Statistically lacunary convergence of generalized difference sequences in probabilistic normed spaces

R. Haloi, M. Sen and B. C. Tripathy

Abstract. In this paper, we introduce the notions of μ-statistically lacunary convergence of generalized difference sequence in probabilistic normed spaces and investigate some characterizations. Furthermore, the notion of μ-statistically lacunary Cauchy for generalized difference sequences has been developed in the settings of probabilistic norm and investigated some of its properties.

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1 Introduction

In numerous branches of mathematics, it has been found much convenient to have a idea of distance that is applicable for the members of abstract sets. In context to this, Fréchet [11] introduced the metric space theory in 1906. In this theory, by associating a non-negative real number, he described the concept of distance between two elements of a set satisfying some conditions. But it is not always possible for associating such a unique number to each pair of members of the set. In such type of conditions, it is better to view the distance concept as a statistical instead of a determinate one. In this context, summing up the idea of metric space, Menger [19] presented the notion of statistical metric space, now called probabilistic metric space. Utilizing the idea of statistical metric, and summing up the concept of ordinary normed linear space, Šerstnev [33] presented the idea of probabilistic normed space (in short PN-space) in 1962, in which norm of a vector is expressed by distribution function instead of a positive number. Situations in which the usual norm is not been able to compute the length of a vector precisely, the idea of probabilistic norm [1, 16] happens to be valuable. The concept of statistical convergence was first introduced by Steinhaus [34] as well as by Fast [10] in 1951 and then studied by many authors [13, 27]. In 2007, Karakus [17] has given an analogous extension for the idea of statistical convergence into the probabilistic normed spaces. As an important generalization of statistical convergence [5, 22, 37], Fridy and Orhan [14, 15] presented the idea of lacunary statistical convergence in 1993, which was extended to the idea of probabilistic normed

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spaces by Rafi [26] in 2009. Further, this theory was studied by numerous authors [8, 20, 23, 24, 25, 31, 38] from different aspects. The idea of lacunary strong convergence was introduced by Freedman et al. [12] and investigated by other authors [2, 25, 35]. The concept of difference sequence was first proposed by Kizmaz [18] in 1981 and then in 1995, it was generalized by Et and Çolak [9] to termed as generalized difference sequence. Then Tripathy and Mahanta [36] have studied the concept of generalized difference sequence from lacunary sequence point of view and then the statistical analog of this notions has been examined by numerous authors [2, 21, 32] in different aspects.

An intriguing generalization to the theory of statistical convergence is to think about the idea of statistical convergence employing a complete two valued measure μ which is defined on a field of subsets of natural numbers as introduced by Connor [3, 4]. Some recent works in this field can be found in [6, 7, 28, 29]. As motivated by the literature, we feel that the study of lacunary statistical convergence of generalized difference sequence in PN-spaces using the two valued measure μ will provide a more general framework for the area. In context to that, we present the concept of μ-statistically lacunary convergence of generalized difference sequence in PN-spaces and investigate some results. Further, we introduce (Δ^n, μ)-statistically lacunary Cauchy sequences in PN-spaces and study some properties.

A brief sketch of the article is described as follows : Section 2 gives some preliminary definitions and examples which are going to be used during this investigation. We have defined μ-statistically lacunary convergence of generalized difference sequences in PN-spaces and discussed some of their properties in section 3. In section 4, we introduce the notion of (Δ^n, μ)-statistically lacunary Cauchy sequences in the framework of PN-spaces and investigate some characterizations.

2 Preliminaries

Throughout the article, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}^+$ denote the sets of non-negative real, real, natural numbers and non-negative integers, respectively.

Definition 2.1. [30] "A function $f : \mathbb{R}^+ \to [0, 1]$ is called a distribution function if it is non-decreasing, left-continuous with $\inf_{t\in\mathbb{R}^+} f(t) = 0$ and $\sup_{t\in\mathbb{R}^+} f(t) = 1$. Let D denotes the set of all distribution functions."

Definition 2.2. [30] "A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions, for all $a, b, c, d \in [0, 1]$:

(i) $a * 1 = a$,

(ii) $a * b = b * a$,

(iii) $a * b \le c * d$, whenever $a \le c$ and $b \le d$,

(iv) $(a * b) * c = a * (b * c)$."

Definition 2.3. [1] "A triplet $(Y, M, *)$ is called a probabilistic normed space (in short a PN-space) if Y is a real vector space, M a mapping from Y into D (for $y \in Y$, the distribution function $M(y)$ is denoted by M_y and $M_y(t)$ is the value of M_y at $t \in \mathbb{R}^+$) and $*$ a t-norm satisfying the following conditions:

(i) $M_y(0) = 0$,

(ii) $M_y(t) = 1$, for all $t > 0$ if and only if $y = 0$,

(iii) $M_{\alpha y}(t) = M_y\left(\frac{t}{|\alpha|}\right)$, for all $\alpha \in \mathbb{R} \setminus \{0\}$,

(iv) $M_{x+y}(s+t) \geq M_x(s) * M_y(t)$, for all $x, y \in Y$ and $s, t \in \mathbb{R}^+$."

Example 2.4. [17] "Let $(Y, ||.||)$ be a normed linear space. Let $a * b = \min\{a, b\}$, for all $a, b \in [0,1]$ and $M_y(t) = \frac{t}{t + ||y||}$, $y \in Y$ and $t \geq 0$. Then $(Y, M, *)$ is a PN-space."

Definition 2.5. [9] "For an integer $m \in \mathbb{Z}^+$, the generalized difference operator $\Delta^n x_i$ is defined as $\Delta^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i+1}$, where $\Delta^0 x_i = x_i$ and $\Delta x_i = x_i - x_{i+1}$, for all $i \in \mathbb{N}$."

With the help of above definition, we introduce the following three definitions.

Definition 2.6. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is Δ^n-convergent to $y_0 \in Y$ in terms of the probabilistic norm M, if for every $\lambda \in (0,1)$ and $\varepsilon > 0$, there is an $i_0 \in \mathbb{N}$ such that $M_{\Delta^n y_i - y_0}(\varepsilon) > 1 - \lambda$, whenever $i \geq i_0$. It is denoted by $M - \lim \Delta^n y = y_0$.

Definition 2.7. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is Δ^n-Cauchy sequence in terms of the probabilistic norm M, if for every $\lambda \in (0,1)$ and $\varepsilon > 0$, there is an $i_0 \in \mathbb{N}$ such that $M_{\Delta^n y_i - \Delta^n y_j}(\varepsilon) > 1 - \lambda$, for all $i, j \geq i_0$.

Definition 2.8. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is Δ^n-bounded in terms of the probabilistic norm N, if there exists $\lambda \in (0,1)$ and $\varepsilon > 0$ such that $M_{\Delta^n y_i}(\varepsilon) > 1 - \lambda$, for all i. We denote the collection of all Δ^n-bounded sequence in $(Y, M, *)$ by $\ell_\infty^M(\Delta^n)$.

Throughout the article, μ will mean a complete $\{0,1\}$-valued finitely additive measure defined on Γ, a field of all finite subsets of $\mathbb{N}$ and suppose that $\mu(P) = 0$, if $|P| < \infty$; if $P \subset Q$ and $\mu(Q) = 0$, then $\mu(P) = 0$; and $\mu(\mathbb{N}) = 1$.

Using the above notion of μ, we introduce the next two definitions in the theory of probabilistic normed space keeping in mind that these notions are going to be useful in the next section.

Definition 2.9. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is μ-statistically convergent to y_0 in terms of the probabilistic norm M, if for every $\lambda \in (0,1)$ and $\varepsilon > 0$,

$$\mu(\{i \in \mathbb{N} : M_{y_i - y_0}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

It is denoted by $\mu - stat_M - \lim y = y_0$.

Definition 2.10. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is μ-statistically Cauchy in terms of the probabilistic norm M, provided that for every $\lambda \in (0,1)$ and $\varepsilon > 0$, there is a integer $j \in \mathbb{N}$ satisfying

$$\mu(\{i \in \mathbb{N} : M_{y_i - y_j}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

Definition 2.11. [14] "An increasing sequence $\theta = \{k_r\}, \ r = 0, 1, 2, \ldots$ with $k_0 = 0$ of non-negative integers is said to be a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$ whenever $r \to \infty$. The intervals governed by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r."

In view of the above idea, we define the following notions in a PN-space.

Definition 2.12. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is lacunary convergent to y_0 in terms of the probabilistic norm M, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{i \in I_r} M_{y_i - y_0}(\varepsilon) > 1 - \lambda,$$

for all $r \geq r_0$. It is written as $M^\theta - \lim y = y_0$.

Definition 2.13. We say that a sequence $y = (y_i)$ in a PN-space $(Y, M, *)$ is Δ^n-lacunary convergent to y_0 in terms of the probabilistic norm M, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1 - \lambda,$$

for all $r \geq r_0$. It is written as $M^\theta - \lim \Delta^n y = y_0$.

Definition 2.14. Suppose that θ is a lacunary sequence. Then we say that $y = \{y_i\}$ in a PN-space $(Y, M, *)$ is μ-statistically lacunary convergent to y_0 in terms of the probabilistic norm M, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

It is denoted by $\mu_\theta - stat_M - \lim y = y_0$.

Definition 2.15. Suppose that θ is a lacunary sequence. Then we say that $y = \{y_i\}$ in a PN-space $(Y, M, *)$ is μ-statistically lacunary Cauchy in terms of the probabilistic norm M, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, there is a $j \in \mathbb{N}$ such that

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{y_i - y_j}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

3 μ-statistically lacunary convergence of generalized difference sequences in PN-spaces

In the current section, the idea of μ-statistically lacunary convergence of generalized difference sequences in PN-spaces has been introduced and studied some properties.

Definition 3.1. Suppose that θ is a lacunary sequence. Then we say that $y = \{y_i\}$ in a PN-space $(Y, M, *)$ is (Δ^n, μ)-statistically lacunary convergent to y_0 in terms of the probabilistic norm M, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

It is denoted by $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$.

In view of the Definition 3.1 and other properties of measure, we state the next result without proof.

Lemma 3.1. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then the following are equivalent for every $\lambda \in (0, 1)$ and $\varepsilon > 0$:*

(i) $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$,

(ii) $\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0$,

(iii) $\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1 - \lambda\right\}\right) = 1$,

(iv) $\mu_\theta - stat - \lim M_{\Delta^n y_i - y_0}(\varepsilon) = 1$.

Using Lemma 3.1, the next results can easily be proved. So we omit the proof.

Theorem 3.2. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. If (y_i) in Y is (Δ^n, μ)-statistically lacunary convergent in terms of the probabilistic norm M, then $\mu_\theta - stat_M -$limit is unique.*

Theorem 3.3. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. If $M^\theta - \lim \Delta^n y_i = y_0$, then $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$.*

The other way round of the Theorem 3.3 is not valid in general, which can be shown with the help of succeeding example.

Example 3.2. Suppose that $(\mathbb{R}, \| \cdot \|)$ be the space of all real numbers with standard norm. Let $p * q = pq$, for $p, q \in [0, 1]$ and $M_y(s) = \frac{s}{s + \| y \|}$, where $y \in \mathbb{R}$ and $s \geq 0$. Then we observe that $(\mathbb{R}, M, *)$ is a probabilistic normed space. Let $\theta = \{k_r\}$ be a lacunary sequence and $A = \{i \in \mathbb{N} : k_r - [\sqrt{h_r}] + 1 \leq i \leq k_r, r \in \mathbb{N}\} \subset \mathbb{N}$ be such that $\mu(A) = 0$. We now define $y = (y_i)$ whose elements are given as follows:

$$\Delta^n y_i = \begin{cases} i, & \text{if } k_r - [\sqrt{h_r}] + 1 \leq i \leq k_r,\ r \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Now, for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, let

$$A_r(\lambda, \varepsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i}(\varepsilon) \leq 1 - \lambda\right\}.$$

Then

$$\begin{aligned}
A_r(\lambda,\varepsilon) &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} \frac{\varepsilon}{\varepsilon + \parallel \Delta^n y_i \parallel} \leq 1-\lambda \right\} \\
&= \left\{ r \in \mathbb{N} : h_r \sum_{i\in I_r} \frac{\varepsilon + \parallel \Delta^n y_i \parallel}{\varepsilon} \geq \frac{1}{1-\lambda} \right\} \\
&= \left\{ r \in \mathbb{N} : \sum_{i\in I_r} \parallel \Delta^n y_i \parallel \geq \frac{1-h_r^2(1-\lambda)}{1-\lambda} \cdot \frac{\varepsilon}{h_r} > 0 \right\} \\
&= \{ r \in \mathbb{N} : \Delta^n y_i = i \} \\
&= \left\{ i \in \mathbb{N} : k_r - [\sqrt{h_r}] + 1 \leq i \leq k_r,\ r \in \mathbb{N} \right\}.
\end{aligned}$$

Thus $\mu(A_r(\lambda,\varepsilon)) = 0$ and consequently $\mu_\theta - stat_M - \lim \Delta^n y_i = 0$. On the other hand, the sequence $\{\Delta^n y_i\}$ is not lacunary convergent to 0 in terms of the probabilistic norm M since

$$\begin{aligned}
M_{\Delta^n y_i}(\varepsilon) &= \frac{\varepsilon}{\varepsilon + \parallel \Delta^n y_i \parallel} \\
&= \begin{cases} \frac{\varepsilon}{\varepsilon + \|i\|}, & \text{for } k_r - [\sqrt{h_r}] + 1 \leq i \leq k_r,\ r \in \mathbb{N} \\ 1, & \text{otherwise,} \end{cases}
\end{aligned}$$

and so $\frac{1}{h_r} \sum_{i\in I_r} M_{\Delta^n y_i}(\varepsilon) \leq 1$, which completes the rest of the proof.

Lemma 3.4. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then*

(a) If $\mu_\theta - stat_M - \lim \Delta^n x_i = x_0$ and $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$, then $\mu_\theta - stat_M - \lim \Delta^n (x_i + y_i) = x_0 + y_0$.

(b) If $\mu_\theta - stat_M - \lim \Delta^n x_i = x_0$ and $\alpha \in \mathbb{R}$, then $\mu_\theta - stat_M - \lim \Delta^n (\alpha x_i) = \alpha x_0$.

(c) If $\mu_\theta - stat_M - \lim \Delta^n x_i = x_0$ and $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$, then $\mu_\theta - stat_M - \lim \Delta^n (x_i - y_i) = x_0 - y_0$.

Theorem 3.5. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$ iff there is an increasing index sequence of natural numbers $P = \{i_k\}$ such that $\mu(P) = 1$ and $M^\theta - \lim \Delta^n y_{i_k} = y_0$.*

Proof. First we prove the necessary part. Let $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$. For any $\varepsilon > 0$ and $\gamma = 1, 2, \ldots$, we consider the succeeding two sets:

$$A_M(\gamma,\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1 - \frac{1}{\gamma} \right\},$$

$$B_M(\gamma,\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \frac{1}{\gamma} \right\}.$$

Then $\mu(B_M(\gamma,\varepsilon)) = 0$, by hypothesis. Also for $\varepsilon > 0$ and $\gamma \in \mathbb{N}$, we observe that

$$A_M(\gamma+1,\varepsilon) \subset A_M(\gamma,\varepsilon), \tag{3.1}$$

and

$$\mu(A_M(\gamma,\varepsilon)) = 1. \tag{3.2}$$

Now we need to show that $M^\theta - \lim \Delta^n y_{i_k} = y_0$, for any $r \in A_M(\gamma,\varepsilon)$. Suppose that $M^\theta - \lim \Delta^n y_i \neq y_0$, for some $r \in A_M(\gamma,\varepsilon)$. Then for all $r_0 \in \mathbb{N}$, there exists $\lambda \in (0,1)$ and $\varepsilon > 0$ such that

$$\frac{1}{h_r}\sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1-\lambda, \quad \text{for } r \geq r_0.$$

Suppose

$$\frac{1}{h_r}\sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1-\lambda, \quad \text{for } r < r_0.$$

Then

$$\mu\left(\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1-\lambda\right\}\right) = 0.$$

Since $\lambda > 1/\gamma$, so $\mu(A_M(\gamma,\varepsilon)) = 0$, which is a contradiction to (3.2). Thus we must have $M^\theta - \lim \Delta^n y_{i_k} = y_0$.

Conversely, suppose that there is an increasing index sequence $P = \{i_k\}$ of natural numbers with $\mu(P) = 1$ and $M^\theta - \lim \Delta^n y_{i_k} = y_0$. Then for every $\lambda \in (0,1)$ and $\varepsilon > 0$, there is $r_0 \in \mathbb{N}$ so that

$$\frac{1}{h_r}\sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1-\lambda, \quad \text{for all } r \geq r_0.$$

Now, we define the following set as

$$\begin{aligned} B_M(\lambda,\varepsilon) &= \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1-\lambda\right\} \\ &\subseteq \mathbb{N} - \{i_{k+1}, i_{k+2}, \dots\}. \end{aligned}$$

Then $\mu(B_M(\lambda,\varepsilon)) \leq 1-1 = 0$. Hence $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$. □

Theorem 3.6. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$ iff there is a sequence $x = \{x_i\}$ such that $M^\theta - \lim \Delta^n x_i = y_0$ and $\mu(\{i\in\mathbb{N} : \Delta^n x_i = \Delta^n y_i\}) = 1$.*

Proof. Suppose $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$. Then, by Theorem 3.5, we obtain an increasing index sequence $P = \{i_k\}$ of natural numbers so that $\mu(P) = 1$ and $M^\theta - \lim \Delta^n y_{i_k} = y_0$. Now we define x whose terms are given as

$$\Delta^n x_i = \begin{cases} \Delta^n y_i, & \text{if } i \in P \\ y_0, & \text{otherwise} \end{cases} \tag{3.3}$$

serves our purpose.

Conversely, suppose that $x = (x_i)$ and $y = (y_i)$ are two sequences so that $M^\theta - \lim \Delta^n x_i = y_0$ and $\mu(\{i \in \mathbb{N} : \Delta^n x_i = \Delta^n y_i\}) = 1$. Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n x_i - y_0}(\varepsilon) \leq 1 - \lambda\right\} \cup \{i \in \mathbb{N} : x_i \neq y_i\}.$$

Thus,

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) \leq \mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n x_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) + \mu\left(\{i \in \mathbb{N} : x_i \neq y_i\}\right).$$

Since $M^\theta - \lim \Delta^n x_i = y_0$, so the set

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n x_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}$$

contains at most finite numbers of terms. Thus we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n x_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

Also by hypothesis, $\mu(\{i \in \mathbb{N} : \Delta^n x_i \neq \Delta^n y_i\}) = 0$. Thus, we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0$$

and consequently, $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$. □

Theorem 3.7. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then $\mu_\theta - stat_M - \lim \Delta^n x_i = L$ iff there exist sequences $\{y_i\}$ and $\{z_i\}$ in Y such that $\Delta^n x_i = \Delta^n y_i + \Delta^n z_i$ for all $i \in \mathbb{N}$, where $M^\theta - \lim \Delta^n y_i = L$ and $\mu_\theta - stat_M - \lim \Delta^n z_i = 0$.*

Proof. Let $\mu_\theta - stat_M - \lim \Delta^n x_i = L$. Then by Theorem 3.5, there is an increasing index sequence $P = \{i_k\}$ of natural numbers such that $\mu(P) = 1$ and $M^\theta - \lim \Delta^n x_{i_k} = L$. We define $\{y_i\}$ and $\{z_i\}$ whose terms are given as follows:

$$\Delta^n y_i = \begin{cases} \Delta^n x_i, & \text{if } i \in P \\ L, & \text{otherwise,} \end{cases}$$

and

$$\Delta^n z_i = \begin{cases} 0, & \text{if } i \in P \\ \Delta^n x_i - L, & \text{otherwise.} \end{cases}$$

Then $\{y_i\}$ and $\{z_i\}$ serve our purpose.

Conversely, suppose that $\{y_i\}$ and $\{z_i\}$ are two sequences so that $\Delta^n x_i = \Delta^n y_i + \Delta^n z_i$ for all $i \in \mathbb{N}$, where $M^\theta - \lim \Delta^n y_i = L$ and $\mu_\theta - stat_M - \lim \Delta^n z_i = 0$. Then by Theorem 3.3, we have $\mu_\theta - stat_M - \lim \Delta^n y_i = L$. Also from Lemma 3.4(a), we have

$$\begin{aligned}\mu_\theta - stat_M - \lim \Delta^n x_i &= \mu_\theta - stat_M - \lim \Delta^n (y_i + z_i)\\ &= L + 0 = L.\end{aligned}$$

Hence the result. □

4 (Δ^n, μ)-statistically lacunary Cauchy sequences in PN-spaces

In this section, we develop the concepts of (Δ^n, μ)-statistically lacunary Cauchy sequences in PN-spaces and study some properties.

Definition 4.1. Suppose that $(Y, M, *)$ is a PN-space. Then a sequence $y = (y_n)$ in Y is (Δ^n, μ)-statistically lacunary Cauchy in terms of the probabilistic norm M, if there is a subsequence $\{y_{i(r)}\}$ with $i(r) \in I_r$, for each r such that $M - \lim_r \Delta^n y_{i(r)} = y_0$ and for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - \Delta^n y_{i(r)}}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

Theorem 4.1. *Suppose that θ is a lacunary sequence and let $(Y, M, *)$ be a PN-space. Then $y = \{y_i\} \in Y$ is (Δ^n, μ)-statistically lacunary convergent in terms of the probabilistic norm M iff it is (Δ, μ)-statistically lacunary Cauchy in terms of the probabilistic norm M.*

Proof. Suppose that $\mu_\theta - stat_M - \lim \Delta^n y_i = y_0$. For each j, let

$$K_j = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) > 1 - \frac{1}{j}\right\}.$$

Then for each j, $K_{j+1} \subseteq K_j$ and $\mu(K_j \cap I_r) = 1$. So there exists q_1 such that $q_1 \leq r$ implies $K_1 \cap I_r \neq \phi$. Again we choose $q_2 > q_1$ such that $q_2 \leq r$ gives $K_2 \cap I_r \neq \phi$. Then for each r with $q_1 \leq r \leq q_2$, we select $i(r) \in I_r$ so that $i(r) \in K_1 \cap I_r$. In general, we select $k_{j+1} > p_j$ so that $p_{j+1} < r$ with $i(r) \in K_j \cap I_r$. Therefore, $i(r) \in I_r$ for each r and

$$\frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_{i(r)} - y_0}(\varepsilon) > 1 - \frac{1}{j}.$$

Consequently, $M^\theta - \lim \Delta^n y_{i(r)} = y_0$. Then by Theorem 3.3 and Lemma 3.4(c) we obtain

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - \Delta^n y_{i(r)}}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

Conversely, we assume that $y = \{y_i\}$ be (Δ^n, μ)-statistically lacunary Cauchy in Y. For $\lambda > 0$, we select $\gamma \in (0,1)$ so that $(1-\gamma) * (1-\gamma) > 1-\lambda$. Then for any $\varepsilon > 0$, we define the following two sets:

$$K_{N,1} = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - \Delta^n y_{i(r)}}(\varepsilon/2) \le 1-\gamma \right\},$$

$$K_{N,2} = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_{i(r)} - y_0}(\varepsilon/2) \le 1-\gamma \right\}.$$

Let $K_N = K_{N,1} \cap K_{N,2}$. Then $\mu(K_N) = 1$. Now for $k \in K_N$,

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) &\ge \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - \Delta^n y_{i(r)}}(\varepsilon/2) * \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_{i(r)} - \Delta^n y_0}(\varepsilon/2) \\ &> (1-\gamma) * (1-\gamma) \\ &> 1-\lambda. \end{aligned}$$

Hence,

$$\mu\left(\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} M_{\Delta^n y_i - y_0}(\varepsilon) \le 1-\lambda \right\} \right) = 0.$$

Consequently, $y = \{y_i\}$ is a (Δ^n, μ)-statistically lacunary convergent in Y. □

Corollary 4.2. *Any (Δ^n, μ)-statistically lacunary convergent sequence in a PN-space $(Y, M, *)$ has a Δ^n-convergent subsequence in it.*

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Authors' addresses:

Rupam Haloi
Department of Mathematics,
National Institute of Technology Silchar, Silchar, 788010, Assam, India.
E-mail: rupam.haloi15@gmail.com

Mausumi Sen
Department of Mathematics,
National Institute of Technology Silchar, Silchar, 788010, Assam, India.
E-mail: senmausumi@gmail.com

Binod Chandra Tripathy
Department of Mathematics,
Tripura University, Agartala, 799022, Tripura, India.
E-mail: tripathybc@rediffmail.com , tripathybc@gmail.com