# Nonlinear PDE-based models for photon-limited image restoration

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**Abstract.** An overview of the state of the art mathematical models for photon-limited image restoration is provided in this research article. The most important partial differential equation (PDE) - based Poisson denoising techniques are presented here. Thus, quantum noise filtering algorithms using Total Variation (TV) regularization - based models are described first. Then, some PDE variational filtering schemes for mixed Poisson - Gaussian noise removal are discussed. Our own contributions in this domain, representing a variational restoration approach and some non-variational anisotropic diffusion-based photon-limited image denoising solutions using nonlinear parabolic and hyperbolic PDE models, are also described in this paper and compared to the state of the art methods.

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**Key words**: photon-limited image; nonlinear PDE-based model; variational Poisson denoising scheme; mixed Poisson-Gaussian noise removal; non-variational anisotropic diffusion-based quantum denoising.

## 1 Introduction

The photon image devices capture the images by counting the photon detections at various spatial locations over a certain period of observation. Because this photon emission and detection process has an inherently discrete nature, the signal-dependent errors of those image acquisition mechanisms generate a special type of noise that is deteriorating seriously the captured images, not only quantitatively, but also qualitatively, producing the so-called photon-limited images.

It is called quantum or *shot noise* and is modelled as a Poisson process, according to a Poisson law [12]. For this reason, it is also called *Poison noise*, being characterized by the following Poisson probability distribution:

$$P(n) = \frac{e^{-\mu}\mu^n}{n!}, \quad n \geq 0. \tag{1.1}$$

The medical images are often corrupted by Poisson noise, because of the photon-counting devices used by various medical imaging operations, like the positron-emission

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tomography and microscopy imaging procedures. Various photon-limited image filtering algorithms were proposed in the last years. Some classic nonlinear filters [11], such as the two-dimension median filter, can be applied successfully for quantum denoising. More effective shot noise removal approaches include the Non-Local Mean Poisson filter [7], Poisson Reducing Bilateral Filter [17], Multi Scale Variance Stabilizing Transform [21], moving average and Wavelet-based filters [19].

The partial differential equations (PDE) have been widely used for image filtering in the last 35 years, in both variational and non-variational form. While most of them are applied for additive white 2D Gaussian noise (AWGN) reduction [1-3, 20], numerous PDE-based Poisson noise removal algorithms have been also elaborated in the last years.

An overview on the state of the art PDE-based mathematical models for photon-limited image restoration is provided in this work. The total variation regularization models for Poisson denoising are surveyed first. Then, some variational filtering approaches for mixed Poisson-Gaussian noise are described.

Our own contributions in the PDE-based photon-limited image restoration domain are also described briefly in this paper. This article ends with a section of conclusions and a list of references.

## 2 Total Variation regularization models for Poisson denoising

Numerous Total Variation (TV)-based Poisson denoising models have been derived from the total variation regularization schemes that remove properly the white additive noise [8, 10, 15], in the last years. Thus, the well-known TV Denoising (ROF) model and its versions could be adapted for Poisson noise removal by constructing a data-fidelity term that is suitable for quantum noise. So, some well-known second- and fourth-order variational PDE-based regularization models for Gaussian noise removal have been successfully adapted for the shot noise filtering task.

Thus, a TV-based variational Poisson denoising model was proposed by Le et al. in 2007 [13]. Their variational model consists of the following minimization of an energy cost functional:

$$u_{\text{rest}} = \underset{u \in BV(\Omega)}{\arg\min} \int_{\Omega} (|\nabla u| + \lambda(u - u_0 \log u)) d\Omega, \tag{2.1}$$

where $u_0$ represents the photon-limited observed image, $BV(\Omega)$ is the space of the bounded variation functions on the image domain $\Omega \subseteq R^2$ and $\lambda$ represents a penalty term. Next, the variational scheme (2.1) leads to a nonlinear second-order parabolic PDE model [13]. This PDE-based technique achieves a good trade-off between noise removal and edge preservation, but it could also generate the undesired staircase effect.

For this reason, some fourth-order PDE regularization schemes that suppress the Poisson noise have been introduced. Such a variational quantum denoising framework has been proposed by Zhou and Li in 2012 [22]. It restores the photon-limited images using the following minimization of the energy functional $E$:

$$u_{\min} = \min_{u} \left\{ E(u) = \int_{\Omega} (|D^2 u| + \lambda(u - u_0 \log u)) d\Omega \right\}, \tag{2.2}$$

where $Du$ represents the distributional derivative of $u$. A nonlinear parabolic fourth-order PDE is next obtained from the variational scheme (2.2). This PDE model is then solved numerically by applying the alternating minimization (AM) algorithm [22].

The fourth-order PDE-based approach removes successfully the quantum noise, while overcoming the staircasing. Thus, it achieves better restoration results than second-order PDE shot denoising solutions. Some Poisson denoising method comparison results are presented in the Table 1 and Fig. 1, where one displays the number of iterations, running time and relative error values for the variational models given by (2.1) and (2.2).

**Table 1. Results achieved by the two TV-based shot denoising schemes on *Panda* image**

| | **Iteration** | **Running time (sec)** | **Relative Error** |
|---|---|---|---|
| $2^{\text{nd}}$ order PDE model | 450 | 17:21 | 0.0541 |
| $4^{\text{th}}$ order PDE model | 250 | 10:04 | 0.0530 |

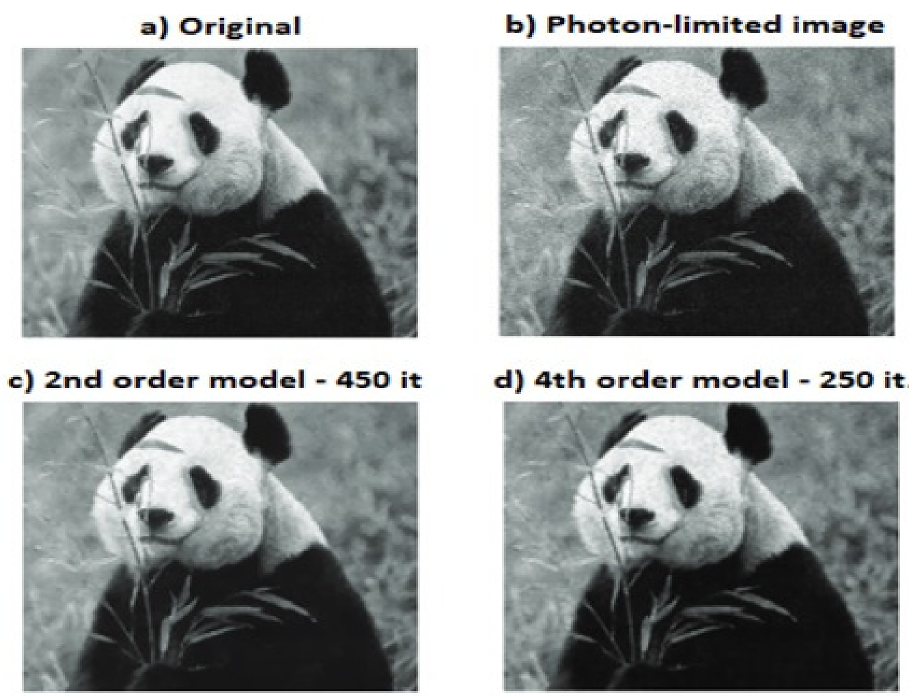


**Figure 1. Poisson denoising results achieved by $2^{\text{nd}}$ and $4^{\text{th}}$ order PDE models**

Another variational Poisson denoising technique was introduced by Sawatzky et. al in 2009 [16]. Their PDE-based restoration model is using the following total variation-based minimization:

$$u_{\min} = \underset{u \in BV(\Omega)}{\arg\min} \left\{ \frac{1}{2} \int_{\Omega} \left( \frac{(u-u_0)^2}{u_0} + \alpha |u|_{BV} \right) d\Omega \right\}, \tag{2.3}$$

where $u_{\min}$ represents the restored image, $u_0$ is the observed image, corrupted by quantum noise, $\alpha \geq 0$ and the image domain $\Omega \subseteq R^2$.

This PDE variational denosing approach provides effective Poisson noise removal results. A filtering exampe is displayed in Fig. 2. The original image in a) is corrupted by a high amount of shot noise in b) and is restored in c).

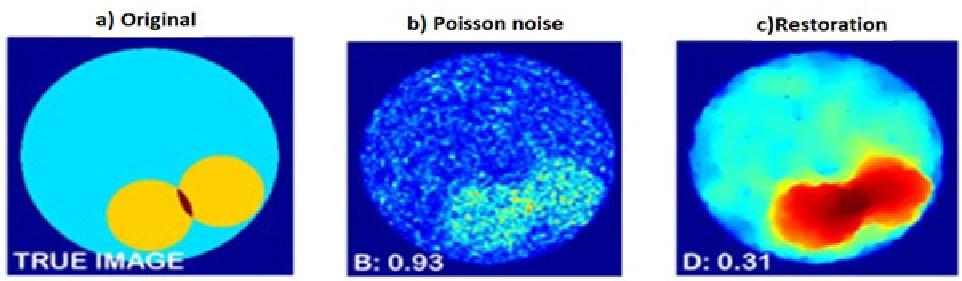


**Figure 2. Poisson denoising example**

The *fractional-order total variation-based Poisson denoising* (FOTV) model was proposed by M.R. Chowdhury et al. in 2020 [9]. This PDE variational approach is based on higher order total variation.

Thus, the FOTV-regularized quantum denoising framework is characterized by the following minimization:

$$\min_{u \in BV^{\alpha}\Omega)} TV^{\alpha}(u) + \beta \int_{\Omega} (u - u_0 \log u) d\Omega, \tag{2.4}$$

where $TV^{\alpha}(u)$ represents the $\alpha$-order total variation and $\alpha > 0$ is the fractional order [9]. The corresponding PDE model is well-posed under some certain conditions. The existence and uniqueness of a solution of the proposed model is demonstrated in [9] for the parameter value $\beta > 0$.

The experimental results prove the effectiveness of this variational filtering technique [9]. It also outperforms some state of the art Poisson denoising techniques that are not based on PDEs [9].

Some quantum noise removal results are displayed in the next figure. In Fig. 3, the first image represents the original, the second one represents the photon-limited image and the next three images represent the FOTV Poisson denoising results for various values of the fractional order parameter.

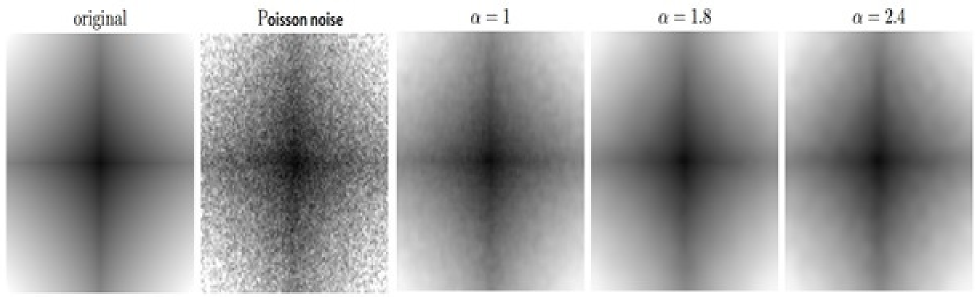


**Figure 3. FOTV Poisson denoising example**

We have also developed a novel variational PDE model for photon-limited image restoration that is disseminated in [4]. Our variational approach is based on the following energy functional minimization:

$$u_{\min} = \underset{u \in BV(\Omega)}{\arg\min} \left[ F(u) = \int_{\Omega} (\alpha\psi(|\nabla^2 u|) - \xi(u - u_0 \ln u)) d\Omega \right], \tag{2.5}$$

where

$$\psi(s) = \int_0^s \tau\lambda \left( \frac{\beta}{\nu |\ln(s^k + \beta)^3| + \gamma} \right)^{\frac{1}{2}} d\tau,$$

$\alpha, \xi \in (0,1]$, $\nu, \lambda, \gamma \in (0,5]$, $\beta \geq 25$, $k \geq 2$.

By applying the Euler-Lagrange equation and then the steepest descent method on (2.5), one obtains this well-posed nonlinear fourth-order PDE model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\alpha \nabla^2 (\varphi(\|\Delta u\|) \nabla^2 u) + \dfrac{\xi(u - u_0)}{u}, \ \forall (x,y) \in \Omega, \\ u(x,y,0) = u_0(x,y), \quad \forall (x,y) \in \Omega \subseteq R^2, \\ u(x,y,t) = 0, \quad \forall (x,y) \in \partial\Omega, \\ \dfrac{\partial u}{\partial \vec{n}} \, (x,y,t) = 0, \quad \forall (x,y) \in \partial\Omega, \end{cases} \tag{2.6}$$

where

$$\varphi(s) = \frac{1}{s} \, \frac{\partial \psi(s)}{\partial s}.$$

The obtained fourth-order PDE-based denoising model is then solved numerically by applying a consistent and fast converging finite difference-based numerical approximation algorithm on it [4]. Given the fourth-order of (2.6), this approximation scheme removes effectively the Poisson noise, overcomes the blurring and staircasing, and preserves the essential image features. See a filtering example in Fig. 4 and some method comparison results in Table 2.

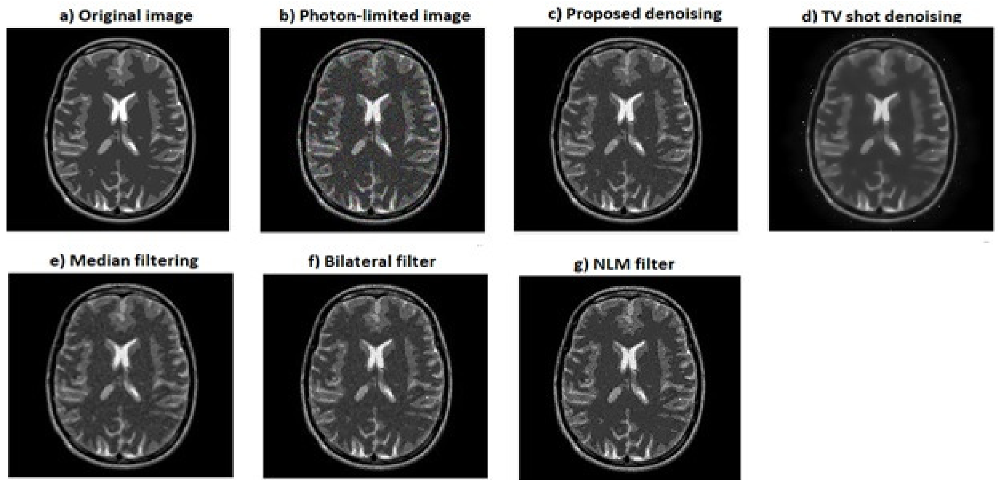


**Figure 4. Photon-limited image restored by several methods**

**Table 2. Average PSNR and SSIM values of several methods**

| **Restoration algorithm** | **Avg. PSNR** | **Avg. SSIM** |
|---|---|---|
| The proposed method | 32.5082 (dB) | 1.2496 |
| 2D Median filter | 28.8973 (dB) | 1.1121 |
| TV-based Poisson denoising | 308456 (dB) | 1.1872 |
| NLM filter | 29.1354 (dB) | 1.1205 |
| Bilateral 2D filter | 31.0938 (dB) | 1.2003 |

# 3 Variational filtering models for mixed Poisson-Gaussian noise

The photon-limited images are often affected by white additive 2D Gaussian noise (AWGN), too. The mixed noise, representing a combination of Poisson noise and Gaussian noise, is corrupting biomedical images, like the electron microscope images. Some total variation regularization-based image filtering solutions have been proposed for mixed noise removal.

Thus, C.T. Pham et al. introduced the following variational mixed denoising model in 2018 [14]:

$$u^* = \arg\min_u \int |\nabla u| dxdy + \frac{\lambda}{2}\int (u-u_0)^2 dxdy + \beta \int (u - u_0 \log u) dxdy, \tag{3.1}$$

where $\lambda, \beta > 0$ represent the weight parameters of the Gaussian and Poisson noise, respectively.

A nonlinear PDE model is then obtained from this variational scheme by applying the gradient descent to (3.1). Next, a numerical approximation algorithm is provided for this differential model [14].

The proposed mixed denoising approach removes successfully both the Gaussian and the quantum noise. A Poisson-Gaussian noise filtering example is described in the following figure.


**Figure 5. Mixed noise removal example**

Another variational mixed noise removal technique was introduced by D.N.H. Thanh and S.D. Dvoenko in 2015 [18]. Their approach combines TV-ROF model that removes the Gaussian noise to a modified ROF model that filters the Poisson noise.

So, the following mixed denoising problem with constrained conditions was considered in [18]:

$$\begin{cases} u^* = \arg\min_u \int_\Omega |\nabla u| d\Omega, \\ \frac{\lambda_1}{2\sigma^2} \int_\Omega (u_0 - u)^2 d\Omega + \lambda_2 \int_\Omega (u - u_0 \ln u) d\Omega = \kappa, \end{cases} \tag{3.2}$$

where $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$ and $\kappa$ is a constant parameter.

This optimization problem is solved by applying the Euler-Lagrange equation. Then, a finite difference method-based numerical discretization is performed on the obtained PDE-based model [18].

This variational approach provides effective mixed noise filtering results and outperform other denoising techniques [18]. A mixed noise removal example related to this technique is displayed in Fig. 6.

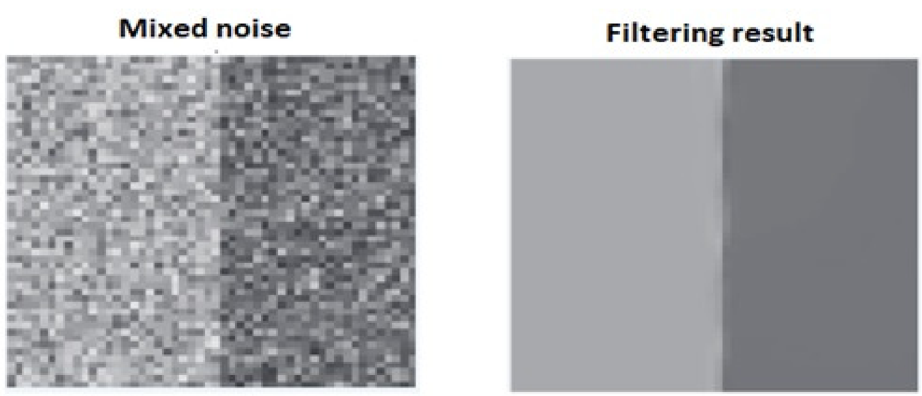


**Figure 6. Poisson-Gaussioan noise filtering example**

Some method comparison results illustrating the effectiveness if this technique are provided in Table 3.

**Table 3. Method comparison results**

| | **PSNR** | **SSIM** | **MSE** |
|---|---|---|---|
| Noisy | 19.5643 | 0.1036 | 718.8782 |
| TV-ROF | 35.1284 | 0.9130 | 19.9635 |
| Median | 31.4844 | 0.7797 | 46.1996 |
| Wiener | 30.1502 | 0.6018 | 62.8146 |
| Proposed method $\lambda_1 = 0.l2,\ \lambda_2 = 0.8,\ \mu = 0.114,\ \sigma = 46.052$ | 29.1325 | 0.5933 | 79.4014 |
| Proposed method $\lambda_1 = 0.5,\ \lambda_2 = 0.5,\ \mu = 1.1429,\ \sigma = 46.052$ | 37.0462 | 0.9453 | 12.8379 |

## 4 Non-variational PDE-based quantum denoising models

While the most PDE-based Poisson denoising models have a variational character, being derived from minimization problems like those described in the previous sections, some photon-limited image restoration techniques use PDE models that cannot be obtained from variational schemes. We developed such non-variational PDE quantum denoising models and disseminated them in some recent journal papers [5, 6].

Thus, in [5] we proposed a second-order anisotropic diffusion-based scheme for Poisson noise removal. It is based on a non-variational second-order nonlinear parabolic PDE model with boundary conditions that is adapted to the Poisson distribution:

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \delta(\|\nabla u\|)\nabla \cdot (\psi(\|\nabla u\|)\nabla u) + \lambda\ \dfrac{u - u_0}{u} = 0 \\ u(x,y,0) = u_0(x,y), \quad \forall (x,y) \in \Omega \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \quad \forall (x,y) \in \partial\Omega, \end{cases}$$

where $\psi(s) = \left(\frac{|\mu(\|\nabla u\|)+\alpha t|}{|\beta \ln(\mu(\|\nabla u\|)+\alpha t)+\eta s^3|}\right)^{\frac{1}{2}}$ and $\delta : [0,\infty) \to [0,\infty)$, $\delta(s) = \dfrac{\sqrt[3]{\gamma s^2 + \zeta}}{\xi}$.

A finite difference method-based numerical approximation was performed for this second-order parabolic PDE-based model [5]. The fast-converging numerical approximation scheme was successfully used in the photon-limited image restoration experiments that prove the effectiveness of the proposed approach.

This Poisson denoising framework removes successfully the shot noise, overcomes the undesired effects, preserves the edges and other image details and outperforms other quantum noise filtering techniques. See some method comparison results in the following table and the next figure.

**Table 4. Poisson denoising method comparison**

| Filtering technique | Average PSNR value |
|---|---|
| The proposed AD method | 34.2536 (dB) |
| Median 2D filter | 29.7369 (dB) |
| NLM filter | 31.3794 (dB) |
| Bilateral 2D filter | 29.8467 (dB) |
| TV for Poisson noise | 33.2542 (dB) |

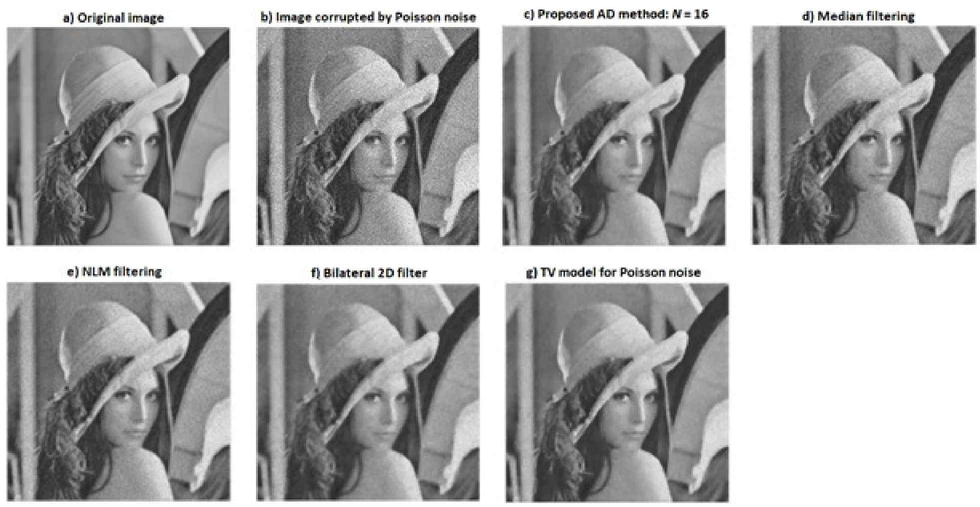


**Figure 7. Poisson denoising output of several filters**

A nonlinear hyperbolic PDE-based photon-limited image restoration framework was proposed in [6]. We constructed the next nonlinear second-order hyperbolic PDE model with boundary conditions:

$$(4.2)\qquad \begin{cases} \lambda\,\dfrac{\partial^2 u}{\partial t^2} + \xi\,\dfrac{\partial u}{\partial t} - \beta\varphi(\|\nabla^2 u\|)\mathrm{div}(\psi(\|\nabla u\|)\nabla u) + \alpha\left(\dfrac{u-u_0}{|u|+\delta}\right) = 0 \\ u(x,y,0) = u_0(x,y), \quad \forall(x,y)\in\Omega\subseteq R^2 \\ u_t(x,y,0) = u_1(x,y), \quad \forall(x,y)\in\Omega \\ u(x,y,t) = 0, \quad \forall(x,y)\in\partial\Omega \\ \dfrac{\partial u}{\partial \vec{n}}\,(x,y,t) = 0, \quad \forall(x,y)\in\partial\Omega, \end{cases}$$

where $\alpha,\beta,\lambda,\xi,\delta\in(0,1]$, $\psi:[0,\infty)\to[0,\infty)$, $\psi(s)=\left(\frac{\eta}{|\zeta s^k+\gamma\log 10(\eta)|}\right)^{\frac{1}{k-1}}$, $\zeta\in(0,1]$, $\gamma\in(0,5,1]$, $k>3$, $\varphi:[0,\infty)\to[0,\infty)$, $\varphi(s)=\frac{1}{\varepsilon}\,(\nu s^{r-1}+\zeta)^{\frac{1}{r}}$ and $\varepsilon,\nu,\zeta,r\in[1,5)$.

The second-time derivative, which provides the hyperbolic character of this nonlinear PDE denoising model, is sharpening the edges, thus enhancing the image details. The PDE model (11) is well-posed, admitting a unique weak solution. A stable and consistent numerical approximation scheme that converges fast to that solution was constructed in [6] by applying the finite difference method.

The proposed hyperbolic PDE-based quantum denoising approach has been tested successfully on hundreds of photon-limited medical microscopy images related to the coronavirus pandemic [6]. It outperforms many existing Poisson denoising filters, as it results from the method comparison results described in Fig. 8 and Table 5.

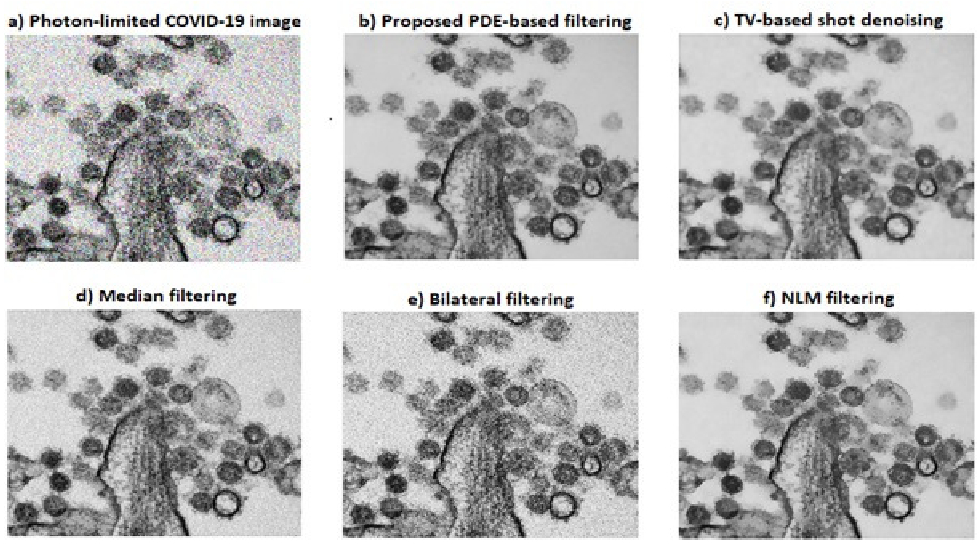


**Figure 8. Photon-limited microscope images filtered by various schemes**

**Table 5. Method comparison: PSNR, MSE and SNR values**

| **Restoration technique** | **Average PSNR** | **Average MSE** | **Average SNR** |
|---|---|---|---|
| Hyperbolic PDE-based filter | 33.3697 (dB) | 29.9301 | 33.0576 |
| Median filter | 28.8361 (dB) | 85.0093 | 28.2003 |
| Bilateral 2D filter | 29.2061 (dB) | 78.0666 | 28.6433 |
| TV-based Poisson denoising | 30.8756 (dB) | 60.5191 | 30.3135 |
| NLM | 31.7068 (dB) | 43.8936 | 31.3238 |

## 5 Conclusions

A survey of the state of the art PDE-based photon-limited image restoration techniques has been presented in this work. Most of them represent PDE variational models based on total variation regularizations of second and higher orders. They deal successfully with Poisson noise and mixed (quantum-Gaussian) noise.

While most of these shot denoising techniques are based on variational schemes that lead to nonlinear parabolic PDE models, we proposed some effective non-variational PDE-based models, one of them having also a hyperbolic character.

Experimental results and denoising method comparisons have been also discussed in this paper. Since the Poisson noise affects often the microscopy images, the described quantum denoising techniques could be successfully applied in the medical imaging domain. Various photon-limited medical images, including those related to the current pandemic, have been successfully filtered by the proposed shot denoising techniques.

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