Decomposition of groups and the wreath product of permutation groups

N. Ghadbane

Abstract. In this work, we study decomposition of groups. Let $G$ be a group, and let $N$ be a normal subgroup of $G$; we shall show how multiplication in $G$ can be viewed as a two-step process consisting of multiplication in the quotient $G/N$ followed by multiplication in $N$. The object of wreath product of permutation groups is defined by the actions on Cartesian product of two sets. In this paper we consider $S(\Gamma)$ and $S(\Delta)$ - the permutation groups on $\Gamma$ and $\Delta$ respectively, and $S(\Gamma)^\Delta$ - the set of all maps of $\Delta$ into the permutations group $S(\Gamma)$, to provide the wreath product $W$ of $S(\Gamma)$ by $S(\Delta)$, and the action of $W$ on $\Gamma \times \Delta$.

M.S.C. 2010: 20AXX, 20BXX.
Key words: group; decomposition of group; acts of group in a set; morphism of groups; semidirect product of groups; wreath product of groups.

1 Introduction

In Mathematics, the wreath product in group theory is a specialized product of two groups. The wreath product is an important tool in the classification of permutation groups, and also provides a way of constructing interesting examples of groups. The wreath product and its generalizations play an important role in algebraic theory. For example, it can be used to prove the theorem on the decomposition of every finite semi-group automation into a step wise combination of flip-flop and simple group automata.

The remainder of this paper is organized as follows. In Section 2 we introduce the concept of decomposition of groups. In Section 3, we provide some mathematical preliminaries. In Section 4, we give a proposition on wreath product of groups. In Section 5, we introduce the wreath product of permutation groups and the notion of group actions on a set, and adjacent concepts like the orbit and the stabilizer. Finally, we draw our conclusions in Section 6.
2 Preliminaries

Let $S(X)$ be the set of one to one and onto functions on the $n$-element set $X$, with multiplication to composition of functions. The elements of $S(X)$ are called permutations and $S(X)$ is called the symmetric group on $X$.

A group homomorphism is a well-defined map $\varphi : G \rightarrow H$ between two groups $G$ and $H$, which preserves the multiplicative structure. In other words, $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x,y \in G$. A bijective homomorphism is called an isomorphism. When there is an isomorphism between two groups $G$ and $H$, we say $G$ and $H$ are isomorphic and we write $G \cong H$.

Let $G$ and $H$ be group and $\varphi : G \rightarrow H$ be a homomorphism. Then $N = \ker \varphi$ is a normal subgroup of $G$ and the induced map $\bar{\varphi} : G/N \rightarrow \text{Im}(\varphi) \leq H, Ng \mapsto \varphi(g)$ is an isomorphism between the quotient group $G/N$ and the image $\text{Im}(\varphi)$.

Let $G$ be a group and $X$ be a non empty set. We say that $G$ acts on the set $X$ if to each $g$ in $G$ and each $x$ in $X$, there corresponds a unique point $g \cdot x$ in $X$ such that, for all $x$ in $X$ and $g_1, g_2$ in $G$, we have that

$$(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \text{ and } 1_G \cdot x = x.$$ 

To be explicit, we assume that $G$ acts on the set $X$ on the left. The stabilizer of an element $x \in X$ under the action of $G$ is defined by:

$$G_x = \{g \in G : g \cdot x = x\}.$$

The kernel of an action $G \times X \rightarrow X, (g,x) \mapsto g \cdot x$ is given by:

$$Ker = \{g \in G : g \cdot x = x \text{ for all } x \in X\}.$$

We define the orbit containing $x \in X$ to be $G \cdot x = \{g \cdot x, g \in G\}$.

Let $G$ be a group acting on a set $X$. Then, for all $x \in X$, we have $|G_x| |G \cdot x| = |G|$.

Let $G$ and $K$ be two groups. We say that $G$ acts on $K$ as a group if to each $k$ in $K$ there corresponds a unique element $k^g$ in $K$ such that for $g_1, g_2, g$ in $G$ and $k_1, k_2, k$ in $K$ we have that

$$(k^{g_1})^{g_2} = k^{g_1g_2}, k^{1_G} = k \text{ and } (k_1k_2)^g = k_1^g k_2^g.$$ 

Given any groups $G$ and $H$ and a morphism $\theta : G \rightarrow \text{Aut}(H)$, we denote the automorphism $\theta(g)$ by $\theta_g$, and then $G \times H$ is a group with the multiplication $(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1\theta_{g_1}(h_2))$, where $g_1, g_2 \in G$ and $h_1, h_2 \in H$. The group $(G \times H, \cdot)$ is called the semidirect product of $G$ and $H$ with respect to $\theta$.

3 Decomposition of groups

In this section, we introduce the concept of decomposition of groups.

**Proposition 3.1.** Let $(G, \cdot)$ be a group, and let $N$ be a normal subgroup of $G$. Let $S = \{g_i : i \in I\}$ be a complete set of coset representatives of $N$. If $g \in G$, then we denote by $[g]$ the chosen representative of $Ng$. Thus,
1. For all \( n_1, n_2 \in N, g_i, g_j \in S : (n_1 g_i) (n_2 g_j) = n g \), where \( g = [g_i, g_j] \) and \( n = n_1 (g_i n_2 g_j^{-1}) \).

2. Consider the set \( N \times G/N \) the Cartesian product of \( N \) and \( G/N \), and define a multiplication "\(*\)" on \( N \times G/N \) as follows:

\[
(n_1, N g_i) * (n_2, N g_j) = (n_1 g_i n_2 g_j^{-1}, N g_i g_j)
\]

The result is a group.

3. \((N \times G/N, \ast )\) and \((G, \cdot )\) are isomorphic.

Proof. 1) For all \( n_1, n_2 \in N, g_i, g_j \in S \), we have \( n_1 g_i \in N g_i, n_2 g_j \in N g_j \) and \( N g_i, N g_j = N g_i, N g_j \), then \((n_1 g_i) (n_2 g_j) \in N g_i, N g_j \). This implies that there exists \( n \in N \) such that

\[
(n_1 g_i) (n_2 g_j) = n g \quad \text{and consequently} \quad g = [g_i, g_j] \quad \text{and} \quad n = n_1 (g_i n_2 g_j^{-1}).
\]

2) We will prove that the set \( N \times G/N \) forms a group such that for any \((n_1, N g_i), (n_2, N g_j) \in N \times G/N \), \((n_1, N g_i) * (n_2, N g_j) = (n_1 g_i n_2 g_j^{-1}, N g_i g_j) \).

a) \( N \times G/N \) is non-empty and is closed with respect to multiplication.

b) We will prove that "\(*\)" is associative on \( N \times G/N \):

let \((n_1, N g_i), (n_2, N g_j), (n_3, N g_k) \in N \times G/N \), we have

\[
((n_1, N g_i) * (n_2, N g_j)) * (n_3, N g_k) = (n_1 g_i n_2 g_j^{-1}, N g_i g_j) * (n_3, N g_k) = (n_1 g_i n_2 g_j n_3 g_j^{-1} g_i^{-1}, N (g_i g_j) g_k).
\]

Also we have

\[
(n_1, N g_i) * ((n_2, N g_j) * (n_3, N g_k)) = (n_1 g_i n_2 g_j n_3 g_j^{-1} g_i^{-1}, N g_i (g_j g_k)).
\]

Then "\(*\)" is associative on \( N \times G/N \).

c) For \((n, N g_i) \in N \times G/N \), we have

\[
(n, N g_i) * (1_G, N) = (1_G, N) * (n, N g_i) = (n, N g_i).
\]

The identity element in \((N \times G/N, \ast)\) is \((1_G, N)\).

d) We show that every element of \((N \times G/N, \ast)\) is invertible.

Let \((n, N g_i) \in N \times G/N \). We have

\[
(n, N g_i) * (g_i^{-1} n g_i, N g_i^{-1}) = (g_i^{-1} n g_i, N g_i^{-1}) * (n, N g_i) = (1_G, N).
\]

3) We define the mapping \( \varphi : N \times G/N \to G/\varphi (n, N g_i) = n g_i \); the mapping \( \varphi \) is a morphism because for all \((n_1, N g_i), (n_2, N g_j) \in N \times G/N \),

\[
\varphi ((n_1, N g_i) * (n_2, N g_j)) = \varphi (n_1 g_i n_2 g_j^{-1}, N g_i g_j) = n_1 g_i n_2 g_j^{-1} g_i g_j = n_1 g_i n_2 g_j.
\]

It is clear that \( \varphi \) is onto. The mapping \( \varphi \) is one-to-one, because we have for all \((n_1, N g_i), (n_2, N g_j) \in N \times G/N \),

\[
\varphi (n_1, N g_i) = \varphi (n_2, N g_j) \implies n_1 = n_2 \implies n_1^{-1} n_1 = g_j g_i^{-1} \implies g_j g_i^{-1} \in N \implies g_j g_i^{-1} = 1_G \implies (n_1 = n_2 \text{ and } N g_i = N g_j).
\]

\[\square\]

**Example 3.1.** Let \( G = (S_3, \circ) \) be the symmetric group on \( \{1, 2, 3\} \), where

\( S_3 = \{e = (1), \tau_1 = (23), \tau_2 = (13), \sigma_3 = (12), \sigma_3 = (123), \sigma_2 = (132)\}\).

The Cayley table of \((S_3, \circ)\) is defined as follows (see Table 1):

<table>
<thead>
<tr>
<th></th>
<th>( e )</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \tau_3 )</th>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>( e )</td>
<td>( \tau_1 )</td>
<td>( \tau_2 )</td>
<td>( \tau_3 )</td>
<td>( \sigma_1 )</td>
<td>( \sigma_2 )</td>
</tr>
<tr>
<td>( \tau_1 )</td>
<td>( \tau_1 )</td>
<td>( e )</td>
<td>( \sigma_1 )</td>
<td>( \sigma_2 )</td>
<td>( \tau_2 )</td>
<td>( \tau_3 )</td>
</tr>
<tr>
<td>( \tau_2 )</td>
<td>( \tau_2 )</td>
<td>( \sigma_1 )</td>
<td>( e )</td>
<td>( \sigma_1 )</td>
<td>( \tau_3 )</td>
<td>( \tau_1 )</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( \tau_3 )</td>
<td>( \sigma_1 )</td>
<td>( \sigma_2 )</td>
<td>( e )</td>
<td>( \tau_1 )</td>
<td>( \tau_2 )</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>( \sigma_1 )</td>
<td>( \tau_3 )</td>
<td>( \tau_1 )</td>
<td>( \tau_2 )</td>
<td>( e )</td>
<td>( \sigma_2 )</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>( \sigma_2 )</td>
<td>( \tau_1 )</td>
<td>( \sigma_2 )</td>
<td>( e )</td>
<td>( \sigma_1 )</td>
<td>( \tau_3 )</td>
</tr>
</tbody>
</table>
Let \( N = \langle (123) \rangle = \langle \sigma_1 \rangle \) the subgroup of \( (S_3, \circ) \) generated by \( \sigma_1 \).

We have \( N = \{ e, \sigma_1, \sigma_2 \} \). \( N \) is a normal subgroup of \( (S_3, \circ) \) because the index of \( N \) in \( S_3 \) \( [S_3 : N] = 2 \).

The quotient \( S_3/N = \{ N \circ \sigma : \sigma \in S_3 \} = \{ \{ \tau_1, \tau_2, \tau_3 \}, \{ e, \sigma_1, \sigma_2 \} \} = \{ [e], [\tau_1] \} \).

The Cayley table of \( (S_3/N, \cdot) \) is defined as follows (see Table 2):

\[
\begin{array}{ccc}
  \cdot & [e] & [\tau_1] \\
  [e] & [e] & [\tau_1] \\
  [\tau_1] & [\tau_1] & [e] \\
\end{array}
\]

\( N \times S_3/N = \{ (e, [e]), (e, [\tau_1]), (\sigma_1, [e]), (\sigma_1, [\tau_1]), (\sigma_2, [e]), (\sigma_2, [\tau_1]) \} \).

The Cayley table of \( (N \times S_3/N, \ast) \) is defined as follows (see Table 3):

\[
\begin{array}{cccccccc}
  * & (e, [e]) & (e, [\tau_1]) & (\sigma_1, [e]) & (\sigma_1, [\tau_1]) & (\sigma_2, [e]) & (\sigma_2, [\tau_1]) \\
  (e, [e]) & (e, [e]) & (e, [\tau_1]) & (\sigma_1, [e]) & (\sigma_1, [\tau_1]) & (\sigma_2, [e]) & (\sigma_2, [\tau_1]) \\
  (e, [\tau_1]) & (e, [\tau_1]) & (e, [e]) & (\sigma_2, [\tau_1]) & (\sigma_2, [e]) & (\sigma_1, [\tau_1]) & (\sigma_1, [e]) \\
  (\sigma_1, [e]) & (\sigma_1, [e]) & (\sigma_1, [\tau_1]) & (\sigma_2, [e]) & (\sigma_2, [\tau_1]) & (\sigma_1, [e]) & (\sigma_1, [\tau_1]) \\
  (\sigma_1, [\tau_1]) & (\sigma_1, [\tau_1]) & (\sigma_1, [e]) & (e, [\tau_1]) & (e, [e]) & (\sigma_1, [\tau_1]) & (\sigma_1, [e]) \\
  (\sigma_2, [e]) & (\sigma_2, [e]) & (\sigma_2, [\tau_1]) & (e, [e]) & (e, [\tau_1]) & (\sigma_1, [e]) & (\sigma_1, [\tau_1]) \\
  (\sigma_2, [\tau_1]) & (\sigma_2, [\tau_1]) & (\sigma_2, [e]) & (\sigma_1, [\tau_1]) & (\sigma_1, [e]) & (e, [\tau_1]) & (e, [e]) \\
\end{array}
\]

Finally \( (N \times G/N, \ast) \) and \( (S_3, \circ) \) are isomorphic.

4 The wreath product of groups

In this section, we introduce the concept of wreath product of groups.

**Theorem 4.1.** Let \( G \) and \( H \) be two groups. Let \( H^G \) be the set of all functions defined on \( G \) with values in \( H \).

1. The set \( H^G \) forms a group such that for any \( \varphi, \psi \in H^G \), let \( \varphi \psi \in H^G \) be defined for all \( x \in G \) by:

\[
(\varphi \psi)(x) = \varphi(x)\psi(x).
\]

2. The group \( G \) acts on \( H^G \) as a group, in the following way:

\[
\text{if } a \in G, \varphi \in H^G, \text{ then } (a \cdot \varphi)(x) = \varphi^a(x) = \varphi(xa^{-1}) \text{ for } x \in G.
\]

3. The set of all pairs \( (a, \varphi) \) with \( a \in G, \varphi \in H^G \), with the multiplication operation given by:

\[
(a, \varphi)(b, \psi) = (ab, \varphi^b \psi) \text{ where } a, b \in G \text{ and } \varphi, \psi \in H^G
\]

provides as resulting group \( W \), called the wreath product of \( G \) and \( H \), and denoted by \( GW_r H \).
Proof. (1) First we will prove that the set $H^G$ forms a group; for any $\varphi, \psi \in H^G$, let $\varphi \psi \in H^G$ in $H^G$.

(i) $H^G$ is non-empty and is closed with respect to multiplication. If $\varphi, \psi \in H^G$, then $\varphi(x), \psi(x) \in H$, for all $x \in G$. Hence $\varphi(x) \psi(x) \in H$. This implies that $(\varphi \psi)(x) \in H$ and so $\varphi \psi \in H^G$.

(ii) Since multiplication in $H$ is associative, so is the multiplication in $H^G$, as well.

(iii) The identity element in $H^G$ is the map $e : G \to H$ given by $e(x) = 1_H$, for all $x \in G$, where $1_H$ is the identity element of $H$.

(iv) For every element $\varphi \in H^G$, is defined for all $x \in G$ by $\varphi^{-1}(x) = (\varphi(x))^{-1}$.

Thus $H^G$ is a group with respect to the multiplication defined above.

(2) We shall further prove that $G$ acts on $H^G$ as group; assume that $G$ acts on $H^G$ as follows $G \times H^G \to H^G ; (a, \varphi) \to \varphi^a$, such that for $x \in G$ we have $\varphi^a(x) = \varphi(xa^{-1})$, $a \in G, \varphi \in H^G$. Take $\varphi, \psi \in H^G$ and $a, b \in G$; then

(i) $(\varphi^a)^b(x) = \varphi^a(xb^{-1}) = \varphi((xb^{-1})a^{-1}) = \varphi(x(ab)^{-1}) = \varphi^{ab}(x)$.

(ii) $\varphi^1_G(x) = \varphi(x1_G^{-1}) = \varphi(x)$.

(iii) $(\varphi\psi)^b(x) = \varphi\psi(xa^{-1}) = \varphi(xa^{-1})\psi(xa^{-1}) = \varphi^a(x)\psi^a(x)$.

Now we can construct the wreath product $W$ of $G$ and $H$, that is, the semidirect product of $G$ and $H^G$; then we shall prove that $G \times H^G$ is a group with multiplication $(a, \varphi)(b, \psi) = (ab, \varphi^a \psi)$.

Then we have the following:

(i) The closure property follows from the definition of multiplication.

(ii) Take $\varphi, \psi, \eta \in H^G$ and $a, b, c \in G$; then

$$((a, \varphi)(b, \psi))(d, \eta) = (ab, \varphi^b \psi)(d, \eta) = \left((ab)d, (\varphi^b \psi)^d \eta\right).$$

Also we have $(a, \varphi)((b, \psi)(d, \eta)) = (a, \varphi)(bd, \psi^d \eta) = (a(bd), \varphi^{bd} \psi^d \eta) = \left((ab)d, \varphi^{bd} \psi^d \eta\right)$.

Now if $x \in G$, then

$$\varphi^a(xd^{-1})\psi(xd^{-1}) \eta(x) = \varphi(xd^{-1}b^{-1})\psi(xd^{-1}) \eta(x) = \varphi(xbd^{-1})\psi(xd^{-1}) \eta(x) = \varphi^{bd}(x)\psi^d(x) \eta(x).$$

As well, $\varphi^{bd} \psi^d \eta(x) = \varphi^{bd}(x)\psi^d(x) \eta(x)$. Thus we have established the associativity of the multiplication on the set $G \times H^G$.

(iii) We know that for every $\varphi \in H^G, \varphi^{1_G} = \varphi$, for every $g \in G$, the map $\varphi \to \varphi^g$ is an automorphism of $H^G$. Also, if $e$ is the identity element in $H^G$, then $e^g = e$. We have $(a, \varphi)(1_G, e) = (a1_G, \varphi^{1_G} e) = (a, \varphi e) = (a, \varphi)$. Also $(1_G, e)(a, \varphi) = (1_G a, e^a \varphi) = (a, e \varphi) = (a, \varphi)$. Hence the identity element exists.

(iv) We have $(a, \varphi)\left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right) = \left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right)(a, \varphi) = (1_G, e)$. Thus the inverse element of $(a, \varphi)$ is $\left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right)$. Hence $G \times H^G$ is a group with respect to the multiplication defined above.

In the following proposition, we show that the group $H^G$ is a normal subgroup of $W$ and $G$ is a subgroup of $W$.

**Proposition 4.2.**

1. If $G$ and $H^G$ are finite groups, then the wreath product $W$ is a finite group of order $|W| = |G| \cdot |H^G|$.

2. The group $H^G$ is a normal subgroup of $W$ and $G$ is a subgroup of $W$. 


3. $G \cap H^G = (1_G, e)$.

4. $GW_r H^G = G \times H^G$.

Proof. (1) It is clear.

(2) We have the injective maps $\Phi : H^G \to G \times H^G$ given by $f \mapsto (1_G, f)$, and $\Psi : G \to G \times H^G$ given by $a \mapsto (a, e)$. Both of them are homomorphisms, since $\Phi (f_1 f_2) = (1_G, f_1 f_2) = (1_G, f_1) W_r (1_G, f_2) = \Phi (f_1) W_r \Phi (f_2)$. As well, $\Psi (ab) = (ab, e) = (a, e) W_r (b, e) = \Psi (a) W_r \Psi (b)$. Then $H^G \cong \text{Im} (\Phi) \leq G \times H^G$, and $G \cong \text{Im} (\Psi) \leq G \times H^G$. These injective homomorphisms let us think of $H^G$ and $G$ as subgroups of $G \times H^G$. Finally, we must show that $H^G$ is normal in $G \times H^G$; from the calculations, there follows:

$$(a, e) (1_G, f) (a, e)^{-1} = (a, e) (1_G, f) \left( a^{-1}, (e^{-1})^{a^{-1}} \right) = (a, e) (1_G, f) (a^{-1}, e) = (a_1, e^1 f) (a^{-1}, e) = (1_G, f) \left( a_1, e^1 f \right) = (a, f) \text{ for all } (a, f) \in G \times H^G$$

5. **Wreath product of permutation groups**

This section is essentially an upgrade of the results of Ibrahim and Audu (see [2]) on the wreath product of permutation groups. We introduce the notion of group actions on a set and its adjacent concepts, like the orbit and the stabilizer.

**Theorem 5.1.** Let $S(\Gamma)$ and $S(\Delta)$ be permutation groups on $\Gamma$ and $\Delta$ respectively. Let $S(\Gamma)^\Delta$ be the set of all maps of $\Delta$ into the permutations group $S(\Gamma)$. That is, $S(\Gamma)^\Delta = \{ f : \Delta \to S(\Gamma) \}$. For any $f_1, f_2$ in $S(\Gamma)^\Delta$, let $f_1 f_2$ in $S(\Gamma)^\Delta$ be defined for all $\delta$ in $\Delta$ by $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$. With respect to this operation of multiplication, $S(\Gamma)^\Delta$ acquires the structure of a group.

Proof. (i) $S(\Gamma)^\Delta$ is non-empty and is closed with respect to multiplication. If $f_1, f_2 \in S(\Gamma)^\Delta$, then $f_1(\delta), f_2(\delta) \in S(\Gamma)$. Hence $f_1(\delta) f_2(\delta) \in S(\Gamma)$. This implies that $(f_1 f_2)(\delta) \in S(\Gamma)$ and so $f_1 f_2 \in S(\Gamma)^\Delta$.

(ii) Since multiplication is associative, so is the multiplication in $S(\Gamma)^\Delta$.

(iii) The identity element in $S(\Gamma)^\Delta$ is the map $e : \Delta \to S(\Gamma)$ given by $e(\delta) = id_\Gamma$ for all $\delta \in \Delta$ where $id_\Gamma$ is the identity element of $S(\Gamma)$.

(iv) Every element $f \in S(\Gamma)^\Delta$ is defined for all $\delta \in \Delta$ by $f^{-1}(\delta) = (f(\delta))^{-1}$.

Thus $S(\Gamma)^\Delta$ is a group with respect to the multiplication defined above. We denote this group by $P$.

**Proposition 5.2.** Assume that $S(\Delta)$ acts on $P$ as follows:

$S(\Delta) \times S(\Gamma)^\Delta \to S(\Gamma)^\Delta$, $(s, f) \mapsto s \cdot f = f^s$, where $f^s(\delta) = (f \circ s^{-1})(\delta)$ $= (fs^{-1})(\delta)$ for all $\delta \in \Delta$. Then $S(\Delta)$ acts on $P$ as a group.

Proof. Take, $f, f_1, f_2 \in S(\Gamma)^\Delta$ and $s, s_1, s_2 \in S(\Delta)$, then
The set of all ordered \((f, s)\) with \(f \in S(\Gamma)\) and \(s \in S(\Delta)\) acquires the structure of a group, when we define for all \(f_1, f_2 \in S(\Gamma)\) and \(s_1, s_2 \in S(\Delta)\).

\[
(f_1, s_1) (f_2, s_2) = \left( f_1 f_2^{s_1^{-1}}, s_1 s_2 \right).
\]

Thus \(S(\Gamma)^{\Delta} \times S(\Delta)\) is a group with respect to the multiplication defined above. We denote this group by \(W\). The resulting group \(W\) is called the wreath product of \(S(\Gamma)\) by \(S(\Delta)\), and is denoted by \(W = S(\Gamma)^{\Delta} \times S(\Delta)\).

**Proof.**

(i) The closure property follows from the definition of multiplication.

(ii) Take \(f_1, f_2, f_3 \in S(\Gamma)^{\Delta}\) and \(s_1, s_2, s_3 \in S(\Delta)\). Then, \([(f_1, s_1) (f_2, s_2)] (f_3, s_3) = \left( f_1 f_2^{s_1^{-1}} f_3^{s_1 s_2 s_3}, s_1 s_2 s_3 \right) = \left( f_1 f_2^{s_1^{-1}} f_3^{s_1^{-1} s_1^{-1}}, s_1 s_2 s_3 \right).

Also, we get in the same manner that, \((f_1, s_1) [(f_2, s_2) (f_3, s_3)] = (f_1, s_1) \left( f_2 f_3^{s_2^{-1}}, s_2 s_3 \right) = \left( f_1 f_2^{s_1^{-1}} f_3^{s_2^{-1} s_1^{-1}}, s_1 s_2 s_3 \right).

Hence, multiplication is associative.

(iii) We know that for every \(f \in S(\Gamma)^{\Delta}\), \(f^{id_\Delta} = f\). Now for every \(s \in S(\Delta)\), the map \(f \mapsto f^s\) is an automorphism of \(S(\Gamma)^{\Delta}\). Also, if \(e\) is the identity element in \(S(\Gamma)^{\Delta}\), then \(e^s = e\). Also, \((f^{-1})^s = (f^s)^{-1}\). Now, \((f, s) (e, id_\Delta) = \left( f e^{s^{-1}}, s \circ id_\Delta \right) = (f, s).

Also, using the reverse order, we have that, \((e, id_\Delta) (f, s) = \left( e f^{(id_\Delta)^{-1}}, id_\Delta \circ s \right) = (f, s). Thus the identity element exists.

(iv) \((f, s) [(f^{-1})^s, s^{-1}] = ((f^{-1})^s, s^{-1}) (f, s) = (e, id_\Delta)\).}

In the following proposition, we show that the group \(S(\Gamma)^{\Delta}\) is a normal subgroup of \(W\), and \(S(\Delta)\) is a subgroup of \(W\).

**Proposition 5.4.**

1. If \(S(\Delta)\) and \(S(\Gamma)\) are finite groups, then the wreath product \(W\) is a finite group of order \(|W| = |S(\Gamma)|^{|\Delta|} \cdot |S(\Delta)|\).

2. The group \(S(\Gamma)^{\Delta}\) is a normal subgroup of \(W\) and \(S(\Delta)\) is a subgroup of \(W\).

3. \(S(\Gamma)^{\Delta} \cap S(\Delta) = \langle e, id_\Delta \rangle\).

4. \(S(\Gamma)^{\Delta} W_S(\Delta) = S(\Gamma)^{\Delta} \times S(\Delta)\).

5. The action of \(W\) on \(\Gamma \times \Delta\) is given by: \((f, s) (\gamma, \delta) = (f(\delta)(\gamma), s(\delta))\) for all \((f, s) \in S(\Gamma)^{\Delta} \times S(\Delta)\) and \((\gamma, \delta) \in \Gamma \times \Delta\).
Proof. (1) Obvious.

(2) We have the injective maps \( \Phi : S(\Gamma)^{\Delta} \rightarrow S(\Gamma)^{\Delta} \times S(\Delta) \) given by
\[ f \mapsto (f, id_{\Delta}) \], and \( \Psi : S(\Delta) \rightarrow S(\Gamma)^{\Delta} \times S(\Delta) \) given by \( s \mapsto (e, s) \). Both of them are homomorphisms, since
\[ \Phi(f_1 f_2) = (f_1 f_2, id_{\Delta}) = (f_1 f_2, id_{\Delta} \circ id_{\Delta}) \]
\[ = (f_1, id_{\Delta}) W_r (f_2, id_{\Delta}) = \Phi(f_1) W_r \Phi(f_2) . \]
As well, \( \Psi(s_1 \circ s_2) = (e, s_1 \circ s_2) = (e(e(s_1))^{-1}, s_1 \circ s_2) \)
\[ = (e, s_1) W_r (e, s_2) = \Psi(s_1) W_r \Psi(s_2) . \]

Then \( S(\Gamma)^{\Delta} \cong \text{Im}(\Phi) \leq S(\Gamma)^{\Delta} \times S(\Delta) \). And \( S(\Delta) \cong \text{Im}(\Psi) \leq S(\Gamma)^{\Delta} \times S(\Delta) \). These injective homomorphisms let us think of \( S(\Gamma)^{\Delta} \) and \( S(\Delta) \) as subgroups of \( S(\Gamma)^{\Delta} \times S(\Delta) \). Finally, we must show that \( S(\Gamma)^{\Delta} \) is normal in \( S(\Gamma)^{\Delta} \times S(\Delta) \); and the calculation leads to
\[ (e, s) (f, id_{\Delta}) (e, s)^{-1} = (e, s) (f, id_{\Delta}) ((e^{-1})^{s}, s^{-1}) = (e, s) (f, id_{\Delta}) (e, s^{-1}) \]
\[ = \left( e f^{(s)^{-1}, s \circ id_{\Delta}} (e, s^{-1}) \right) \left( e f^{(s)^{-1}, id_{\Delta}} (e, s^{-1}) \right) . \]

(3) It is clear that \( S(\Gamma)^{\Delta} \cap S(\Delta) = (e, id_{\Delta}) \).

(4) We have \( S(\Gamma)^{\Delta} W_r S(\Delta) = S(\Gamma)^{\Delta} \times S(\Delta) \) since \( (f, id_{\Delta}) W_r (e, s) = (f^{e(id_{\Delta})^{-1}, id_{\Delta}} \circ s, (f, s) \) for all \( (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) \).

(5) Take, \( (f_1, s_1), (f_2, s_2) \in S(\Gamma)^{\Delta} \times S(\Delta) \) and \( (\gamma, \delta) \in \Gamma \times \Delta \), then
\[ i) \quad (e, id_{\Delta}) (\gamma, \delta) = (e(\delta)) (\gamma, id_{\Delta}(\delta)) = (id_{\Gamma} (\gamma), \delta) = (\gamma, \delta) . \]
\[ ii) \quad [(f_1, s_1) (f_2, s_2)] (\gamma, \delta) = \left( f_1 f_2^{-(s_1, s_2)} (\gamma, s_1 s_2) \right) (\gamma, \delta) = \left( f_1 f_2^{-(s_1, s_2)} (\gamma, s_1 s_2) \right) . \]
Also, we have in the same manner that,
\[ (f_1, s_1) [(f_2, s_2) (\gamma, \delta)] = (f_1, s_1) (f_2 (\delta) (\gamma) , s_2 (\delta)) = (f_1 (s_2 (\delta)) (f_2 (\delta) (\gamma)) , s_1 s_2 (\delta)) . \]
\[ \square \]

**Proposition 5.5.** Under the action of \( W \) on \( \Gamma \times \Delta \), the stabilizer of any point \( (\gamma, \delta) \in \Gamma \times \Delta \) denoted by \( W_{(\gamma, \delta)} \) is given by \( W_{(\gamma, \delta)} = S(\Gamma)^{\Delta} (\delta) \times S(\Delta) \). Where \( S(\Gamma)^{\Delta} (\delta) \) is the set of all \( f(\delta) \) that stabilize \( \gamma \), and \( S(\Delta) \) is the stabilizer of \( \delta \) under the action of \( S(\Delta) \) on \( \Delta \).

**Proof.** We have \( W_{(\gamma, \delta)} = \{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / (f, s) (\gamma, \delta) = (\gamma, \delta) \} \)
\[ = \{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / (f(\delta) (\gamma) , s(\delta)) = (\gamma, \delta) \} \]
\[ = \{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / f(\delta) (\gamma) , s(\delta) = (\gamma, \delta) \} = S(\Gamma)^{\Delta} (\delta) \times S(\Delta) . \]
\[ \square \]

**Example 5.1.** Consider the permutation groups \( S(\Gamma) = \{(1), (12)\} \) and \( S(\Delta) = \{(1), (12), (13), (23), (123), (132)\} \) on the sets \( \Gamma = \{1, 2\} \) and \( \Delta = \{1, 2, 3\} \) respectively. Let \( S(\Gamma)^{\Delta} = \{ f : \Delta \rightarrow S(\Gamma) \} \), then
\[ |S(\Gamma)^{\Delta}| = 2 \cdot 2 = 4 \] as mappings are:
\[
\begin{align*}
  f_1 : 1 & \mapsto (1), 2 \mapsto (1), 3 \mapsto (1) \\
  f_2 : 1 & \mapsto (1), 2 \mapsto (1), 3 \mapsto (12) \\
  f_3 : 1 & \mapsto (1), 2 \mapsto (12), 3 \mapsto (1) \\
  f_4 : 1 & \mapsto (1), 2 \mapsto (12), 3 \mapsto (12) \\
  f_5 : 1 & \mapsto (12), 2 \mapsto (1), 3 \mapsto (1) \\
  f_6 : 1 & \mapsto (12), 2 \mapsto (1), 3 \mapsto (12) \\
  f_7 : 1 & \mapsto (12), 2 \mapsto (12), 3 \mapsto (1) 
\end{align*}
\]
Decomposition of groups and the wreath product of permutation groups

We have

\[ S(\Gamma) \Delta \times S(\Delta) = \{(f, s) / f \in S(\Gamma) \Delta, s \in S(\Delta)\} \]

\[ = \{(f_1, (1)), (f_2, (12)), (f_3, (12)), (f_1, (23)), (f_2, (132)), (f_1, (132)) \} \leq i \leq 8\). \]

And \[|S(\Gamma) \Delta \times S(\Delta)| = |S(\Gamma) \Delta| \times |S(\Delta)| = 8.6 = 48\]

\[ S(\Gamma) \Delta \times S(\Delta) \]

is a group with respect to the operation \((\varphi, \psi)(\delta, s_2) = (\varphi(\psi(s_1)^{-1}), s_1 s_2)\).

We have \(\Gamma \times \Delta = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} \)

The stabilizer of \((1, 1)\) denoted by

\[ W_{(1,1)} = S(\Gamma) \Delta (1) \times S(\Delta) = \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\} \]

\[ = \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)), (f_3, (23)) \} \]

Then \(W_{(1,1)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

Also, we can infer in the same manner that,

\[ W_{(1,2)} = S(\Gamma) \Delta (2) \times S(\Delta) = \{f_1, f_2, f_5, f_6\} \times \{(1), (13)\} \]

\[ = \{(f_1, (1)), (f_2, (1)), (f_5, (1)), (f_6, (1)), (f_1, (13)), (f_2, (13)), (f_5, (13)), (f_6, (13)) \} \]

Then \(W_{(1,2)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

\[ W_{(1,3)} = S(\Gamma) \Delta (3) \times S(\Delta) = \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\} \]

\[ = \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)), (f_5, (12)), (f_7, (12)) \} \]

Then \(W_{(1,3)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

\[ W_{(2,1)} = S(\Gamma) \Delta (2) \times S(\Delta) = \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\} \]

\[ = \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)), (f_3, (23)), (f_4, (23)) \} \]

Then \(W_{(2,1)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

\[ W_{(2,2)} = S(\Gamma) \Delta (2) \times S(\Delta) = \{f_1, f_2, f_3, f_5\} \times \{(1), (13)\} \]

\[ = \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_5, (1)), (f_1, (13)), (f_2, (13)), (f_3, (13)), (f_5, (13)) \} \]

Then \(W_{(2,2)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

\[ W_{(2,3)} = S(\Gamma) \Delta (3) \times S(\Delta) = \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\} \]

\[ = \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)), (f_5, (12)), (f_7, (12)) \} \]

Then \(W_{(2,3)}\) is a subgroup of \(S(\Gamma) \Delta \times S(\Delta)\) of order 8.

Finally, we have \(W_{(1,1)} = W_{(2,1)}, W_{(1,2)} = W_{(2,2)}, W_{(1,3)} = W_{(2,3)}\).

For \((\gamma, \delta) \in \Gamma \times \Delta\), we have \(|W_{(\gamma, \delta)}| = |W| \times |W(\gamma, \delta)| = |W|\), then

\[ |W(\gamma, \delta)| = \frac{|W|}{|W_{(\gamma, \delta)}|} = \frac{48}{6} = 8 \]

In this example, we have

\[ (f_1, (1)) (1, 1) = (f_2, (1)) (1, 1) = (f_3, (1)) (1, 1) = (f_4, (1)) (1, 1) = (1, 1) \]

\[ (f_1, (12)) (1, 1) = (f_2, (12)) (1, 1) = (f_3, (12)) (1, 1) \]

\[ = (f_4, (12)) (1, 1) = (1, 2) \]

\[ (f_1, (13)) (1, 1) = (f_2, (13)) (1, 1) = (f_3, (13)) (1, 1) \]

\[ = (f_4, (13)) (1, 1) = (1, 1) \]

\[ (f_1, (23)) (1, 1) = (f_2, (23)) (1, 1) = (f_3, (23)) (1, 1) \]

\[ = (f_4, (23)) (1, 1) = (1, 1) \]

\[ (f_1, (123)) (1, 1) = (f_2, (123)) (1, 1) = (f_3, (123)) (1, 1) \]

\[ = (f_4, (123)) (1, 1) = (1, 2) \]

\[ (f_1, (132)) (1, 1) = (f_2, (132)) (1, 1) = (f_3, (132)) (1, 1) \]

\[ = (f_4, (132)) (1, 1) = (1, 3) \]

\[ (f_5, (1)) (1, 1) = (f_6, (1)) (1, 1) = (f_7, (1)) (1, 1) \]

\[ = (f_8, (1)) (1, 1) = (2, 1) \]
\[
(f_5, (12)) (1, 1) = (f_6, (12)) (1, 1) = (f_7, (12)) (1, 1) = (f_8, (12)) (1, 1) = (2, 2)
\]
\[
(f_5, (13)) (1, 1) = (f_6, (13)) (1, 1) = (f_7, (13)) (1, 1) = (f_8, (13)) (1, 1) = (2, 3)
\]
\[
(f_5, (23)) (1, 1) = (f_6, (23)) (1, 1) = (f_7, (23)) (1, 1) = (f_8, (23)) (1, 1) = (2, 1)
\]
\[
(f_5, (123)) (1, 1) = (f_6, (123)) (1, 1) = (f_7, (123)) (1, 1) = (f_8, (123)) (1, 1) = (2, 2)
\]
\[
(f_5, (132)) (1, 1) = (f_6, (132)) (1, 1) = (f_7, (132)) (1, 1) = (f_8, (132)) (1, 1) = (2, 3)
\]
Then the orbit of \((1, 1)\) is
\[
\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} = \Gamma \times \Delta.
\]

6 Conclusions

In this paper, study the decomposition of groups, derive results on wreath product of groups, and provide several illustrative examples.

References


**Author’s address:**

Nacer Ghadbane
Laboratory of Pure and Applied Mathematics, Department of Mathematics
University of M’sila, BP 166 Ichebilia, 28000, M’sila, Algeria.
E-mail: nasser.ghedbane@univ-msila.dz