Strong insertion of
a contra-Baire-1 (Baire-.5) function

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Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that \( F_\sigma \)-kernel of sets are \( F_\sigma \)-sets.


Key words: Insertion; strong binary relation; Baire-.5 function; kernel of sets; lower cut set.

1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [18]. He investigated the sets that can be represented as union of closed sets and called them \( V \)-sets. Complements of \( V \)-sets, i.e., sets that are intersection of open sets are called \( \Lambda \)-sets [18].

Recall that a real-valued function \( f \) defined on a topological space \( X \) is called \( A \)-continuous [25] if the preimage of every open subset of \( \mathbb{R} \) belongs to \( A \), where \( A \) is a collection of subsets of \( X \). Most of the definitions of function used throughout this paper are consequences of the definition of \( A \)-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuity. A considerable number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 8, 9, 10, 12, 13, 23].

The results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that \( F_\sigma \)-kernel of sets are \( F_\sigma \)-sets.

A real-valued function \( f \) defined on a topological space \( X \) is called \textit{contra-Baire-1} (\textit{Baire-.5}) if the preimage of every open subset of \( \mathbb{R} \) is a \( G_\delta \)-set in \( X \) [26].
If \( g \) and \( f \) are real-valued functions defined on a space \( X \), we write \( g \leq f \) in case \( g(x) \leq f(x) \) for all \( x \in X \).

The following definitions are modifications of the conditions considered in [16].

A property \( P \) defined relative to a real-valued function on a topological space is a \( B - .5-\)property provided that any constant function has property \( P \) and provided that the sum of a function with property \( P \) and any Baire-.5 function also has property \( P \). If \( P_1 \) and \( P_2 \) are \( B - .5-\)properties, the following terminology is used: (i) A space \( X \) has the weak \( B - .5-\)insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f \) and \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a Baire-.5 function \( h \) such that \( g \leq h \leq f \). (ii) A space \( X \) has the strong \( B - .5-\)insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f \) and \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a Baire-.5 function \( h \) such that \( g \leq h \leq f \) and such that if \( g(x) < f(x) \) for any \( x \) in \( X \), then \( g(x) < h(x) < f(x) \).

In this paper, for a topological space that \( F_\sigma -\)kernel of sets are \( F_\sigma -\)sets, is given a sufficient condition for the weak \( B - .5-\)insertion property. Also, for a space with the weak \( B - .5-\)insertion property, we give necessary and sufficient conditions for the space to have the strong \( B - .5-\)insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the strong insertion of a contracontinuous function between two comparable real-valued functions has also recently considered by the authors in [21].

## 2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

**Definition 2.1.** Let \( A \) be a subset of a topological space \((X, \tau)\). We define the subsets \( A^A \) and \( A^V \), as follows:
\[
A^A = \cap\{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}.
\]
In [7, 19, 22], \( A^A \) is called the kernel of \( A \).

We also define the subsets \( G_\delta(A) \) and \( F_\sigma(A) \), as follows:
\[
G_\delta(A) = \cup\{O : O \subseteq A, O \text{is } G_\delta \text{-set}\} \quad \text{and} \quad F_\sigma(A) = \cap\{F : F \supseteq A, F \in F_\sigma \text{-set}\}.
\]

\( F_\sigma(A) \) is called the \( F_\sigma -\)kernel of \( A \).

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \) then \( \rho^\circ \) is defined as follows: \( x \rho^\circ y \) if and only if \( y \rho \nu \) implies \( x \rho \nu \) and \( u \rho x \) implies \( u \rho y \) for any \( u \) and \( v \) in \( S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a strong binary relation in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:
1) If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \rho B$.

3) If $A \rho B$, then $F_\rho(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

**Definition 2.4.** If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is a lower indefinite cut set in the domain of $f$ at the level $\ell$.

We now give the following main results:

**Theorem 2.1.** Let $g$ and $f$ be real-valued functions on the topological space $X$, that $F_\rho$-kernel sets in $X$ are $F_\alpha$-sets, with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$, then $A(f, t_1) \rho A(g, t_2)$, then there exists a Baire-5 function $h$ defined on $X$ such that $g \leq h \leq f$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$ such that $g \leq f$. By hypothesis, there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$, then $A(f, t_1) \rho A(g, t_2)$.

Define the functions $F$ and $G$ mapping the rational numbers $\mathbb{Q}$ into the power set of $X$ by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \rho F(t_2), G(t_1) \rho G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [15], it follows that there exists a function $H$ mapping $\mathbb{Q}$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2), H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any $x \in X$, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' < t$; since $x$ is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\alpha(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is a $G_\delta$-set in $X$, i.e., $h$ is a Baire-5 function on $X$.

The above proof used the technique of Theorem 1 of [14].

If a space has the strong $B-5$-insertion property for $(P_1, P_2)$, then it has the weak $B-5$-insertion property for $(P_1, P_2)$.

The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $B-5$-insertion property to satisfy the strong $B-5$-insertion property.

**Theorem 2.2.** Let $P_1$ and $P_2$ be $B-5$-property and $X$ be a space that satisfies the weak $B-5$-insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the strong $B-5$-insertion property for $(P_1, P_2)$ if and only if there exist lower cut...
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sets $A(f - g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of $X$ such that (i) for each $n$, $F_n$ and $A(f - g, 2^{-n})$ are completely separated by Baire-.5 functions, and (ii) $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f - g, 2^{-n}))$ of lower cut sets for $f - g$ and suppose that there is a sequence $(F_n)$ of subsets of $X$ such that

$$\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each $n$, there exists a Baire-.5 function $k_n$ on $X$ into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on $F_n$ and $k_n = 0$ on $A(f - g, 2^{-n})$. The function $k$ from $X$ into $[0, 1/4]$ which is defined by

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)$$

is a Baire-.5 function by the Cauchy condition and the properties of Baire-.5 functions.

1. $k^{-1}(0) = \{x \in X : (f - g)(x) = 0\}$ and (2) if $(f - g)(x) > 0$ then $k(x) < (f - g)(x)$ : In order to verify (1), observe that if $(f - g)(x) = 0$, then $x \in A(f - g, 2^{-n})$ for each $n$ and hence $k_n(x) = 0$ for each $n$. Thus $k(x) = 0$. Conversely, if $(f - g)(x) > 0$, then there exists an $n$ such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n})$$

and that $(A(f - g, 2^{-n}))$ is a decreasing sequence. Thus if $(f - g)(x) > 0$ then either $x \notin A(f - g, 1/2)$ or there exists a smallest $n$ such that $x \notin A(f - g, 2^{-n})$ and $x \in A(f - g, 2^{-j})$ for $j = 1, \ldots, n - 1$.

In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f - g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x).$$

Thus $0 \leq k \leq f - g$ and if $(f - g)(x) > 0$ then $(f - g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since $P_1$ and $P_2$ are $B - .5$-properties, then $g_1$ has property $P_1$ and $f_1$ has property $P_2$. Since by hypothesis $X$ has the weak $B - .5$-insertion property for $(P_1, P_2)$, then there exists a Baire-.5 function $h$ such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Therefore $X$ has the strong $B - .5$-insertion
property for \((P_1, P_2)\). (The technique of this proof is by Lane [16].)

Conversely, assume that \(X\) satisfies the strong \(B - .5\)-insertion for \((P_1, P_2)\). Let \(g\) and \(f\) be functions on \(X\) satisfying \(P_1\) and \(P_2\) respectively such that \(g \leq f\). Thus there exists a Baire-.5 function \(h\) such that \(g \leq h \leq f\) and such that if \(g(x) < f(x)\) for any \(x\) in \(X\), then \(g(x) < h(x) < f(x)\). We follow an idea contained in Powderly [24]. Now consider the functions \(0\) and \(f - h\), \(0\) satisfies property \(P_1\) and \(f - h\) satisfies property \(P_2\). Thus there exists a Baire-.5 function \(h_1\) such that \(0 \leq h_1 \leq f - h\) and if \(0 < (f - h)(x)\) for any \(x\) in \(X\), then \(0 < h_1(x) < (f - h)(x)\). We next show that

\[
\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.
\]

If \(x\) is such that \((f - g)(x) > 0\), then \((f - h)(x) > 0\). Therefore \(g(x) < h(x) < f(x)\). Thus \(f(x) - h(x) > 0\) or \((f - h)(x) > 0\). Hence \(h_1(x) > 0\). On the other hand, if \(h_1(x) > 0\), then since \((f - h) \geq h_1\) and \(f - g \geq f - h\), therefore \((f - g)(x) > 0\).

For each \(n\), let

\[
A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}, \quad F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}
\]

and

\[
k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.
\]

Since \(\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}\), it follows that

\[
\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.
\]

We next show that \(k_n\) is a Baire-.5 function which completely separates \(F_n\) and \(A(f - g, 2^{-n})\). From its definition and by the properties of Baire-.5 functions, it is clear that \(k_n\) is a Baire-.5 function. Let \(x \in F_n\). Then, from the definition of \(k_n, k_n(x) = 2^{-n}\). If \(x \in A(f - g, 2^{-n})\), then since \(h_1 \leq f - h \leq f - g, h_1(x) \leq 2^{-n}\).

Thus \(k_n(x) = 0\), according to the definition of \(k_n\). Hence \(k_n\) completely separates \(F_n\) and \(A(f - g, 2^{-n})\).

\(\square\)

**Theorem 2.3.** Let \(P_1\) and \(P_2\) be \(B - .5\)-properties and assume that the space \(X\) satisfied the weak \(B - .5\)-insertion property for \((P_1, P_2)\). The space \(X\) satisfies the strong \(B - .5\)-insertion property for \((P_1, P_2)\) if and only if \(X\) satisfies the strong \(B - .5\)-insertion property for \((P_1, B - .5)\) and for \((B - .5, P_2)\).

**Proof.** Assume that \(X\) satisfies the strong \(B - .5\)-insertion property for \((P_1, B - .5)\) and for \((B - .5, P_2)\). If \(g\) and \(f\) are functions on \(X\) such that \(g \leq f, g\) satisfies property \(P_1\), and \(f\) satisfies property \(P_2\), then since \(X\) satisfies the weak \(B - .5\)-insertion property for \((P_1, P_2)\) there is a Baire-.5 function \(k\) such that \(g \leq k \leq f\). Also, by hypothesis there exist Baire-.5 functions \(h_1\) and \(h_2\) such that \(g \leq h_1 \leq k \leq h_2 \leq f\) and if \(g(x) < k(x)\) then \(g(x) < h_1(x) < k(x)\) and such that \(k \leq h_2 \leq f\) and if \(k(x) < f(x)\) then \(k(x) < h_2(x) < f(x)\). If a function \(h\) is defined by \(h(x) = (h_2(x) + h_1(x))/2\), then \(h\) is a Baire-.5 function, \(g \leq h \leq f\), and if \(g(x) < f(x)\) then \(g(x) < h(x) < f(x)\). Hence \(X\) satisfies the strong \(B - .5\)-insertion property for \((P_1, P_2)\). The converse is obvious since any Baire-.5 function must satisfy both properties \(P_1\) and \(P_2\). (The technique of this proof is by Lane [17].)  

\(\square\)
3 Applications

**Definition 3.1.** A real-valued function $f$ defined on a space $X$ is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_{\delta}$-set for any real number $t$.

The abbreviations *usc, lsc, cusB*.5 and *clsB*.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

**Remark 1.** [14, 15]. A space $X$ has the weak $c-$insertion property for (*usc, lsc*) if and only if $X$ is normal.

Before stating the consequences of theorem 2.1, 2.2 and 2.3 we suppose that $X$ is a topological space that $F\sigma$-kernel of sets are $F\sigma$-sets.

**Corollary 3.1.** For each pair of disjoint $F\sigma$-sets $F_1, F_2$, there are two $G_{\delta}$-sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if $X$ has the weak $B -.5$-insertion property for ($\text{cusB} -.5, \text{clsB} -.5$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ is *lsB*$_1, g$ is *usB*$_1$, and $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $F\rho(A) \subseteq G_{\delta}(B)$, then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $F\sigma$-set and since $\{x \in X : g(x) < t_2\}$ is a $G_{\delta}$-set, it follows that $F\sigma(A(f, t_1)) \subseteq G_{\delta}(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let $F_1$ and $F_2$ are disjoint $F\sigma$-sets. Set $f = \chi_{F_1}$ and $g = \chi_{F_2}$, then $f$ is *clsB* -.5, $g$ is *cusB* -.5, and $g \leq f$. Thus there exists Baire-.5 function $h$ such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then $G_1$ and $G_2$ are disjoint $G_{\delta}$-sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

**Remark 2.** [27]. A space $X$ has the weak $c-$insertion property for (*lsc, usc*) if and only if $X$ is extremally disconnected.

**Corollary 3.2.** For every $G$ of $G_{\delta}$-set, $F\sigma(G)$ is a $G_{\delta}$-set if and only if $X$ has the weak $B -.5$-insertion property for ($\text{clsB} -.5, \text{cusB} -.5$).

Before giving the proof of this corollary, the necessary lemma is stated.

**Lemma 3.1.** The following conditions on the space $X$ are equivalent:

(i) For every $G$ of $G_{\delta}$-set we have $F\sigma(G)$ is a $G_{\delta}$-set.

(ii) For each pair of disjoint $G_{\delta}$-sets as $G_1$ and $G_2$ we have $F\sigma(G_1) \cap F\sigma(G_2) = \emptyset$.

The proof of Lemma 3.1 is a direct consequence of the definition $F\sigma$-kernel of sets.

We now give the proof of Corollary 3.2.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ is *clsB* -.5, $g$ is *cusB* -.5, and $f \leq g$. If a binary relation $\rho$ is defined by $A \rho B$ in case $F\rho(A) \subseteq
\( G \subseteq F_\sigma(G) \subseteq G_\delta(B) \) for some \( G_\delta \)-set \( g \) in \( X \), then by hypothesis and Lemma 3.1 \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(g, t_1) = \{ x \in X : g(x) < t_1 \} \subseteq \{ x \in X : f(x) \leq t_2 \};
\]

\[
= A(f, t_2);
\]

since \( \{ x \in X : g(x) < t_1 \} \) is a \( G_\delta \)-set and since \( \{ x \in X : f(x) \leq t_2 \} \) is a \( F_\sigma \)-set, by hypothesis it follows that \( A(g, t_1) \rho A(f, t_2) \). The proof follows from Theorem 2.1.

On the other hand, Let \( G_1 \) and \( G_2 \) are disjoint \( G_\delta \)-sets. Set \( f = \chi g_2 \) and \( g = \chi g_1 \), then \( f \) is \( \text{cl} s B - .5, g \) is \( \text{cus} B - .5 \), and \( f \leq g \).

Thus there exists Baire-.5 function \( h \) such that \( f \leq h \leq g \). Set \( F_1 = \{ x \in X : h(x) \leq \frac{1}{3} \} \) and \( F_2 = \{ x \in X : h(x) \geq 2/3 \} \) then \( F_1 \) and \( F_2 \) are disjoint \( F_\sigma \)-sets such that \( G_1 \subseteq F_1 \) and \( G_2 \subseteq F_2 \). Hence \( F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset \).

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

**Lemma 3.2.** The following conditions on the space \( X \) are equivalent:

(i) Every two disjoint \( F_\sigma \)-sets of \( X \) can be separated by \( G_\delta \)-sets of \( X \).

(ii) If \( F \) is a \( F_\sigma \)-set of \( X \) which is contained in a \( G_\delta \)-set \( G \), then there exists a \( G_\delta \)-set \( H \) such that \( F \subseteq H \subseteq F_\sigma(H) \subseteq G \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( F \subseteq G \), where \( F \) and \( G \) are \( F_\sigma \)-set and \( G_\delta \)-set of \( X \), respectively. Hence, \( G^c \) is a \( F_\sigma \)-set and \( F \cap G^c = \emptyset \).

By (i) there exists two disjoint \( G_\delta \)-sets \( G_1, G_2 \) such that \( F \subseteq G_1 \) and \( G^c \subseteq G_2 \). But

\[
G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,
\]

and

\[
G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c
\]

hence

\[
F \subseteq G_1 \subseteq G_2 \subseteq G
\]

and since \( G_2^c \) is a \( F_\sigma \)-set containing \( G_1 \) we conclude that \( F_\sigma(G_1) \subseteq G_2^c \), i.e.,

\[
F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.
\]

By setting \( H = G_1 \), condition (ii) holds.

(ii) \( \Rightarrow \) (i) Suppose that \( F_1, F_2 \) are two disjoint \( F_\sigma \)-sets of \( X \).

This implies that \( F_1 \subseteq F_2^c \) and \( F_2 \) is a \( G_\delta \)-set. Hence by (ii) there exists a \( G_\delta \)-set \( H \) such that, \( F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2 \).

But

\[
H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset
\]

and

\[
F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.
\]

Furthermore, \( (F_\sigma(H))^c \) is a \( G_\delta \)-set of \( X \). Hence \( F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c \) and \( H \cap (F_\sigma(H))^c = \emptyset \). This means that condition (i) holds. \( \Box \)
Lemma 3.3. Suppose that $X$ is the topological space such that we can separate every two disjoint $F_\sigma$-sets by $G_\delta$-sets. If $F_1$ and $F_2$ are two disjoint $F_\sigma$-sets of $X$, then there exists a Baire-.5 function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.

Proof. Suppose $F_1$ and $F_2$ are two disjoint $F_\sigma$-sets of $X$. Since $F_1 \cap F_2 = \emptyset$, hence $F_1 \subseteq F_\sigma^c_2$. In particular, since $F_\sigma^c_2$ is a $G_\delta$-set of $X$ containing $F_1$, by Lemma 3.2, there exists a $G_\delta$-set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_\sigma^c_2.$$ 

Note that $H_{1/2}$ is a $G_\delta$-set and contains $F_1$, and $F_\sigma^c_2$ is a $G_\delta$-set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 3.2, there exists $G_\delta$-sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_\sigma^c.$$ 

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $G_\delta$-sets $H_t$ with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function $h$ on $X$ by $h(x) = \inf \{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., $h$ maps $X$ into $[0,1]$. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that $h$ is a Baire-5 function on $X$.

For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \bigcup \{H_t : t < \alpha\}$, hence, they are $G_\delta$-sets of $X$. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = \{x \in X : h(x) > \alpha\}$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \bigcup \{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a $G_\delta$-set. Consequently $h$ is a Baire-5 function. \qed

Lemma 3.4. Suppose that $X$ is the topological space such that we can separate every two disjoint $F_\sigma$-sets by $G_\delta$-sets. If $F_1$ and $F_2$ are two disjoint $F_\sigma$-sets of $X$ and $F_1$ is a countable intersection of $G_\delta$-sets, then there exists a Baire-5 function $h$ on $X$ into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

Proof. Suppose that $F_1 = \bigcap_{n=1}^\infty G_n$, where $G_n$ is a $G_\delta$-set of $X$. We can suppose that $G_n \cap F_2 = \emptyset$, otherwise we can substitute $G_n$ by $G_n \setminus F_2$. By Lemma 3.3, for every $n \in \mathbb{N}$, there exists a Baire-5 function $h_n$ on $X$ into $[0,1]$ such that $h_n(F_1) = \{0\}$ and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^\infty 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that $h$ is a Baire-5 function from $X$ to $[0,1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$, therefore $h_n(F_2) = \{1\}$ and consequently $h(F_2) = \{1\}$. Since $h_n(F_1) = \{0\}$, hence $h(F_1) = \{0\}$. It suffices to show that if $x \notin F_1$, then $h(x) \neq 0$.

Now if $x \notin F_1$, since $F_1 = \bigcap_{n=1}^\infty G_n$, therefore there exists $n_0 \in \mathbb{N}$ such that $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$. \qed

Lemma 3.5. Suppose that $X$ is the topological space such that we can separate every two disjoint $F_\sigma$-sets by $G_\delta$-sets. The following conditions are equivalent:

(i) For every two disjoint $F_\sigma$-sets $F_1$ and $F_2$, there exists a Baire-5 function $h$ on $X$ into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$. 


(ii) Every $F_\sigma$-set is a countable intersection of $G_\delta$-sets.
(iii) Every $G_\delta$-set is a countable union of $F_\sigma$-sets.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $F$ is a $F_\sigma$-sets. Since $\emptyset$ is a $F_\sigma$-set, by (i) there exists a Baire-.5 function $h$ on $X$ into $[0, 1]$ such that $h^{-1}(0) = F$. Set $G_n = \{ x \in X : h(x) < \frac{1}{n} \}$. Then for every $n \in \mathbb{N}$, $G_n$ is a $G_\delta$-set and $\bigcap_{n=1}^{\infty} G_n = \{ x \in X : h(x) = 0 \} = F$.

(ii) $\Rightarrow$ (i). Suppose that $F_1$ and $F_2$ are two disjoint $F_\sigma$-sets. By Lemma 3.4, there exists a Baire-.5 function $f$ on $X$ into $[0, 1]$ such that $f^{-1}(0) = F_1$ and $f(F_2) = \{ 1 \}$. Set $G = \{ x \in X : f(x) < \frac{1}{2} \}$, $F = \{ x \in X : f(x) = \frac{1}{2} \}$, and $H = \{ x \in X : f(x) > \frac{1}{2} \}$. Then $G \cup F$ and $H \cup F$ are two $F_\sigma$-sets and $(G \cup F) \cap F_2 = \emptyset$. By Lemma 3.4, there exists a Baire-.5 function $g$ on $X$ into $[\frac{1}{2}, 1]$ such that $g^{-1}(1) = F_2$ and $g(G \cup F) = \{ \frac{1}{2} \}$. Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. $h$ is well-defined and a Baire-.5 function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0, 1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every $F_\sigma$-set is a $G_\delta$-set and complement of every $G_\delta$-set is a $F_\sigma$-set, the equivalence is hold.

**Remark 3.** [20] A space $X$ has the strong $c-$insertion property for $(usc, lsc)$ if and only if $X$ is perfectly normal.

**Corollary 3.3.** For every two disjoint $F_\sigma$-sets $F_1$ and $F_2$, there exists a Baire-.5 function $h$ on $X$ into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ if and only if $X$ has the strong $B-.5-$insertion property for $(uscB-.5, lscB-.5)$.

**Proof.** Since for every two disjoint $F_\sigma$-sets $F_1$ and $F_2$, there exists a Baire-.5 function $h$ on $X$ into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{ x \in X : h(x) < \frac{1}{2} \}$ and $G_2 = \{ x \in X : h(x) > \frac{1}{2} \}$. Then $G_1$ and $G_2$ are two disjoint $G_\delta$-sets that contain $F_1$ and $F_2$, respectively. This means that, we can separate every two disjoint $F_\sigma$-sets by $G_\delta$-sets. Hence by Corollary 3.1, $X$ has the weak $B-.5-$insertion property for $(uscB-.5, lscB-.5)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ is $uscB-.5$ and $f$ is $lscB-.5$. Since $f - g$ is $lscB-.5$, therefore the lower cut set $A(f - g, 2^{-n}) = \{ x \in X : (f - g)(x) \leq 2^{-n} \}$ is a $F_\sigma$-set. By Lemma 3.5, we can choose a sequence $\{ F_n \}$ of $F_\sigma$-sets such that $\{ x \in X : (f - g)(x) > 0 \} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}$, $F_n$ and $A(f - g, 2^{-n})$ are disjoint. By Lemma 3.3, $F_n$ and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, $X$ has the strong $B-.5-$insertion property for $(uscB-.5, lscB-.5)$.

On the other hand, suppose that $F_1$ and $F_2$ are two disjoint $F_\sigma$-sets. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. Set $g = \chi_{F_2}$ and $f = \chi_{F_1^c}$. Then $f$ is $lscB-.5$ and $g$ is $uscB-.5$ and furthermore $g \leq f$. By hypothesis, there exists a Baire-.5 function $h$ on $X$ such that $g \leq h \leq f$ and whenever $g(x) < f(x)$ we have $g(x) < h(x) < f(x)$. By definitions of $f$ and $g$, we have $h^{-1}(1) = F_2 \cap F_1^c = F_2$ and $h^{-1}(0) = F_1 \cap F_1^c = F_1$.

**Remark 4.** [2] A space $X$ has the strong $c-$insertion property for $(lsc, usc)$ if and only if each open subset of $X$ is closed.
Corollary 3.4. Every $G_\delta$-set is a $F_\sigma$-set if and only if $X$ has the strong $B-\sigma$.5-insertion property for ($\text{cls}B-.5, \text{cus}B-.5$).

Proof. By hypothesis, for every $G$ of $G_\delta$-set, we have $F_\sigma(G) = G$ is a $G_\delta$-set. Hence by Corollary 3.2, $X$ has the weak $B-\sigma$.5-insertion property for ($\text{cls}B-.5, \text{cus}B-.5$).

Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ is $\text{cls}B-.5$ and $f$ is $B-.5$. Set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) < 2^{-n}\}$. Then, since $f - g$ is $\text{cus}B-.5$, we can say that $A(f - g, 2^{-n})$ is a $G_\delta$-set. By hypothesis, $A(f - g, 2^{-n})$ is a $F_\sigma$-set. Set $G_n = X \setminus A(f - g, 2^{-n})$. Then $G_n$ is a $G_\delta$-set. This means that $G_n$ and $A(f - g, 2^{-n})$ are disjoint $G_\delta$-sets and also are two disjoint $F_\sigma$-sets. Therefore $G_n$ and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Now, we have $\bigcup_{n=1}^{\infty} G_n = \{x \in X : (f - g)(x) > 0\}$. By Theorem 2.2, $X$ has the strong $B-\sigma$.5-insertion property for ($\text{cls}B-.5, B-.5$). By an analogous argument, we can prove that $X$ has the strong $B-\sigma$.5-insertion property for ($B-.5, \text{cus}B-.5$). Hence, by Theorem 2.3, $X$ has the strong $B-\sigma$.5-insertion property for ($\text{cls}B-.5, \text{cus}B-.5$).

On the other hand, suppose that $X$ has the strong $B-\sigma$.5-insertion property for ($\text{cls}B-.5, \text{cus}B-.5$). Also, suppose that $G$ is a $G_\delta$-set. Set $f = 1$ and $g = \chi_G$. Then $f$ is $\text{cus}B-.5$, $g$ is $\text{cls}B-.5$ and $f \leq g$. By hypothesis, there exists a Baire-.5 function $h$ on $X$ such that $g \leq h \leq f$ and whenever $g(x) < f(x)$, we have $g(x) < h(x) < f(x)$. It is clear that $h(G) = \{1\}$ and for $x \in X \setminus G$ we have $0 < h(x) < 1$. Since $h$ is a Baire-.5 function, therefore $\{x \in X : h(x) \geq 1\} = G$ is a $F_\sigma$-set, i.e., $G$ is a $F_\sigma$-set. □

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References


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