The Fundamental Theorem of Algebra and Liouville's Theorem geometrically revisited

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Abstract. Abstract. If $f(z)$ is either a polynomial with no zeroes or a bounded entire function, then a Riemannian metric g_f is constructed on the complex plane $\mathbb{C}$. This metric g_f is shown to be flat and geodesically complete. Therefore, the Riemannian manifold $(\mathbb{C}, g_f)$ must be isometric to $(\mathbb{C}, |dz|^2)$, which implies that $f(z)$ is a constant.

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1 Introduction

In [2] we gave the first proof of the Fundamental Theorem of Algebra from Riemannian geometry. Other arguments, within this geometry, also led to new proofs [3]. In both papers, the two dimensional sphere $\mathbb{S}^2$ played a central role. The main aim of this short note is to give a new argument, this time using the complex plane $\mathbb{C}$ instead of $\mathbb{S}^2$. Moreover, the new argument also provides a new proof of Liouville's theorem. Our approach is a consequence of the following result, which can be thought of as a Riemannian geometric generalization of Liouville's theorem since it gives a sufficient condition, formulated in terms of Riemannian geometry, for an entire function to be a constant:

Theorem 1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Assume f has no zeroes. If the Riemannian metric*

$$g_f := |f(z)|^2 |dz|^2, \tag{1.1}$$

defined on $\mathbb{C}$, is geodesically complete, then f is a constant.

We deduce first the Fundamental Theorem of Algebra from this result. If a polynomial $P(z) \in \mathbb{C}[z]$ has no zeroes, we can define on $\mathbb{C}$ the corresponding Riemannian metric:

$$g_P := |P(z)|^2 |dz|^2.$$

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Thus, we only have to check that the Riemannian manifold $(\mathbb{C}, g_{_P})$ is geodesically complete.

It is well-known that Liouville's theorem leads to an easy proof of the Fundamental Theorem of Algebra. The argument follows from the fact that if $P(z)$ has no zeroes, then $1/P(z)$ is a bounded entire function [1, p. 122]. Thus, our proof to obtain the Fundamental Theorem of Algebra is a Riemannian geometric alternative to Liouville's theorem.

The remainder of the content of this note is organized as follows: Section 2 recalls some facts on geodesically completeness in Riemannian geometry that will be used later. In Section 3 we prove the Fundamental Theorem of Algebra from Theorem 1.1. Section 4 is devoted to obtain Liouville's theorem's also from Theorem 1.1. Finally, the proof of Theorem 1.1 is given in Section 5.

2 Preliminaries

Recall a Riemannian manifold is said geodesically complete if each of its inextensible geodesics is defined on all $\mathbb{R}$. The classical Hopf-Rinow theorem asserts that a Riemannian manifold (M, g) is geodesically complete if and only if the metric space (M, d_g) is complete, where d_g is the canonical distance on M associated to the Riemannian metric g (see for instance [4, Chap. 7]). Even more, for a non-compact Riemannian manifold, geodesic completeness is equivalent to the following property [4, p. 153]:

> Every divergent curve in M, starting at any point, has infinite g-length, i.e., for any (smooth) curve $\gamma : [0, \infty) \longrightarrow M$, such that for every compact subset C of M there exists $t_0 \in (0, \infty)$ such that $\gamma(t) \notin C$ for any $t > t_0$, we have
>
> $$\lim_{T\to\infty} \text{length}_g\big(\gamma_{|_{[0,T]}}\big) = \lim_{T\to\infty} \int_0^T \sqrt{g(\gamma'(t), \gamma'(t))}\, dt = \infty\,.$$

3 Proof of the Fundamental Theorem of Algebra

Assume $P(z) \in \mathbb{C}[z]$ is not a constant. Then $\deg P(z) \geq 1$ and $\lim_{z\to\infty} P(z) = \infty$. Hence, given $\epsilon > 0$, there exists $R > 0$ such that

$$|P(z)| > \epsilon \quad \text{whenever} \quad |z| > R\,.$$

Let us consider a divergent curve $\gamma : [0, \infty) \longrightarrow \mathbb{C}$ such that $|\gamma'(t)| = 1$ for any $t \in [0, \infty)$. There is no lack of generality in this assumption because the length of a curve is independent of re-parametrizations.

Consider the compact disc $C_R = \{z \in \mathbb{C} : |z| \leq R\}$ and let $t_0 \in (0, \infty)$ such that $|\gamma(t)| > R$ for any $t > t_0$. We have

$$\text{length}_{g_{_P}}(\gamma_{\|_{[0,T]}}) \geq \epsilon \int_{t_0}^T dt = \epsilon\,(T - t_0),$$

which implies

$$\lim_{T\to\infty} \text{length}_{g_P}\left(\gamma_{|[0,T]}\right) = \infty .$$

This shows that the Riemannian manifold $(\mathbb{C}, g_P)$ is geodesically complete. Therefore, $P(z)$ must be a constant and the proof ends by contradiction with Theorem 1.1. □

4 Proof of Liouville's theorem

The same argument also gives a proof of Liouville's theorem on entire functions. Indeed, assume that $f \in H(\mathbb{C})$ and $|f(z)| \leq C$ for all z. Take $h(z) = f(z) - 2C$. Then $h \in H(\mathbb{C})$ and $|h(z)| = |f(z) - 2C| \geq ||f(z)| - 2C| \geq C > 0$ for all z. Thus, we can define on $\mathbb{C}$ the Riemannian metric $g_h = |h(z)|^2|dz|^2$ and any curve $\gamma : [0,\infty) \longrightarrow \mathbb{C}$ such that $|\gamma^{'}(t)| = 1$ for any $t \in [0,\infty)$ will satisfy that

$$\text{length}_{g_h}(\gamma_{|[0,T]}) \geq C \int_0^T dt = CT \to \infty \text{ when } T \to \infty.$$

Henceforth, $(\mathbb{C}, g_h)$ is geodesically complete and $h(z)$ is constant, which obviously implies that $f(z)$ is also constant. □

5 Proof of Theorem 1.1

We end this note giving a proof of the announced result on entire holomorphic functions.

Assume that $f : \mathbb{C} \to \mathbb{C}$ is an holomorphic function with no zeroes. Taking into account that the Riemannian metric g_f given by (1.1) is point-wise conformally related to the usual flat metric $|dz|^2$ on $\mathbb{C}$, its Gauss curvature K_f satisfies

$$K_f = -\frac{1}{2\,|f(z)|^2}\,\Delta \log |f(z)|^2\,, \tag{5.1}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the usual Laplacian. On the other hand, $\log|f(z)|^2$ is harmonic. Consequently, this fact and formula (5.1) lead to

$$K_f = 0.$$

Summing up, we have just obtained that the Riemannian manifold $(\mathbb{C}, g_f)$ is flat. Moreover, by hypothesis, it is also geodesically complete and, clearly, $\mathbb{C}$ is simply-connected. The classical Killing-Hopf classification theorem of real space forms [4, Th. 8.4.1] can be called now to assure the existence of a preserving orientation (global) isometry

$$F : (\mathbb{C}, g_f) \to (\mathbb{C}, |dz|^2)\,.$$

In particular, we have

$$F^*|dz|^2 = |f(z)|^2|dz|^2\,. \tag{5.2}$$

But (5.2) means that F is a conformal transformation of $\mathbb{C}$. Therefore, $|f(z)|$ must be constant and, consequently, f is also constant. □

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