# Study of ring structure from multiset context

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**Abstract.** The concept of multiset is a generalization of the Cantor set. In this paper we have introduced the notion of multirings (in short mrings) and study some of their important properties. It is shown that the intersection of two mrings is again a mring but their union may not be a mring and a mring over a non-commutative ring may be commutative.

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## 1 Introduction

A multiset (mset) is an unordered collection of objects, unlike a standard Cantorian set, elements are allowed to repeat. It is observed from the survey of available literature on multiset and its application that the idea of multiset was hinted by R. Dedekind in 1988. The multiset theory which contains set theory as a special case was introduced by Cerf et al [5]. The term multiset, as Knuth [8] notes was first suggested by N.G. de Bruijn in a private communication to him. Further study was carried out by Peterson [10], Yager [13]. Blizard [3, 4]gave a new dimension to the multiset theory. From a practical point of view multiset are very useful structures arising in many areas of mathematics and computer science. The prime factorization of an integer $n > 0$ is an example of a multiset. The terminal string of a non-circular context-free grammer form a multiset which is a set if and only if the grammar is unambiguous.

Research on the multiset theory has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of multisets. It is possible to extend some of the main notion and results of sets to the setting of multisets. In 2009, Girish et al. [6] introduced the concepts of relation, function, composition and equivalence in multiset context.

Tella and Daniel [11] have considered set of mappings between multisets and studied about symmetric groups from multiset perspective. Nazmul et. al. [9] have considered the initial universe set to be a group. Then they have defined a group on the multiset derived from the initial universal set. Recently, these concepts have been extended by many authors [1, 2, 7, 12]. Therefore the study of ring structure

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in multisets context is very natural. In this paper we have introduced the notion of multirings (in short mrings) and study some of their important properties. We have shown that the intersection of two multirings is again a multiring but their union may not be a mring. Also shown that a multiring over a non-commutative ring may be commutative.

## 2 Definitions and Preliminaries

We procure the following definitions available in the existing literatures, those will be used in this article.

**Definition 2.1.** [6]: A collection of elements which are allowed to repeat is called a multiset. Formally if $X$ is a set of elements, a multiset $A$ drawn from the set $X$ is represented by a function $C_A$ defined by $C_A : X \to N$, where $N$ represents the set of non-negative integers.

For each $x \in X$, $C_A(x)$ is the characteristic value of $x$ in $A$ and indicates the number of occurrences of the element $x$ in $A$. A multiset $A$ is a set if $C_A(x) = 0$ or 1 for all $x \in X$.

The word "multiset", often written as "mset".

**Definition 2.2.** : Let $A$ be a mset. Then $A^*$ is called the root set of $A$ if for each $x \in A$ such that $C_A(x) > 0$ implies $x \in A^*$, and for every $x$ such that $C_A(x) = 0$ implies $x \notin A^*$.

i.e, the characteristic function of $A^*$, $C_{A^*}(x) = \begin{cases} 1, & if\ C_A(x) > 0 \\ 0, & if\ C_A(x) = 0 \end{cases}$.

i.e, $A^*$ is an ordinary set.

**Definition 2.3.** [6]: Let $A_1$ and $A_2$ be two msets drawn from a set $X$. $A_1$ is a sub-mset of $A_2$ ($A_1 \subseteq A_2$) if $C_{A_1}(x) \leq C_{A_2}(x)\ \forall x \in X$. $A_1$ is a proper sub-mset of $A_2$ ($A_1 \subset A_2$) if $C_{A_1}(x) \leq C_{A_2}(x)\ \forall x \in X$ and there exists at least one $x \in X$ such that $C_{A_1}(x) < C_{A_2}(x)$.

**Definition 2.4.** [6]: The cardinality of an mset $A$ drawn from a set $X$ is defined by card $A = \sum_{x \in X} C_A(x)$. It is also denoted by $|A|$.

**Definition 2.5.** [6]: The addition of two msets $A_1$ and $A_2$ drawn from a set $X$ is a mset $A$ denoted by $A = A_1 \oplus A_2$ such that $\forall x \in X$, $C_A(x) = C_{A_1}(x) + C_{A_2}(x)$.

**Definition 2.6.** [6]: The subtraction of two msets $A_1$ and $A_2$ drawn from a set $X$ is a mset $A$ denoted by $A = A_1 \ominus A_2$ such that $\forall x \in X$, $C_A(x) = Max\{C_{A_1}(x) - C_{A_2}(x), 0\}$.

**Definition 2.7.** [6]: The union of two msets $A_1$ and $A_2$ drawn from a set $X$ is a mset $A$ denoted by $A = A_1 \cup A_2$ such that $\forall x \in X$, $C_A(x) = max\{C_{A_1}(x), C_{A_2}(x)\}$.

**Definition 2.8.** [6]: The intersection of two msets $A_1$ and $A_2$ drawn from a set $X$ is a mset $A$ denoted by $A = A_1 \cap A_2$ such that $\forall x \in X$, $C_A(x) = min\{C_{A_1}(x), C_{A_2}(x)\}$.

**Remark 2.9.** Let $A$ be a mset from $X$ with $x$ appearing $n$ times in $A$. It is denoted by $x \in^n A$.

**Definition 2.10.** [6]: Let $A_1$ and $A_2$ be two msets drawn from a set $X$. Then the Cartesian product of $A_1$ and $A_2$ is defined by $A_1 \times A_2 = \{mn/(x, y) : x \in^m A_1, y \in^n A_2\}$.

**Definition 2.11.** [9]: Let $X$ be a group. A multiset $G$ over $X$ is said to be a multigroup over $X$ if the count function of $G$ i.e $C_G$ satisfies the following two conditions:

(i) $C_G(xy) \geq min\{C_G(x), C_G(y)\}$, $\forall x, y \in X$
(ii) $C_G(x^{-1}) \geq C_G(x), \forall x \in X$.

**Example 2.12.** [9]:Let $X = \{e, a, b, c\}$be Klein's 4-group and $G = \{e, e, e, a, a, b, b, b, c, c\}$ be a multiset over $X$. Then it can be easily shown that $G$ is a multigroup over $X$.

# 3 Main Results

Throughout the section, let $X$ be a ring and 0 be the additive identity of $X$. Also the rest of the paper we assume that msets are taken from $[X]^w$ =set of all msets whose elements are in $X$ such that no element in the mset occurs more than $w$ times.

**Definition 3.1.** : Let $X$ be a ring. A multiset $A$ over $X$ is said to be a multiring over $X$, if the count function of $A$ i.e $C_A$ satisfies the following conditions:
(i) $C_A(x + y) \geq min\{C_A(x), C_A(y)\}$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\}$, $\forall x, y \in X$.
(ii) $C_A(-x) \geq C_A(x)$, $\forall x \in X$.
The set of all multirings over $X$ is denoted by $MR(X)$.

**Example 3.2.** : Let $X = Z_4 = \{0, 1, 2, 3\}$ be a ring with respect to addition and multiplication under congruence modulo 4. Also consider $A = \{0, 0, 0; 1, 1; 2, 2, 2; 3, 3\}$ and $A_1 = \{0, 0, 0; 1, 1; 2, 2, 2, 2; 3, 3, 3\}$ be two multiset over $X$. Then it can be shown that $A$ is a multiring over $X$ but $A_1$ is not a multiring over $X$.

**Definition 3.3.** : Let $A$ be a multiring over $X$. Then $A$ is said to be commutative multirings iff $C_A(x.y) = C_A(y.x)$, $\forall x, y \in X$.

From the definition, it is obvious that a multiring over a commutative ring is always commutative.

**Remark 3.4.** : A multiring over a non-commutative ring may be commutative.

**Example 3.5.** : Let $X = M_2(Z_2)$ be a non-commutative ring. Then the multiset $A = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$ is a commutative multiring over $X$.

**Proposition 3.1.** *: Let $A \in MR(X)$, then*
*(i) $C_A(0) \geq C_A(x)$, $\forall x \in X$.*
*(ii) $C_A(nx) \geq C_A(x)$, $\forall x \in X$.*
*(iii) $C_A(-x) = C_A(x)$, $\forall x \in X$.*

*Proof.* Let $x, y \in X$.
(i) $C_A(0) = C_A(-x + x) \geq min\{C_A(-x), C_A(x)\} = C_A(x)$, Since $A \in MR(X)$.

(ii) $C_A(nx) = C_A(x + (n-1)x) \geq min\{C_A(x), C_A((n-1)x)\}$
$\geq min\{C_A(x), C_A(x + (n-2)x)\}$
$\geq$ ..................................................................
$\geq$ ..................................................................
$\geq min\{C_A(x), C_A(x), ............., C_A(x), C_A(x)\}$
$= C_A(x)$

(iii) $C_A(x) = C_A(0 + x) = C_A(0 + (-(-x)))$
$\geq min\{C_A(0), C_A(-(-x))\}$
$\geq C_A(-(-x)) \geq C_A(-x)$
i.e $C_A(x) \geq C_A(-x)$
Therefore, $C_A(x) = C_A(-x)$ □

**Proposition 3.2.** *: Let $A \in MR(X)$. If $C_A(x) < C_A(y)$ for some $x, y \in X$, then $C_A(x + y) = C_A(x)$.*

*Proof.* Let $C_A(x) < C_A(y)$ for some $x, y \in X$. Since $A \in MR(X)$ then $C_A(x + y) \geq min\{C_A(x), C_A(y)\} = C_A(x)$. Again $C_A(x) = C_A(x + y - y) \geq min\{C_A(x + y), C_A(y)\} = C_A(x + y)$. Therefore $C_A(x + y) = C_A(x)$ □

**Proposition 3.3.** *: Let $A \in MR(X)$. Then $C_A(x - y) = C_A(0)$ implies $C_A(x) = C_A(y)$.*

*Proof.* Let $A \in MR(X)$ and $C_A(x-y) = C_A(0)$ for $x, y \in X$. Then $C_A(x) = C_A((x-y)+y) \geq min\{C_A(x-y), C_A(y)\} = min\{C_A(0), C_A(y)\} = C_A(y)$, i.e $C_A(x) \geq C_A(y)$.
Again $C_A(y) = C_A(-y) = C_A(-x + (x - y)) \geq min\{C_A(-x), C_A(x - y)\} = min\{C_A(x), C_A(0)\} = C_A(x)$ i.e $C_A(y) \geq C_A(x)$. Hence $C_A(x) = C_A(y)$. □

**Proposition 3.4.** *: Let A be an m-set. Then $A \in MR(X)$, if and only if*
$C_A(x-y) \geq min\{C_A(x), C_A(y)\}$ *and* $C_A(x.y) \geq min\{C_A(x), C_A(y)\}$, $\forall x, y \in X$.

*Proof.* Let $A \in MR(X)$, then $C_A(x+(-y)) \geq min\{C_A(x), C_A(-y)\} = min\{C_A(x), C_A(y)\}$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\}$, $\forall x, y \in X$. Therefore the given condition is satisfied.

Conversely, let the given condition be satisfied. Then, $C_A(0) = C_A(x-x) \geq min\{C_A(x), C_A(x)\} = C_A(x)$, $\forall x \in X$....................(1)

So, $C_A(-x) = C_A(0-x)) \geq min\{C_A(0), C_A(x)\} = C_A(x)$, $\forall x \in X$....................(2) (using (1))

Also $C_A(x+y) = C_A(x+(-(-y))) \geq min\{C_A(x), C_A(-y)\} \geq min\{C_A(x), C_A(y)\}$, $\forall x, y \in X$....................(3)(using (2))

Therefore from (2) and (3) and last given condition we have $A \in MR(X)$. □

**Proposition 3.5.** *: Let $A \in MR(X)$, then $A_n, n \in N$ are subrings of X, where $A_n = \{x \in X : C_A(x) \geq n\}$, $n \in N$.*

*Proof.* Let $x, y \in A_n$. Then $C_A(x) \geq n$ and $C_A(y) \geq n$. Since $A \in MR(X)$, it follows that $C_A(x-y) \geq min\{C_A(x), C_A(y)\} \geq n$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\} \geq n$. Thus $x - y \in A_n$ and $x.y \in A_n$. Hence $A_n, n \in N$ are subrings of $X$. □

**Proposition 3.6.** *: If $A_n, n \in N$ are subrings of X, then the m-set A, defined in Theorem 2.9 ([9], page-645), is a mring over X.*

*Proof.* Let $x, y \in X$ and $C_A(x) = p$, $C_A(y) = q$. Then $x \in A_p$, $p = 1, 2, ....., p$, $x \notin A_{p+n}, n \in N$ and $y \in A_q$, $q = 1, 2, ....., q$, $y \notin A_{q+n}, n \in N$. Let $\min\{p, q\} = p$. Since $A_n, n \in N$ are subrings of $X$, we have $x - y \in A_p$ and $x.y \in A_p$, for $p = 1, 2, ....., p$. Hence $C_A(x-y) \geq p = min\{p, q\} = min\{C_A(x), C_A(y)\}$and $C_A(x.y) \geq p = min\{p, q\} = min\{C_A(x), C_A(y)\}$. Therefore $A \in MR(X)$. □

**Proposition 3.7.** *: Let $A \in MR(X)$. Then $A^* = \{x \in X : C_A(x) = C_A(0)\}$ and $A_* = \{x \in X : C_A(x) > 0\}$ are subrings of X.*

*Proof.* Let $x, y \in A^*$. Then $C_A(x) = C_A(y) = C_A(0)$.

Now $C_A(x-y) \geq min\{C_A(x), C_A(y)\} = min\{C_A(0), C_A(0)\} = C_A(0) \geq C_A(x-y)$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\} = min\{C_A(0), C_A(0)\} = C_A(0) \geq C_A(x.y)$.

Therefore $C_A(x-y) = C_A(0) = C_A(x.y)$.

Thus $x, y \in A^*$implies $x - y \in A^*$and $x.y \in A^*$. Therefore $A^*$ is a subring of $X$.

Again let $x, y \in A_*$. Then $C_A(x) > 0$ and $C_A(y) > 0$.

Now $C_A(x-y) \geq min\{C_A(x), C_A(y)\} \succ 0$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\} > 0$. Thus $x, y \in A_*$ implies $x - y \in A_*$ and $x.y \in A_*$.

Therefore $A_*$ is a subring of $X$. □

**Proposition 3.8.** *: Let $A, B \in MR(X)$, then $A \cap B \in MR(X)$.*

*Proof.* Let $A, B \in MR(X)$
then $C_A(x-y) \geq min\{C_A(x), C_A(y)\}$ and $C_A(x.y) \geq min\{C_A(x), C_A(y)\}$,
also $C_B(x-y) \geq min\{C_B(x), C_B(y)\}$ and $C_B(x.y) \geq min\{C_B(x), C_B(y)\}$

Now, $C_{A\cap B}(x-y) = min\{C_A(x-y), C_B(x-y)\} \geq min\{min\{C_A(x), C_A(y)\}, min\{C_B(x), C_B(y)\}\}$
$= min\{C_A(x), C_B(x), C_A(y), C_B(y)\}$
$= min\{min\{C_A(x), C_B(x)\}, min\{C_A(y), C_B(y)\}\}$
$= min\{C_{A\cap B}(x), C_{A\cap B}(y)\}$

and $C_{A\cap B}(x.y) = min\{C_A(x.y), C_B(x.y)\} \geq min\{min\{C_A(x), C_A(y)\}, min\{C_B(x), C_B(y)\}\}$
$= min\{C_A(x), C_B(x), C_A(y), C_B(y)\}$
$= min\{min\{C_A(x), C_B(x)\}, min\{C_A(y), C_B(y)\}\}$
$= min\{C_{A\cap B}(x), C_{A\cap B}(y)\}$ □

**Remark 3.6.** : If $\{A_i : i \in I\}$ be a family of multirings over a ring $X$, then their intersection $\bigcap_{i\in I} A_i$ is a multiring over $X$.

**Remark 3.7.** : If $A, B \in MR(X)$, then the following example shows that their union $A \cup B$ is not a multiring over $X$.

**Example 3.8.** : Let $X = Z_4 = \{0, 1, 2, 3\}$be a ring with respect to addition and multiplication modulo 4. Also consider $A = \{0, 0, 0; 1, 1\}$ and $B = \{0, 0; 2\}$ be two multirings over $X$. Clearly $A \cup B = \{0, 0, 0; 1, 1; 2\}$ and $C_{A\cup B}(3) = C_{A\cup B}(1 +_4 2) = 0 \ngeq \min\{C_{A\cup B}(1), C_{A\cup B}(2)\} = 1$. Therefore $A\cup B$ is not a multiring over $X$ in general.

**Proposition 3.9.** *: Let $A, B \in MR(X)$ such that $A \subseteq B$ or $B \subseteq A$ then $A \cup B \in MR(X)$.*

*Proof.* The proof is straightforward. □

**Definition 3.9.** : Let $A$ and $B$ be two mrings over a ring $X$. Then $A$ is said to be a sub mring of $B$ if $A \subseteq B$.

**Example 3.10.** : Let $X = Z_4 = \{0, 1, 2, 3\}$be a ring with respect to addition and multiplication modulo 4. Also consider $A = \{0, 0, 0; 1, 1\}$ and $B = \{0, 0, 0; 1, 1; 2, 2, 2; 3, 3\}$ be two multirings over $X$. Then clearly $A, B \in MR(X)$ and $A \subseteq B$. Therefore $A$ is a sub mring of $B$.

# 4 Conclusion

In this paper we have introduced the notion of mring for the first time and studied its important properties. The theory of mring may be very useful in many areas like coding theory, cryptography etc. In future we will study the deeper properties of mring such as ideal, isomorphism theorem etc.

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# References

[1] J. A. Awolola, P. A. Ejegwa, *On some algebraic properties of order of an element of a multigroup*. Quasigroups and Related System 25,(2017),21-26.

[2] J. A. Awolola, A. M. Ibrahim, *Some results on multigroups*. Quasigroups and Related System 24,(2016),169-177.

[3] W. D. Blizard, *Multiset Theory*. Notre Dame Journal of logic, 31,(1989),36-65.

[4] W. D. Blizard, *The development of multiset theory*. Modern Logic, 1(4),(1991),319-352.

[5] V. Cerf, E. Fernandez, K. Gostelow, S. Volausky, *Formal control and low properties of a model of computation*, Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, P-81,(1971).

[6] K. P. Girish, S. J. John, *Relations and functions in multiset context*. Information Sciences, 179,(2009),758-768.

[7] A. M. Ibrahim, P. A. Ejegwa, *Multigroup actions on multiset*. Ananals Fuzzy Math. Inform, 14(5),(2017),515-526.

[8] D. E. Knuth, *The Art of computer programming*. V-2 (Seminumerical Algorithms), 2nd ed. Addison-Wesley: Reading Mass(1981).

[9] S. K.Nazmul, P. Majumdar, S. K. Samanta, *On multisets and multigroups*. Anals of Fuzzy Math. Inform, 6(3),(2013),643-656.

[10] J. Peterson, *Computation sequence sets*. J. Computer Syst. Sci, 13(1),(1976),1-24.

[11] Y. Tella, S. Daniel, *A Study of Group Theory in the Context of Multiset Theory*. Int. J. Sci. Techno, 2(8),(2013),609-615.

[12] B. C. Tripathy, S. Debnath, D. Rakshit, *On Multiset Group*. Proyecciones J. Math., 37(3),(2018),479-489.

[13] R. R. Yager, *On the theory of bags*. Int. J. General Syst, 13,(1986),23-37.

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