Riemann-Lagrange geometry for starfish/coral dynamical system

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Abstract. In this paper we develop the Riemann-Lagrange geometry, in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy, associated with the dynamical system concerning social interaction in colonial organisms. Some possible trophodynamic interpretations are derived.

Key words: tangent spaces; least squares Lagrangian functions; Riemann-Lagrange geometry; starfish/coral dynamics.

1 Social interactions in colonial organisms

Let $m \geq 2$ be an integer. We introduce social interactions for starfish/coral dynamics as follows (see Antonelli et al. [1]):

\[
\begin{align*}
\frac{dN^1}{dt} &= \lambda_1 N^1 - \alpha_1 (N^1)^2 - \alpha_2 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \\
&\quad + \frac{\alpha_1}{m-1} \left( \frac{N^2}{N^1} \right)^{m-2} \cdot (N^2)^2 - \delta_1 F N^1 \\
\frac{dN^2}{dt} &= \lambda_2 N^2 - \alpha_2 (N^2)^2 - \alpha_1 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \\
&\quad + \frac{\alpha_2}{m-1} \left( \frac{N^1}{N^2} \right)^{m-2} \cdot (N^1)^2 - \delta_2 F N^2 \\
\frac{dF}{dt} &= \beta F (N^1 + N^2) + \gamma F^2 - \rho F,
\end{align*}
\]

where

- $\alpha_1, \alpha_2, \lambda_1, \lambda_2, \delta_1, \delta_2, \beta, \gamma, \rho$ are positive coefficients;
- $N^1, N^2$ are coral densities;
• $F$ is the starfish density;
• $\lambda_1$ and $\lambda_2$ are growth rates;
• $\frac{\lambda_1}{\alpha_1}$ and $\frac{\lambda_2}{\alpha_2}$ are single species carrying capacities;
• $\beta$, $\delta_1$, and $\delta_2$ are the interaction coefficients for starfish preying on corals;
• $\gamma$ is the coefficient of starfish aggregation.

Note that $m$ is the effect of increasing the social parameter. If we set $m = 2$, we obtain the (2 corals/1 starfish)-model of Antonelli and Kazarinoff [2], in which every term of degree greater than one is quadratic. It is $m \geq 3$ which forces the social interaction terms to be nonquadratic.

By differentiation, the dynamical system (1.1) can be extended to a dynamical system of order two coming from a first order Lagrangian of least squares type. This extension is called in the literature in the field as geometric dynamical system (see Udrişte [7]).

2 The Riemann-Lagrange geometry

The system (1.1) can be regarded on the tangent space $T\mathbb{R}^3$, whose coordinates are
\[
\begin{align*}
  x^1 &= N^1, \quad x^2 = N^2, \quad x^3 = F, \quad y^1 = \frac{dN^1}{dt}, \quad y^2 = \frac{dN^2}{dt}, \quad y^3 = \frac{dF}{dt}.
\end{align*}
\]

Remark 2.1. We recall that the transformations of coordinates on the tangent space $T\mathbb{R}^3$ are given by
\begin{equation}
(2.1)
\begin{align*}
  \tilde{x}^i &= \tilde{x}^i(x^j), \\
  \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j,
\end{align*}
\end{equation}

where $i, j = 1, 3$.

In this context, the solutions of class $C^2$ of the system (1.1) are the global minimum points of the least squares Lagrangian function (see [7], [6])
\begin{equation}
(2.2)
\begin{align*}
  L &= (y^1 - X^1 (N^1, N^2, F))^2 + (y^2 - X^2 (N^1, N^2, F))^2 + \\
  &\quad + (y^3 - X^3 (N^1, N^2, F))^2 \geq 0,
\end{align*}
\end{equation}

where
\begin{align*}
  X^1 (N^1, N^2, F) &= \lambda_1 N^1 - \alpha_1 (N^1)^2 - \alpha_2 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \\
  &\quad + \frac{\alpha_1}{m-1} \left( \frac{N^2}{N^1} \right)^{m-2} \cdot (N^2)^2 - \delta_1 F N^1,
\end{align*}
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\[ X^2 (N^1, N^2, F) = \lambda_2 N^2 - \alpha_2 (N^2)^2 - \alpha_1 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \]
\[ + \frac{\alpha_2}{m-1} \left( \frac{N^1}{N^2} \right)^{m-2} \cdot (N^1)^2 - \delta_2 F N^2, \]
\[ X^3 (N^1, N^2, F) = \beta F (N^1 + N^2) + \gamma F^2 - \rho F, \]

Remark 2.2. The solutions of class \( C^2 \) of the system (1.1) are solutions of the Euler-Lagrange equations attached to the least squares Lagrangian (2.2), namely (geometric dynamics, in Udrişte’s terminology)

\[
\begin{align*}
\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) &= 0, \quad y^i = \frac{dx^i}{dt}, \quad \forall \ i = 1, 3, \Leftrightarrow \\
\frac{d^2 x^i}{dt^2} + 2G^i(x^k, y^k) &= 0 \Leftrightarrow \frac{d^2 x^i}{dt^2} + \frac{1}{2} \left( \frac{\partial^2 L}{\partial y^j \partial x^i} y^k - \frac{\partial L}{\partial x^i} \right) = 0 \Leftrightarrow \\
\frac{d^2 x^i}{dt^2} &= \left( \frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i} \right) y^k + \frac{\partial X^k}{\partial x^i} X^k,
\end{align*}
\]

where

\[
G^i(x^k, y^k) = \frac{1}{4} \left( \frac{\partial^2 L}{\partial y^j \partial x^i} y^k - \frac{\partial L}{\partial x^i} \right) = \]
\[= -\frac{1}{2} \left[ \left( \frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i} \right) y^k + \frac{\partial X^k}{\partial x^i} X^k \right]
\]
is endowed with the geometrical meaning of semispray of \( L \) (for more geometrical details, see Miron and Anastasiei book [5] and Udrişte’s book [7]).

But, the least squares Lagrangian (2.2), together with its Euler-Lagrange equations (2.3), provide us with an entire Riemann-Lagrange geometry on the tangent space \( T\mathbb{R}^3 \), in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy. These geometrical objects are naturally associated with the trophodynamical system (1.1).

Let us recall the main geometrical ideas developed in the Miron and Anastasiei book [5]. The canonical nonlinear connection \( \mathbb{N} = (N^i_{j}, i,j=1,3) \) produced by the semispray (2.4) is given by the components

\[ N^i_{j} = \frac{\partial G^i}{\partial y^j} = -\frac{1}{2} \left( \frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i} \right). \]

Remark 2.3. We recall that, under a transformation of coordinates (2.1), the local components of the nonlinear connection obey the rules [4], [5]

\[ \tilde{N}^k_i = N^i_{j} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{y}^k}{\partial x^j}. \]

From a geometrical point of view, we point out that the coefficients \( N^i_{j} \) of the above nonlinear connection have not a global character on \( T\mathbb{R}^3 \).
Remark 2.4. Using the well-known Cartan-Kosambi-Chern (KCC) theory, used also in the paper of Böhmer, Harko and Sabău [3], we can remark that the deviation curvature tensor associated with the dynamical system (1.1) is given by the formula

\[ P^i_j = -2 \frac{\partial G^i}{\partial x^j} - 2G^i \frac{\partial N^j}{\partial y^l} + \frac{\partial N^i}{\partial x^l} y^l + N^i_l N^l_j. \]

It is important to note that the solutions of the Euler-Lagrange equations (2.3) are Jacobi stable iff the real parts of the eigenvalues of the deviation tensor \( P^i_j \) are strictly negative everywhere, and Jacobi unstable, otherwise. For more details, see [3] and references therein.

The canonical nonlinear connection defines the adapted bases of vector fields and covector fields on the tangent space \( T\mathbb{R}^3 \), namely

\[ \{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j} \} \subset \mathcal{X}(T\mathbb{R}^3), \]

\[ \{ dx^i, \delta y^i = dy^i + N^i_j dx^j \} \subset \mathcal{X}^*(T\mathbb{R}^3). \]

The adapted local components of the Cartan \( N \)-linear connection \( C\Gamma(N) = (L^i_{jk}, C^i_{jk}) \) are given by the formulas

\[ L^i_{jk} = \frac{g^{ir}}{2} \left( \frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{rj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^r} \right), \quad C^i_{jk} = \frac{g^{ir}}{2} \left( \frac{\partial g_{rk}}{\partial y^j} + \frac{\partial g_{rj}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^r} \right), \]

where

\[ g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = \delta_{ij}. \]

The only non-vanishing d-torsion adapted component associated with the Cartan \( N \)-linear connection \( C\Gamma(N) \) is given by the coefficient

\[ R^r_{ij} = \frac{\delta N^r_i}{\delta x^j} - \frac{\delta N^r_j}{\delta x^i}, \]

At the same time, all the adapted components of the curvature attached to the Cartan \( N \)-linear connection \( C\Gamma(N) \) are zero (for all curvature formulas, see [5]).

The electromagnetic-like distinguished 2-form attached to the Lagrangian \( L \), defined via its deflection d-tensors (for more details, see Miron and Anastasiei book [5]), is given by \( F = F_{ij} \delta y^i \wedge dx^j \), where

\[ F_{ij} = \frac{1}{2} \left( g_{ir} N^r_j - g_{jr} N^r_i \right) = \frac{1}{2} \left( N^r_j - N^r_i \right) = N^r_i. \]

In this context, let us use the notation

\[ J(X) = \left( \frac{\partial X^i}{\partial x^j} \right)_{i,j=1,3} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}, \]
where

\[ J_{11} = \lambda_1 - 2\alpha_1 N^1 - \alpha_2 \left( \frac{m}{m-1} \right) \cdot N^2 - \alpha_1 \left( \frac{m-2}{m-1} \right) \frac{(N^2)^m}{(N^1)^{m-1}} - \delta_1 F, \]

\[ J_{12} = -\alpha_2 \left( \frac{m}{m-1} \right) \cdot N^1 + \alpha_1 \left( \frac{m}{m-1} \right) \frac{(N^2)^{m-1}}{(N^1)^{m-2}}, \]

\[ J_{13} = -\delta_1 N^1, \quad J_{21} = -\alpha_1 \left( \frac{m}{m-1} \right) \cdot N^2 + \alpha_2 \left( \frac{m}{m-1} \right) \frac{(N^1)^{m-1}}{(N^2)^{m-2}}, \]

\[ J_{22} = \lambda_2 - 2\alpha_2 N^2 - \alpha_1 \left( \frac{m}{m-1} \right) \cdot N^1 - \alpha_2 \left( \frac{m-2}{m-1} \right) \frac{(N^1)^m}{(N^2)^{m-1}} - \delta_2 F, \]

\[ J_{23} = -\delta_2 N^2, \quad J_{31} = \beta F, \quad J_{32} = \beta F, \quad J_{33} = \beta (N^1 + N^2) + 2\gamma F - \rho. \]

Following the above Miron and Anastasiei’s geometrical ideas, we obtain the following geometrical results:

**Theorem 2.1.** (i) The canonical nonlinear connection on \( T\mathbb{R}^3 \), produced by the system (1.1), has the local components

\[ N = \frac{1}{2} \left[ J(X) - T J(X) \right] = \begin{pmatrix} N_1^1 & N_1^2 & N_1^3 \\ N_2^1 & N_2^2 & N_2^3 \\ N_3^1 & N_3^2 & N_3^3 \end{pmatrix}, \]

where

\[ N_1^1 = N_2^2 = N_3^3 = 0, \]

\[ N_2^1 = -N_1^2 = -\frac{1}{2} \left\{ \left( \frac{m}{m-1} \right) (\alpha_1 N^2 - \alpha_2 N^1) + \left( \frac{m}{m-1} \right) \left[ \alpha_2 \left( \frac{(N^1)^{m-1}}{(N^2)^{m-2}} - \alpha_1 \left( \frac{(N^2)^{m-1}}{(N^1)^{m-2}} \right) \right] \right\}, \]

\[ N_3^1 = -N_1^3 = \frac{1}{2} \left( \beta F + \delta_1 N^1 \right), \quad N_3^2 = -N_3^2 = -N_3^2 = \frac{1}{2} \left( \beta F + \delta_2 N^2 \right). \]

(ii) All adapted components of the canonical Cartan connection \( C\Gamma(N) \), produced by the system (1.1), are zero.

(iii) The effective adapted components \( R_{ijk} \) of the torsion d-tensor \( T \) of the canonical Cartan connection \( C\Gamma(N) \), produced by the system (1.1), are the entries of the following skew-symmetric matrices:

\[ R_i = (R_{jik})_{i,j,k=1,3} = \frac{\partial N}{\partial N^i} = \begin{pmatrix} 0 & \frac{\partial N^2}{\partial N^1} & \frac{\delta_1}{2} \\ -\frac{\partial N^1}{\partial N^1} & 0 & 0 \\ -\frac{\delta_1}{2} & 0 & 0 \end{pmatrix}. \]
where

\[
\frac{\partial N_2}{\partial N^1} = \frac{1}{2} \left( \frac{m}{m-1} \right) \left[ \alpha_2 - \alpha_2 (m-1) \left( \frac{N^1}{N^2} \right)^{m-2} - \alpha_1 (m-2) \left( \frac{N_2}{N^1} \right)^{m-1} \right];
\]

\[
R_2 = (R'_{j2})_{i,j=1,3} = \frac{\partial N}{\partial N^2} = \begin{pmatrix}
0 & \frac{\partial N^1_2}{\partial N^2} & 0 \\
-\frac{\partial N^1_2}{\partial N^2} & 0 & \frac{\delta_2}{2} \\
0 & -\frac{\delta_2}{2} & 0
\end{pmatrix},
\]

where

\[
\frac{\partial N_2}{\partial N^2} = \frac{1}{2} \left( \frac{m}{m-1} \right) \left[ -\alpha_1 + \alpha_2 (m-2) \left( \frac{N^1}{N^2} \right)^{m-1} + \alpha_1 (m-1) \left( \frac{N_2}{N^1} \right)^{m-2} \right];
\]

\[
R_3 = (R'_{j3})_{i,j=1,3} = \frac{\partial N}{\partial F} = \begin{pmatrix}
0 & 0 & \beta \\
0 & 0 & \beta \\
-\frac{\beta}{2} & -\frac{\beta}{2} & 0
\end{pmatrix}.
\]

(iv) All adapted components of the curvature d-tensor \( R \) of the canonical Cartan connection \( \Gamma_C(N) \), produced by the system (1.1), vanish.

(v) The geometric electromagnetic-like distinguished 2-form, produced by the system (1.1), is given by \( F = F_{ij} \delta y^i \wedge dx^j \), where the adapted components \( F_{ij} \) are the entries of the skew-symmetric matrix \( F = (F_{ij})_{i,j=1,3} = N \).

(vi) The geometric Yang-Mills-like energy, produced by the system (1.1), is given by the formula

\[
\mathcal{E}_{YM}(t) = F_{12}^2 + F_{13}^2 + F_{23}^2 = \frac{1}{4} \left( \frac{m}{m-1} \right)^2 \left[ \alpha_1 N^2 - \alpha_2 N^1 + \alpha_2 \left( \frac{N^1}{N^2} \right)^{m-1} - \alpha_1 \left( \frac{N_2}{N^1} \right)^{m-1} \right]^2 +
\]

\[
+ \frac{1}{4} \left( \beta F + \delta_1 N^1 \right)^2 + \frac{1}{4} \left( \beta F + \delta_2 N^2 \right)^2.
\]

Remark 2.5. In the author’s opinion, from a trophodynamic point of view the zero level of the jet geometric Yang-Mills energy produced by the system (1.1) is important. The jet geometric Yang-Mills trophodynamical energy produced by the system (1.1) is zero iff

\[
\beta F + \delta_1 N^1 = 0, \quad \beta F + \delta_2 N^2 = 0,
\]

\[
(\alpha_1 N^2 - \alpha_2 N^1) + \left[ \alpha_2 \left( \frac{N^1}{N^2} \right)^{m-1} - \alpha_1 \left( \frac{N_2}{N^1} \right)^{m-1} \right] = 0.
\]
If \( \delta_1 \neq \delta_2 \), these conditions imply the impossible fact that \( F = N^1 = N^2 = 0 \), and if \( \delta_1 = \delta_2 = \delta \), then we obtain \( N^1 = N^2 = -\beta F/\delta \). In this last case, we find a Bernoulli differential equation as the last equation of the system (1.1), namely
\[
\frac{dF}{dt} = -\rho F + \left( \gamma - 2 \frac{\beta^2}{\delta} \right) F^2.
\]

This equation can be integrated by using the changing of variable \( z = F^{-1} \). The solution of the above Bernoulli differential equation is
\[
F(t) = \frac{1}{a \exp(\rho t) + b},
\]
where \( a \in \mathbb{R} \) is an arbitrary constant, and we have
\[
b = \frac{1}{\rho} \left( \gamma - 2 \frac{\beta^2}{\delta} \right).
\]

At the same time, we consider that the constant level surfaces of the jet geometric Yang-Mills trophodynamical energy \( \mathcal{E}_{YM}(t) = C, C > 0 \), could contain important trophodynamic connotations. Consequently, the graphical representation of these surfaces in the system of axes \( OFN^1N^2 \) could be a fruitful and open problem in trophodynamics.

**Remark 2.6.** The deviation curvature tensor components \( P^i_j \) can be obtained by contracting with \( y^k \) the nonzero components of the torsion tensor \( R^i_{jk} \), that is \( P^i_j = R^i_{jk}y^k = (\partial N^i_j/\partial x^k) y^k \). Consequently, the matrix of the deviation curvature tensor is given by
\[
P = R_ky^k = \begin{pmatrix}
0 & \frac{\partial N^1_1}{\partial N^1} - \frac{\delta_1}{2} & \frac{\delta_1}{2} \\
-\frac{\partial N^1_1}{\partial N^1} & 0 & 0 \\
-\frac{\delta_1}{2} & 0 & 0
\end{pmatrix} y^1 + \begin{pmatrix}
0 & \frac{\partial N^1_2}{\partial N^2} & 0 \\
-\frac{\partial N^1_2}{\partial N^2} & 0 & \frac{\delta_2}{2} \\
0 & \frac{\delta_2}{2} & 0
\end{pmatrix} y^2 + \begin{pmatrix}
0 & \frac{\beta}{2} & \frac{\beta}{2} \\
0 & 0 & \frac{\beta}{2} \\
-\frac{\beta}{2} & -\frac{\beta}{2} & 0
\end{pmatrix} y^3 = \begin{pmatrix}
a & b & 0 \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix},
\]
where
\[
a = \frac{\partial N^1_2}{\partial N^1} y^1 + \frac{\partial N^1_1}{\partial N^2} y^2, \quad b = \frac{\delta_1}{2} y^1 + \frac{\beta}{2} y^2, \quad c = \frac{\delta_2}{2} y^2 + \frac{\beta}{2} y^3.
\]

The eigenvalues of the matrix \( P \) are the real values
\[
\lambda_1 = 0, \quad \lambda_{2,3} = \pm \sqrt{a^2 + b^2 + c^2}.
\]

In conclusion, the behavior of neighboring solutions of the Euler-Lagrange equations (2.3) is Jacobi unstable.
Open problem. The trophodynamic interpretations associated with the geometrical objects constructed in this paper still represent an open problem.

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References


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