

$$\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} > \sqrt{(b+\varepsilon)^2 + (c-1)^2}\right\} = 0 \text{ and conversely.}$$

Proof. The proof can be made straightforward. □

Proposition 3.5. *Let* $mx = (x_i/c_i)$ *be a multisequence of* $m\mathbb{R}$*. If* a/c_1 = *st-lim inf* mx *is finite, then for every positive number* ε,

$$\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} < \sqrt{(a+\varepsilon)^2 + (c_1-1)^2}\right\} \neq 0$$

and

$$\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} < \sqrt{(a-\varepsilon)^2 + (c_1-1)^2}\right\} = 0 \text{ and conversely.}$$

Proof. The proof can be made straightforward. □

From the definition of statistical cluster point in (3.11), we see that propositions (3.4) and (3.5) can be interpreted as saying that $st-lim\ sup\ mx$ and $st-lim\ inf\ mx$ are the greatest and least cluster points of mx. The next theorem reinforces that observation.

Proposition 3.6. *For any multisequence* $mx = (x_i/c_i)$ *in* $m\mathbb{R}$,

$$st - lim\ inf\ mx \leq st - lim\ sup\ mx.$$

Proof. First consider the case in which $st - lim\ sup\ mx = -\infty$. This implies that $B_{mx} = \phi$, so for every b/c in $m\mathbb{R}$, $\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} > \sqrt{b^2 + (c-1)^2}\right\} = 0$. This implies that $\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} \leq \sqrt{b^2 + (c-1)^2}\right\} = 1$, so for every α/β in $m\mathbb{R}$, $\delta\left\{\sqrt{x_i^2 + (c_i-1)^2} < \sqrt{\alpha^2 + (\beta-1)^2}\right\} \neq 0$. Hence, $st - lim\ inf\ mx = -\infty$.

The case in which $st - lim\ sup\ mx = +\infty$, needs no proof, so we next assume that $\beta/c = st - lim\ sup\ mx$ is finite, and let $\alpha/c' = st - lim\ inf\ mx$. Given $\varepsilon > 0$ we show that $(\beta+\varepsilon)/c \in A_{mx}$, so that $\sqrt{\alpha^2 + (c'-1)^2} < \sqrt{(\beta+\varepsilon)^2 + (c-1)^2}$. By theorem (3.23), $\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} > \sqrt{(\beta+\varepsilon)^2 + (c-1)^2}\right\} = 0$, because $\beta/c = st-lim\ sup\ mx$. This implies that $\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} \leq \sqrt{(\beta+\varepsilon)^2 + (c-1)^2}\right\} = 1$ which, in turn, implies that $\delta\left\{i : \sqrt{x_i^2 + (c_i-1)^2} < \sqrt{(\beta+\varepsilon)^2 + (c-1)^2}\right\} = 1$ Hence, $(\beta+\varepsilon)/c \in A_{mx}$. By definition, $\alpha/c' = st - lim\ inf\ mx$, so we conclude that $\sqrt{\alpha^2 + (c'-1)^2} \leq \sqrt{(\beta+\varepsilon)^2 + (c-1)^2}$; and since ε is arbitrary this gives us $\sqrt{\alpha^2 + (c'-1)^2} \leq \sqrt{\beta^2 + (c-1)^2}$, i.e., $st - lim\ inf\ mx \leq st - lim\ sup\ mx$. □

Proposition 3.7. *The statistically bounded multisequence $mx = (x_i/c_i)$ of $m\mathbb{R}$ is statistically convergent if and only if*

$$st - lim\ inf\ mx = st - lim\ sup\ mx.$$

Proof. Let $a/c_1 = st - lim\ inf\ mx$ and $b/c_2 = st - lim\ sup\ mx$. First assume that $st - lim\ mx = l/c$ and $\varepsilon > 0$. Then $\delta\left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} = 0$, so $\delta\left\{i : \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon\right\} = 1$, which implies that $\delta\left\{i : |x_i - l| < \frac{\varepsilon}{\sqrt{2}} = \varepsilon_1,\ |c_i - c| < \frac{\varepsilon}{\sqrt{2}} = \varepsilon_1\right\} = 1$.
So, $\delta\{i : |x_i - l| \geq \varepsilon_1,\ |c_i - c| \geq \varepsilon_1\} = 0$, i.e., $\delta\{i : |x_i - l| > \varepsilon_1,\ |c_i - c| > \varepsilon_1\} = 0$.
Therefore, $\delta\{i : x_i > l + \varepsilon_1,\ c_i - 1 > c + \varepsilon_1 - 1\} = 0$ and so,
$\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{(l + \varepsilon_1)^2 + (c + \varepsilon_1 - 1)^2}\right\} = 0$
which implies that $\sqrt{b^2 + (c_2 - 1)^2} < \sqrt{(l + \varepsilon_1)^2 + (c + \varepsilon_1 - 1)^2}$, i.e., $\sqrt{b^2 + (c_2 - 1)^2} \leq \sqrt{l^2 + (c - 1)^2}$. Hence, $st - lim\ sup\ mx \leq st - lim\ mx$. We also have
$\delta\{i : x_i < l - \varepsilon_1, c_i - 1 < c - 1 - \varepsilon_1\} = 0$,
i.e., $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(l - \varepsilon_1)^2 + (c - \varepsilon_1 - 1)^2}\right\} = 0$
which implies that $\sqrt{(l - \varepsilon_1)^2 + (c - \varepsilon_1 - 1)^2} < \sqrt{a^2 + (c_1 - 1)^2}$, i.e., $\sqrt{l^2 + (c - 1)^2} \leq \sqrt{a^2 + (c_1 - 1)^2}$. Hence, $st - lim\ mx \leq st - lim\ inf\ mx$ which we combine with theorem (3.25) to conclude that $st - lim\ inf\ mx = st - lim\ sup\ mx$.

Next assume $a/c_1 = b/c_2$ and define $l/c = a/c_1 = b/c_2$. Now $st - lim \sup\ mx = l/c$. Therefore, given for any $\varepsilon_1 > 0$, $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{(l + \varepsilon_1)^2 + (c - 1)^2}\right\} = 0$. It is possible to find a real number > 0, say $\varepsilon^{'}$ so that $\sqrt{(l + \varepsilon_1)^2 + (c - 1)^2} = \sqrt{l^2 + (c - 1)^2} + \varepsilon^{'}$, i.e., $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{l^2 + (c - 1)^2} + \varepsilon^{'}\right\} = 0$. Again, $st - lim \sup\ mx = l/c$. So, given for any $\varepsilon_2 > 0$,
$\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(l - \varepsilon_2)^2 + (c - 1)^2}\right\} = 0.$

Similarly as before we can find a real number > 0, say $\varepsilon^{''}$, so that $\sqrt{(l - \varepsilon_2)^2 + (c - 1)^2} = \sqrt{l^2 + (c - 1)^2} - \varepsilon^{''}$, i.e., $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{l^2 + (c - 1)^2} - \varepsilon^{''}\right\} = 0$. Hence if $\varepsilon = max\left(\varepsilon^{'}, \varepsilon^{''}\right)$, then we have $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{l^2 + (c - 1)^2} + \varepsilon\right\} = 0$ and $\delta\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{l^2 + (c - 1)^2} - \varepsilon\right\} = 0$, i.e., $st - lim\ mx = l/c$. □

Proposition 3.8. *A statistically convergent multisequence cannot converge more than one statistical limit point.*

Proof. Let, if possible, a multisequence $mx = (x_i/c_i)$ converge to two statistical limit points, say l_1/c_1 and l_2/c_2. Then, for any $\varepsilon > 0$, both of the sets

$$I_1(\varepsilon) = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} \geq \frac{\varepsilon}{2} \right\} \text{ and } I_2(\varepsilon) = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l_2)^2 + (c_i - c_2)^2} \geq \frac{\varepsilon}{2} \right\}$$

have natural density zero.

Now, $\delta\left(\left\{ i \in \mathbb{N} : \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} \geq \frac{\varepsilon}{2} \right\}\right) = 0$

$\Rightarrow \delta\left(\left\{ k \in \mathbb{N} : \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} < \frac{\varepsilon}{2} \right\}\right) \neq 0$

Similarly, $\Rightarrow \delta\left(\left\{ k \in \mathbb{N} : \sqrt{(x_i - l_2)^2 + (c_i - c_2)^2} < \frac{\varepsilon}{2} \right\}\right) \neq 0$.

Now, $\sqrt{(l_1 - l_2)^2 + (c_1 - c_2)^2} = \sqrt{\{(l_1 - x_i) + (x_i - l_2)\}^2 + \{(c_1 - x_i) + (x_i - c_2)\}^2}$

$\leq \sqrt{\{(l_1 - x_i) + (x_i - l_2)\}^2} + \sqrt{\{(c_1 - x_i) + (x_i - c_2)\}^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for infinitely many $i \in \mathbb{N}$ so that the density of the set taking all such i is non-zero.

i.e., $\sqrt{(l_1 - l_2)^2 + (c_1 - c_2)^2} < \varepsilon$ for infinitely many $i \in \mathbb{N}$ so that the density of the set taking all such i is non-zero.

Hence, $(l_1 - l_2)^2 + (c_1 - c_2)^2 = 0$ i.e., $l_1 = l_2$ and $c_1 = c_2$ This completes the proof of the theorem. □

4 Conclusions

In this paper we have introduced the notion of statistical convergence of multisequences for the first time which is a generalization of usual statistical convergence of real sequences and studied some of its important properties. The theory of statistical convergence of multisequences may be useful in many areas of mathematics specially summability theory.

Acknowledgements. The authors would like to thank to the reviewers for their valuable suggestions and comments for the improvement of the paper.

References

[1] J.A. Awolola, P.A. Ejegwa, *On some algebraic properties of order of an element of a multigroup*, Quasigroups and Related System 25 (2017), 21-26.

[2] J.A. Awolola, A.M. Ibrahim, *Some results on multigroups*, Quasigroups and Related System 24 (2016), 169-177.

[3] W.D. Blizard, *Multiset Theory*, Notre Dame Journal of Logic, 31 (1989), 36-65.

[4] W.D. Blizard, *The development of Multiset Theory*, Modern Logic, 1(4) (1991), 319-352.

[5] V. Cerf, E. Fernandez, K. Gostelow, S. Volausky, *Formal control and low properties of a model of computation*, Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, P-81, 1971.

[6] H.Fast, *Sur la convergence statistique*, Colloq. Math. 2 (1951), 241-244.
[7] J.A. Fridy, *On statistical convergence*, Analysis 5 (1985), 301-313.
[8] J.A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. 4 (1993), 1187-1192.
[9] J.A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. 12 (1997), 3625-3631.
[10] K. P. Girish, S. J. John, *Relations and functions in multiset context*, Information Sciences 179 (2009), 758-768.
[11] A. M. Ibrahim, P. A. Ejegwa, *Multigroup actions on multisets*, Ananals Fuzzy Math. Inform, 14(5) (2017), 515-526.
[12] D.E. Knuth, *The Art of computer programming*, V-2 (Seminumerical Algorithms), 2nd ed. Addison-Wesley: Reading Mass(1981).
[13] S.K. Nazmul, P. Majumdar, S.K. Samanta, *On multisets and multigroups*, Anals of Fuzzy Math. Inform, 6(3) (2013), 643-656.
[14] J. Peterson, *Computation sequence sets*, J. Computer Syst. Sci., 13(1) (1976), 1-24.
[15] R. Roy, S. Das, S.K. Samanta, *On multi normed linear spaces*, International Journal of Mathematics, Trends and Technology (IJMTT), 48(2) (2017), 111-119.
[16] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66 (1959), 361-375.
[17] Y. Tella, S. Daniel, *A study of group theory in the context of multiset theory*, Int. J. Sci. Techno., 2(8) (2013), 609-615.
[18] B.C. Tripathy, S. Debnath, D. Rakshit, *On multiset group*, Proyecciones J. Math. 37(3) (2018), 479-489.
[19] R.R. Yager, *On the theory of bags*, Int. J. Gen. Syst., 13 (1986), 23-37.
[20] A. Zygmund, *Trigonometric series*, 2nd ed., vol. II, Cambridge Univ. Press, London and New York, 1979.

Authors' address:

Shyamal Debnath and Amaresh Debnath
Department of Mathematics, Tripura University,
Suryamaninagar, Agartala, Tripura, 799022, India.
E-mail: shyamalnitamath@gmail.com , debnathamaresh91@gmail.com