Projective limits of local shift morphisms

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Abstract. We define the notion of projective limit of local shift morphisms of type (r,s) and endow the space of such mathematical objects with an adapted differential structure. The notion of shift Poisson tensor P on a Hilbert tower corresponds to such a morphism which is antisymmetric and whose Schouten bracket with itself $[P,P]$ vanishes. We illustrate this notion with the example of the famous KdV equation on the circle $\mathbb{S}^1$ for which one can associate a pair of such compatible Poisson tensors on the Hilbert tower $\left(H^n(\mathbb{S}^1)\right)_{n\in\mathbb{N}^*}$.

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1 Introduction

In Mathematical Physics, different frameworks exist in the litterature for interesting evolution equations (KdV, Burgers, ...). The notion of projective limits of shift Poisson tensors on Hilbert towers (whose set can be endowed with a Fréchet structure) introduced in this paper is a new framework for such equations.

The paper is organized as follows. Section 2 introduces the basic notions and results on projective limits of Banach spaces and an adapted notion of differentiability on such spaces. Section 3 introduces shift operators on direct limits of Banach spaces whose set is endowed with a Fréchet structure (Theorem 3.5). Section 4 is devoted to the notion of local shift morphism and is concerned with the smoothness of projective limits of such operators (Theorem 4.1). In section 5, we consider the particular case of Hilbert towers that appears as an adapted framework to describe some PDEs. Section 6 is devoted to the notion of shift Hilbert Poisson tensors P, corresponding to a projective limit of antisymmetric local shift morphisms defined on a Hilbert tower whose Schouten brackets $[P,P]$ vanishes. As a fundamental example, we consider the KdV equation on the circle $\mathbb{S}^1$ (cf. [KapMak]) for which there exists a pair of compatible shift Hilbert Poisson tensors on the projective limit of the Sobolev spaces $H^n\left(\mathbb{S}^1\right)$.

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2 Projective limits of Banach spaces and differentiability

In a lot of situations in global analysis and Mathematical Physics, the framework of Banach or Hilbert spaces is not adapted any more. In some cases, the projective limits of such spaces must be adopted. For such Fréchet spaces, the differentiation method proposed by J.A. Leslie fits well to the requirements of this geometrical situation.
We can remark that the convenient setting, defined by A. Frölicher and A. Kriegl (see [FroKri] and [KriMic]), could have been used. This framework is adapted to various structures (e.g. for convenient partial Poisson structures as defined in [Pel]).

2.1 Projective limits of topological spaces

Definition 2.1. $\left\{\left(X_i,\delta_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ is a projective sequence of topological spaces if we have the following properties:

(PSTS 1) For all $i\in\mathbb{N}$, X_i is a topological space;

(PSTS 2) For all i, $j\in\mathbb{N}$, such that $j\geq i$, $\delta_i^j : X_j\to X_i$ is a continuous mapping;

(PSTS 3) For all $i\in\mathbb{N}$, $\delta_i^i = Id_{X_i}$;

(PSTS 4) For all integers $i\leq j\leq k$, $\delta_i^j\circ\delta_j^k=\delta_i^k$.

Definition 2.2. An element $(x_i)_{i\in\mathbb{N}}$ of the product $\prod\limits_{i\in N} X_i$ is called a thread if for all $j\geq i$, $\delta_i^j\left(x_j\right)=x_i$.
The set $X=\varprojlim X_i$ of such elements, endowed with the finest topology for which all the projections $\delta_i : X\to X_i$ are continuous, is called projective limit of the sequence $\left\{\left(X_i,\delta_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i}$.

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i^j is open).

Definition 2.3. Let $\left\{\left(X_i,\delta_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ and $\left\{\left(Y_i,\gamma_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ be two projective systems whose respective projective limits are X and Y.
A sequence $(f_i)_{i\in\mathbb{N}}$ of continuous mappings $f_i : X_i\to Y_i$, satisfying for all $i,j\in\mathbb{N}$, $j\geq i$, the condition

$$\gamma_i^j\circ f_j = f_i\circ\delta_i^j$$

is called a projective system of mappings.

The projective limit of this sequence is the mapping

$$\begin{array}{cccc} f: & X & \to & Y \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & (f_i(x_i))_{i\in\mathbb{N}} \end{array}$$

The mapping f is continuous and is a homeomorphism if all the f_i are homeomorphisms (cf. [AbbMan]).

2.2 Differentiability

We first introduce the notion of differentiability *à la Leslie* between Hausdorff locally convex vector spaces E and F which corresponds to a particular case of the Gâteaux derivative. For full details, the reader is referred to [Les] and [DoGaVa]. Unlike the classical framework of Banach spaces, the derivative does not involve the space of continuous linear maps $\mathcal{L}(E,F)$ which has no reasonable structure.

Definition 2.4. Let E and F be two Hausdorff locally convex vector spaces and let U be an open subset of E. A continuous map $f : U \longrightarrow F$ is said to be differentiable at $x \in U$ if there exits a continuous linear map $Df_x : E \longrightarrow F$ such that

$$R(t,v) = \left| \begin{array}{cc} \dfrac{f(x+tv) - f(x) - Df_x(tv)}{t}, & t \neq 0 \\ 0, & t = 0 \end{array} \right.$$

is continuous at every $(0,v) \in \mathbb{R} \times F$. The map Df_x is called the derivative (or differential) of f at x.

The map is said to be differentiable if it is differentiable at every $x \in U$.

Note that, in this case, Df_x is uniquely determined.

Definition 2.5. A continuous map $f : U \longrightarrow F$ from an open subset U of a Hausdorff locally convex vector space E to a space of the same type F is called C^1-differentiable if it is differentiable at every $x \in U$, and if the derivative

$$\begin{array}{cccc} Df : & U \times E & \longrightarrow & F \\ & (x,v) & \mapsto & Df_x(v) \end{array}$$

is continuous.

The notion of C^n-differentiability ($n \geq 2$) can be defined by induction (cf. [DoGaVa], Definition 2.2.3) and allows to define the C^∞-differentiability *à la Leslie* which corresponds to the C^∞-differentiability in the ordinary case.

We then have the following properties:

(PDL 1) Every continuous linear map $f : E \longrightarrow F$ is Leslie C^∞ and $Df = F$;

(PDL 2) The differential at x satisfies the relation

$$Df_x(h) = \lim_{t \longrightarrow 0} \frac{f(x+th) - f(x)}{t}$$

(PDL 3) The chain rules holds.

2.3 Differentiability on projective limits

The connection between projective limits of maps and differentiation is given by the following result ([DoGaVa], Propositions 2.3.11 and 2.3.12).

Proposition 2.1. *Let* $\mathbb{F}_1 = \varprojlim \mathbb{E}_1^i$ *and* $\mathbb{F}_2 = \varprojlim \mathbb{E}_2^i$ *projective limits of Banach spaces. Let also* $f^i : U^i \longrightarrow \mathbb{E}_2^i$ *be where, for all* $i \in \mathbb{N}$, U^i *is an open set of* $\mathbb{E}_1^i$. *We assume that* $U = \varprojlim U^i$ *exists and is a non empty open subset of* $\mathbb{F}_1$*; we also assume that* $f = \varprojlim f^i : U \longrightarrow \mathbb{F}_2$ *exists. Then we have:*

If each f^i *is differentiable (resp. smooth), then so is* f *and*

$$\forall x = \left(x^i\right) \in U,\ Df_x = \varprojlim Df_{x^i}.$$

3 Shift operators

In Analysis and Mathematical Physics, Banach representations break down. By weakening the topological requirement, replacing the norm by a sequence of semi-norms, one gets the notion of Fréchet space. For the subsections 3.1 (resp. 3.2), the reader is referred to [Bour], [RobRob] and [Tre] (resp. [DoGaVa]).

3.1 Fréchet spaces

Definition 3.1. A Fréchet space is a Hausdorff, locally convex topological vector space that is metrizable and complete.

The topology of a Fréchet space $\mathbb{F}$ can be induced by a sequence of semi-norms $(\nu_n)_{n\in\mathbb{N}}$ that is complete with respect to such a sequence.

Recall that $\mathbb{F}$ is complete with respect to this topology if and only if every sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{F}$ is such that

$$\forall n \in \mathbb{N}, \forall \varepsilon > 0, \exists i_\varepsilon \in \mathbb{N} : \forall (j,k) \in \mathbb{N}^2, k \geq j \geq i_\varepsilon, \nu_n (x_k - x_j) < \varepsilon$$

converges in $\mathbb{F}$ where the convergence in this Fréchet space is controlled by all the semi-norms ν_n:

$$\lim_{i \longrightarrow +\infty} x_i = x \quad \Longleftrightarrow \quad \forall n \in \mathbb{N}, \lim_{i \longrightarrow +\infty} \nu_n (x_i - x) = 0$$

Example 3.2. The space of real sequences $\mathbb{R}^{\mathbb{N}} = \prod\limits_{n\in\mathbb{N}} \mathbb{R}^n$ endowed with the usual topology is a Fréchet space where the corresponding sequence of semi-norms is given by

$$\nu_n \left((x_i)_{i\in\mathbb{N}}\right) = \sum_{k=0}^{n} |x_k|$$

Metrizability is defined from d as follows

$$d(x,y) = \sum_{k=0}^{+\infty} \frac{|y_k - x_k|}{2^k (1 + |y_k - x_k|)}$$

and the completeness is inherited from that of each $\mathbb{R}$ of the infinite product.

The notion of Fréchet space is closely related with the projective limit of Banach spaces.

If $\{(\mathbb{B}_n, \| \ \|_n)\}_{n\in\mathbb{N}}$ is a projective sequence of Banach spaces, then $\varprojlim \mathbb{B}_n$ is a Fréchet space (cf. [DoGaVa], Theorem 2.3.7) where the sequence $(\nu_n)_{n\in\mathbb{N}}$ of semi-norms is given by

$$\forall x = (x_n)_{n\in\mathbb{N}} \in \varprojlim \mathbb{B}_n,\ \nu_n(x) = \sum_{i=0}^{n} \|x_n\|_n$$

Conversely, if $\mathbb{F}$ is a Fréchet space with associated semi-norms ν_n, the completion $\mathbb{F}_n$ of the normed space $\mathbb{F}/\ker \nu_n$ is a Banach space called the *local Banach space associated to the semi-norm* ν_n. It will be denoted by $(\mathbb{F}_n, \| \ \|_n)$ where $\| \ \|_n$ is the norm associated to ν_n. We then get a projective system $\left\{\left(\mathbb{F}_i, \pi_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ of Banach spaces whose bonding maps are

$$\begin{array}{cccc} \pi_i^j : & \mathbb{F}_j & \longrightarrow & \mathbb{F}_i \\ & [x + \ker \nu_j]_j & \longmapsto & [x + \ker \nu_i]_i \end{array}$$

where the bracket $[\ ]_n$ corresponds to the associated equivalence class. $\mathbb{F}$ will be identified with the projective limit $\varprojlim \mathbb{F}_i$ (cf. [DoGaVa], Theorem 2.3.8).

The representation of Fréchet spaces as projective limits of Banach spaces is very interesting: Issues arising in the Fréchet framework can be solved by considering their components in the Banach factors of the associated projective sequence. So different pathological entities in the Fréchet framework can be replaced by approximations compatible with the inverse limits, e.g. ILB-Lie groups ([Omo]) or projective limits of Banach Lie groups ([Gal1]), manifolds ([AbbMan]), bundles ([Gal2], [AghSur]), algebroids ([Cab]), connections and differential equations ([ADGS]).

3.2 The Fréchet space $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$

Let $\mathbb{F}_1$ (resp. $\mathbb{F}_2$) be a Fréchet space and let $(\nu_1^n)_{n\in\mathbb{N}}$ (resp. $(\nu_2^n)_{n\in\mathbb{N}}$) be the sequence of semi-norms of $\mathbb{F}_1$ (resp. $\mathbb{F}_2$).

Recall ([Vog], 2.) that a linear map $L : \mathbb{F}_1 \longrightarrow \mathbb{F}_2$ is *continuous* if

$$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in \mathbb{F}_1, \nu_2^n(L.x) \leq C_n \nu_1^{k_n}(x)$$

The space $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms $\{p_{n,B}\}$:

$$p_{n,B}(L) = \sup\{\nu_2^n(L.x), x \in B\}$$

where $n \in \mathbb{N}$ and B is any bounded subset of $\mathbb{F}_1$ containing $0_{\mathbb{F}_1}$. This topology is not metrizable since the family $\{p_{n,B}\}$ is not countable.

So $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [Gal2].

If we denote by $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$ the space of linear continuous maps (or equivalently bounded linear maps because $\mathbb{B}_1^n$ and $\mathbb{B}_2^n$ are normed spaces), we then have the following result ([DoGaVa], Theorem 2.3.10).

Theorem 3.1. *The space of all continuous linear maps between* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *which can be represented as projective limits*

$$\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n) : \varprojlim L_n \text{ exists} \right\}$$

is a Fréchet space.

For this sequence (L_n) of linear maps, for any integer $0 \leq i \leq j$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{B}_1^i & \overset{{}_1\delta_i^j}{\longleftarrow} & \mathbb{B}_1^j \\ L_i \downarrow & & \downarrow L_j \\ \mathbb{B}_2^i & \overset{{}_2\delta_i^j}{\longleftarrow} & \mathbb{B}_2^j \end{array}$$

3.3 Shift operators

We assume that $\mathbb{F}_1 = \varprojlim \mathbb{B}_1^n$ (resp. $\mathbb{F}_2 = \varprojlim \mathbb{B}_2^n$) is a Fréchet space where $\left\{ \left(\mathbb{B}_1^i, {}_1\delta_i^j \right), \| \; \|_1^i \right\}_{(i,j) \in \mathbb{N}^2, \; j \geq i}$ (resp. $\left\{ \left(\mathbb{B}_2^i, {}_2\delta_i^j \right), \| \; \|_2^i \right\}_{(i,j) \in \mathbb{N}^2, \; j \geq i}$) is a projective sequence of Banach spaces.

Definition 3.3. A linear map $L : \mathbb{B}_1^{n+r} \longrightarrow \mathbb{B}_2^{n-s}$ is called a shift operator of base n and type $(r, s) \in \mathbb{N} \times \mathbb{N}$ where $n \geq s$, if there exists $C_n > 0$ such that:

$$\forall x \in \mathbb{B}_1^{n+r}, \; \|L.x\|_2^{n-s} \leq C_n \|x\|_1^{n+r}$$

$\mathcal{L}_n^{r,s}(\mathbb{F}_1, \mathbb{F}_2)$ denotes the set of shift operators of base n and type (r, s).

Lemma 3.2. $\mathcal{L}_n^{r,s}(\mathbb{F}_1, \mathbb{F}_2)$ *endowed with the norm* $\| \; \|_{L_n^{r,s}}$ *defined by*

$$\|L\|_{L_n^{r,s}} = \sup_{\|x\|_1^{n+r}} \|L.x\|_2^{n-s}$$

is a Banach space.

A linear operator of base n and type (r, s) is continuous.

Example 3.4. ([Ham], 1.1.2, Examples (4) and 1.2.3 Examples (3)). Let X be a compact manifold. Then $C^\infty(X)$ is a Fréchet space and for any linear partial differential operator L of degree r, we have $\|L.f\|_n \leq \|f\|_{n+r}$; So L is a shift operator of base n and type $(r, 0)$ (tame operator in Hamilton's terminology).

3.4 Projective limit of shift operators

Lemma 3.3. *For any integer* $n \geq s$*, the following set*

$$\mathcal{L}_{s,n}^{r,s}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ \begin{array}{c} (L_s, \ldots, L_n) \in \mathcal{L}_s^{r,s}(\mathbb{F}_1, \mathbb{F}_2) \times \cdots \times \mathcal{L}_n^{r,s}(\mathbb{F}_1, \mathbb{F}_2) : \\ \forall (i,j) \in \mathbb{N}^2 : n \geq j \geq i \geq s, {}_2\delta_{i-s}^{j-s} \circ L_j = L_i \circ {}_1\delta_{i+r}^{j+r} \end{array} \right\}$$

can be endowed with a structure of Banach space relatively to the norm $\| \ \|_{s,n}^{r,s}$ defined by

$$\|(L_s,\ldots,L_n)\|_{s,n}^{r,s}=\sum_{i=s}^{n}\|L_i\|_{L_i^{r,s}}$$

Proof. Since $\mathcal{L}_{s,n}^{r,s}(\mathbb{F}_1,\mathbb{F}_2)$ is a closed subspace of the Banach space $\mathcal{L}_s^{r,s}(\mathbb{F}_1,\mathbb{F}_2)\times\cdots\times\mathcal{L}_n^{r,s}(\mathbb{F}_1,\mathbb{F}_2)$, it is also a Banach space. □

Lemma 3.4. *For $j\geq i\geq s$, the canonical projections*

$$\begin{array}{cccc}\pi_i^j: & \mathcal{L}_{s,j}^{r,s}(\mathbb{F}_1,\mathbb{F}_2) & \longrightarrow & \mathcal{L}_{s,i}^{r,s}(\mathbb{F}_1,\mathbb{F}_2)\\ & (L_s,\ldots,L_j) & \longmapsto & (L_s,\ldots,L_i)\end{array}$$

are linear and continuous.

Proof. For $j\geq i\geq s$, the linearity of π_i^j is obvious.
The continuity of π_i^j is a consequence of

$$\begin{aligned}\left\|\pi_i^j(L_s,\ldots,L_j)\right\|_{s,i}^{r,s} &= \|(L_s,\ldots,L_i)\|_{s,i}^{r,s}\\ &= \sum_{k=s}^{i}\|L_k\|_{L_k^{r,s}}\\ &\leq \sum_{k=s}^{j}\|L_k\|_{L_k^{r,s}}\\ &= \|(L_s,\ldots,L_j)\|_{s,j}^{r,s}\end{aligned}$$

□

We then have the following result.

Theorem 3.5. $\left\{\left(\mathcal{L}_{s,i}^{r,s}(\mathbb{F}_1,\mathbb{F}_2),\pi_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i\geq s}$ *is a projective sequence of Banach spaces whose projective limit $\mathcal{L}^{r,s}(\mathbb{F}_1,\mathbb{F}_2)$ can be endowed with a Fréchet structure.*

Proof. For $k\geq j\geq i\geq s$, it is obvious that $\pi_i^k=\pi_i^j\circ\pi_j^k$. Thus, according to Lemma 3.3 and Lemma 3.4, $\left\{\left(\mathcal{L}_{s,i}^{r,s}(\mathbb{F}_1,\mathbb{F}_2),\pi_i^j\right)\right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i\geq s}$ is a projective sequence of Banach spaces. So its projective limit can be endowed with a structure of Fréchet space (cf. 3.1). □

3.5 Inductive dual

Because the dual of a Fréchet space generally drops out of the Fréchet category, it will be replaced by the inductive dual which is defined as a projective limit of Banach spaces.

Let $\mathbb{F}$ be a graded Fréchet space and let $(\mathbb{F}_n)_{n\in\mathbb{N}}$ be the sequence of associated Banach spaces. We then consider, for $n\in\mathbb{N}$, the following space

$$\mathbb{F}_n^0=\left\{\widehat{\omega_n}=(\omega_0,\ldots\omega_n)\in\prod_{i=0}^{n}\mathbb{F}_i^{'}\right\}$$

where $\mathbb{F}_i^{'}$ is the topological dual of the Banach space $\mathbb{F}_i$. Then $\mathbb{F}_n^0$ is a Banach space for the norm $\| \ \|^n$ defined by

$$\|\widehat{\omega_n}\|^n = \sum_{i=0}^{n} \max_{\|x_i\|_i = 1} |\omega_i(x_i)|$$

Definition 3.5. The projectif limit of the sequence $\left\{ \left(\mathbb{F}_n^0, \Pi_n^{n+1} \right) \right\}_{n \in \mathbb{N}^*}$, where $\Pi_n^{n+1} : \mathbb{F}_{n+1}^0 \longrightarrow \mathbb{F}_n^0$ is the natural projection, is called the inductive dual of $\mathbb{F}$ et denoted by $\mathbb{F}^0$.

The inductive dual $\mathbb{F}^0$ is a graded Fréchet space.

The *inductive cotangent bundle* $T^0\mathbb{F}$ is defined as the trivial bundle of base $\mathbb{F}$ and fiber $\mathbb{F}^0$ and appears as as the projective limit of $\left(\mathbb{F}_n \times \mathbb{F}_n^0, \| \ \|_n + \| \ \|^n \right)$. An *inductive differential form* is a smooth section of this bundle.

4 Projective sequence of local shift morphisms

4.1 Local shift morphisms

Let $\mathbb{F}_1$ (resp. $\mathbb{F}_2, \mathbb{F}_3$) be a graded Fréchet space.
Let $\left(\mathbb{F}_1^n, \| \ \|_n^1 \right)_{n \in \mathbb{N}}$ (resp. $\left(\mathbb{F}_2^n, \| \ \|_n^2 \right)_{n \in \mathbb{N}}$, $\left(\mathbb{F}_3^n, \| \ \|_n^3 \right)_{n \in \mathbb{N}}$) be the sequence of associated local Banach spaces.

Definition 4.1. Let $n \in \mathbb{N}$ such that $n - s \geq 0$. A smooth map

$$\varphi : U_n \longrightarrow \mathcal{L}\left(\mathbb{F}_2^{n+r}, \mathbb{F}_3^{n-s} \right)$$

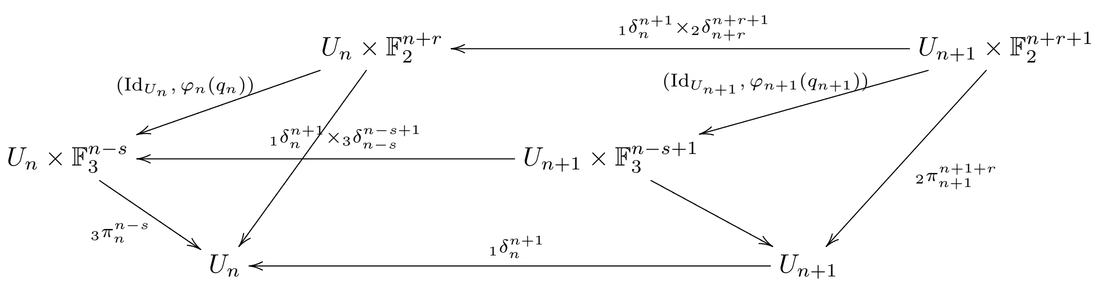
where U_n is an open set of $\mathbb{F}_1^n$, is called a local shift morphism of base n and type $(r, s) \in \mathbb{N} \times \mathbb{N}$ above U_n.

4.2 Projective sequence of local shift morphisms

Definition 4.2. A sequence $(\varphi_n)_{n \in \mathbb{N},\ n \geq s}$ of local shift morphisms φ_n of type $(r, s) \in \mathbb{N} \times \mathbb{N}$ above U_n is said to be a projective sequence of local shift morphisms if

(PSLSM 1) $U_s \supset U_{s+1} \supset \cdots \supset U_n \supset U_{n+1} \supset \cdots$ and $U = \bigcap_{n=s}^{+\infty} U_n$ is a non empty open set of $\mathbb{F}_1$;

(PSLSM 2) For any $q = (q_n)_{n \in \mathbb{N}} \in U$, we have the following commutative diagram:



Theorem 4.1. *The projective limit* $\varprojlim \varphi_n$ *of a projective sequence of local shift morphisms* φ_n *of type* $(r,s) \in \mathbb{N} \times \mathbb{N}$ *above* U_n *is a smooth map from the open set* $U = \bigcap\limits_{n=s}^{+\infty} U_n$ *of the Fréchet space* $\mathbb{F}_1$ *to the Fréchet space* $\mathcal{L}^{r,s}(\mathbb{F}_2, \mathbb{F}_3)$*.*

Proof. Since $\mathcal{L}^{r,s}(\mathbb{F}_2, \mathbb{F}_3)$ is the projective limit of the Banach spaces $\mathcal{L}_n^{r,s}(\mathbb{F}_2, \mathbb{F}_3)$ (cf. Theorem 3.5) the smoothness of $\varprojlim \varphi_n$ results from the smoothness of the maps φ_n and the Proposition 2.1. □

5 Hilbert towers

In this section, we consider the particular case where the Fréchet spaces $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ are all equal to a same projective limit of Hilbert spaces.

5.1 Definition. Example

In this subsection and the following one, the reader is referred to [KapMak].

Definition 5.1. The sequence $(H_n)_{n\in\mathbb{N}}$ is a Hilbert tower if

(HT 1) $(H_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Hilbert spaces: $H_0 \supset H_1 \supset \cdots$;

(HT 2) $\forall n \in \mathbb{N}$, $\overline{H_{n+1}} = H_n$;

(HT 3) There exists a basis of $H_\infty = \bigcap\limits_{n\in\mathbb{N}} H_n$, i.e. an orthonormal basis $(e_m)_{m\in\mathbb{N}}$ of H_0, where $e_m \in H_\infty$, such that $(e_m)_{m\in\mathbb{N}}$ is a basis of any H_N (with $N \in \mathbb{N}$).

A Hilbert tower can be seen as an IHL space as defined in [Omo].

Example 5.2. The sequence of Sobolev spaces $\left(H^n\left(\mathbb{S}^1\right)\right)_{n\in\mathbb{N}}$ where

$$H^n\left(\mathbb{S}^1\right) = \left\{q \in L^2\left(\mathbb{S}^1\right) : \forall k \in \{0,\dots,n\}, q^{(k)} \in L^2\left(\mathbb{S}^1\right)\right\}$$

is a Hilbert tower where the orthonormal basis is $(e_0, e_1, e_{-1}, \dots, e_k, e_{-k}, \dots)$, $(k \in \mathbb{N})$ where $e_k : x \mapsto e^{i2k\pi x}$.

Let $(H_n)_{n\in\mathbb{N}}$ be a Hilbert tower where $\iota_n^{n+1} : H_{n+1} \longrightarrow H_n$ is the natural injection and let us denote $\langle.,.\rangle_n$ the inner product of H_n and $\|\ \|_{H_n}$ the associated norm.

The projective limit H_∞ of the Hilbert tower $(H_n)_{n\in\mathbb{N}}$ is perfectly defined and can be endowed with a structure of Fréchet space.

5.2 Local shift Hilbert morphisms

In the sequel, we reformulate some of the precedent results in the particular case of a Hilbert tower $(H_n)_{n\in\mathbb{N}}$, that is for all $n \in \mathbb{N}, \mathbb{F}_1^n = \mathbb{F}_2^n = \mathbb{F}_2^n = H_n$, where the norm $\|\ \|_1^n = \|\ \|_2^n = \|\ \|_3^n = \sqrt{\langle.,.\rangle_n}$ are associated to the inner product of H_n.

Definition 5.3. A local shift Hilbert morphism of base n and type (r,s) is a smooth map

$$\varphi_n : U_n \longrightarrow \mathcal{L}\left(H_{n+r}, H_{n-s}\right)$$

where U_n is an open set of H_n.

Example 5.4. On the Sobolev tower $\left(H_n = H^n\left(\mathbb{S}^1\right)\right)_{n\in\mathbb{N}}$ (cf. Example 5.2), we consider the operator

$$\begin{array}{cccc} \partial_x : & U \cap H_n & \longrightarrow & \mathcal{L}\left(H_{n+1}, H_n\right) \\ & q & \longmapsto & \left(\partial_x\right)_q \end{array}$$

which corresponds to the first Poisson structure for the KdV equation (cf. Example 6.2.) where $U = H_0 = H^0\left(\mathbb{S}^1\right)$ and

$$\begin{array}{cccc} \left(\partial_x\right)_q : & H_{n+1} & \longrightarrow & H_n \\ & u & \longmapsto & \partial_x u \end{array}.$$

So ∂_x is a local shift Hilbert morphism of type $(1,0)$ above any $H_n = H^n\left(\mathbb{S}^1\right)$.

Example 5.5. On the Sobolev tower $\left(H^n\left(\mathbb{S}^1\right)\right)_{n\in\mathbb{N}}$, the operator

$$\begin{array}{cccc} L_n : & U \cap H_n & \longrightarrow & \mathcal{L}\left(H_{n+2}, H_{n-1}\right) \\ & q & \longmapsto & \left(L_n\right)_q \end{array}$$

corresponds to the second Poisson structure for the KdV equation where $U = H_0 = H^0\left(\mathbb{S}^1\right)$ and

$$\begin{array}{cccc} \left(L_n\right)_q : & H_{n+2} & \longrightarrow & H_{n-1} \\ & u & \longmapsto & -\frac{1}{2}\partial_x^3 u + q.\partial_x u + \partial_x q.u \end{array}.$$

L_n is then a local shift morphism of type $(2,1)$ above any $H_n = H^n\left(\mathbb{S}^1\right)$.

In particular, we have, for $q \in H^n\left(\mathbb{S}^1\right)$,

$$\left(L_n\right)_q \in \mathcal{L}\left(H^{n+2}\left(\mathbb{S}^1\right), H^{n-1}\left(\mathbb{S}^1\right)\right)$$

because

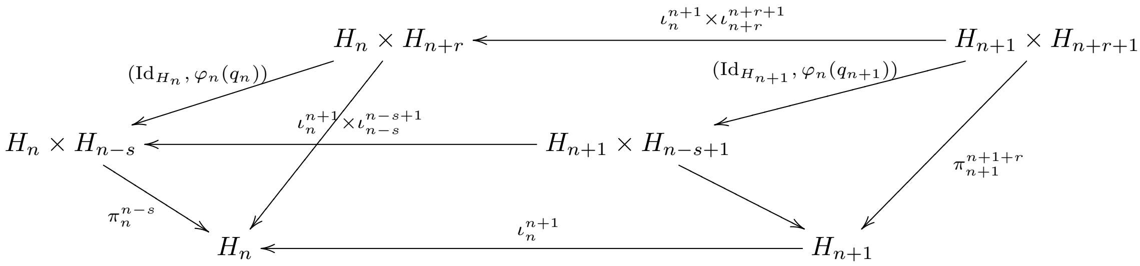
$$\forall u \in H^{n+2}\left(\mathbb{S}^1\right), \ \left\|\left(L_n\right)_q(u)\right\|_{n-1} \leq c_n \|u\|_{n+2}$$

where the norm $\| \ \|_n$ is given by

$$\|v\|_n = \sqrt{\sum_{k=0}^{n} \int_{\mathbb{S}^1} \left[\left(\partial_x^k v\right)(x)\right]^2 dx}.$$

5.3 Projective limits of local shift Hilbert morphisms

Definition 5.6. Let $(H_n)_{n\in\mathbb{N}}$ be a Hilbert tower. A sequence $(\varphi_n)_{n\in\mathbb{N},\, n\geq s}$ of local shift morphisms φ_n of type $(r,s) \in \mathbb{N}\times\mathbb{N}$ above H_n is said to be a projective sequence of local shift Hilbert morphisms if, for any $q = (q_n) \in \prod\limits_{n \in N} H_n$, we have the following commutative diagram:



Let $(H_n)_{n\in\mathbb{N}}$ be a Hilbert tower and consider $H_\infty = \bigcap\limits_{n\in\mathbb{N}} H_n = \varprojlim H_n$. For $n \geq s$, the space

$$\mathcal{H}_{s,n}^{r,s}(H_\infty) = \left\{ \begin{array}{c} (L_s, \dots, L_n) \in \prod\limits_{i=s}^{n} \mathcal{L}(H_{i+r}, H_{i-s}) : \\ \forall (i,j) \in \mathbb{N}^2 : n \geq j \geq i \geq s, \iota_{i-s}^{j-s} \circ L_j = L_i \circ \iota_{i+r}^{j+r} \end{array} \right\}$$

is a Banach space. We then get a projective sequence $\left\{ \left(\mathcal{H}_{s,i}^{r,s}(H_\infty), \pi_i^j \right) \right\}_{(i,j)\in\mathbb{N}^2,\ j\geq i\geq s}$ where

$$\pi_i^j : (L_s, \dots, L_j) \mapsto (L_s, \dots, L_i).$$

Its projective limit $\mathcal{H}^{r,s}(H_\infty)$ can be endowed with a structure of Fréchet space

For a projective sequence of local shift Hilbert morphisms $(\varphi_n)_{n\in\mathbb{N},\ n\geq s}$ of type (r,s), we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_{s,i}^{r,s}(H_\infty) & \overset{\pi_i^j}{\longleftarrow} & \mathcal{H}_{s,j}^{r,s}(H_\infty) \\ (\varphi_s, \dots, \varphi_i) \uparrow & & \uparrow (\varphi_s, \cdots, \varphi_j) \\ U \cap H_s \times \cdots \times U \cap H_i & \overset{p_i^j}{\longleftarrow} & U \cap H_s \times \cdots \times U \cap H_j \end{array}$$

where the maps $(\varphi_s, \dots, \varphi_n) : U \cap H_s \times \cdots \times U \cap H_n \longrightarrow \mathcal{H}_{s,n}^{r,-s}(H_\infty)$ are smooth.

We can define the projective limit

$$\varphi = \varprojlim (\varphi_s, \dots, \varphi_n) : U \cap H_\infty \longrightarrow \mathcal{H}^{r,s}(H_\infty)$$

and this limit is smooth.

Example 5.7. The sequence $(L_n)_{n\in\mathbb{N}}$ of Example 5.5 is a projective sequence of local shift morphisms of type $(2,1)$.

6 Shift Hilbert Poisson tensors

The notion of Poisson tensor is relevant in Mechanics and Mathematical Physics. It corresponds to a tensor field P twice contravariant whose Schouten bracket with itself $[P,P]$ vanishes. Bihamiltonian structures corresponding to a pair of compatible

Poisson tensors is a fundamental tool in the resolution of some dynamical systems because the recursion operator linking both structures gives rise to a hierarchy of conservation laws.

In the framework of Hilbert towers, thanks to the identification of a Hilbert space with its dual (Riesz Theorem), the morphism P from the cotangent bundle to the tangent bundle can be seen as a projective limit of local shift Hilbert morphisms. Such objects are adapted to the description of different evolution equations such as the KdV equation on the circle $\mathbb{S}^1$.

We adapt the notion of Poisson tensor of type (r,s) given in [KapMak], Definition 1.2 using a countable basis in a more intrinsic way.

Definition 6.1. Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of local shift morphisms of type (r,s) on the Hilbert tower $(H_n)_{n\in\mathbb{N}}$ whose projective limit is $P = \varprojlim P_n$.
P is said to be a shift Hilbert Poisson tensor of type (r,s) on $H_\infty = \varprojlim H_n$ if, for any $q = \varprojlim q_n$, $f = \varprojlim f_n$, $g = \varprojlim g_n$ and $h = \varprojlim h_n$, it fulfils the following conditions:

(SHPT 1) P is antisymmetric,
i.e. for all $n \in \mathbb{N}$ such that $n - s \geq 0$,

$$\left\langle (P_n)_{q_n}(f_{n+r}), g_{n-s} \right\rangle_{H_{n-s}} = -\left\langle (P_n)_{q_n}(g_{n+r}), f_{n-s} \right\rangle_{H_{n-s}}$$

(SHPT 2) The Schouten bracket vanishes: $[P,P] = 0$,
where for all $n \in \mathbb{N}$ such that $n + r - 2s \geq 0$,

$$[P_n, P_n]_{q_n}(f_{n+r}, g_{n+r}, h_{n+r}) = \sigma \left\langle f_{n+r-2s}, P'_{q_{n-s}}\left(g_{n+r-s}, (P_n)_{q_n} h_{n+r} \right) \right\rangle$$

In this definition, the differentiabity of P at q is given by:

$$P'_q(f,g) = \frac{d}{dt} P_{q+tg} f_{\,|t=0}$$

Example 6.2. The Korteweg-de Vries (KdV) equation ([KorVri]) is an evolution equation in one space dimension which was proposed as a model to describe waves on shallow water surfaces. This nonlinear and dispersive PDE was first introduced by J. Boussinesq ([Bous]) and rediscovered by D. Korteweg and G. de Vries ([KorVri]) in order to modelize natural phenomena discovered by Russel ([Rus]).

In [Arn], V.Arnold suggested a general framework for the Euler equations on an arbitrary group that describe a geodesic flow with respect to a suitable one-sided invariant Riemannian metric on the group. This approach works for the Virasoro group and provides a natural geometric setting for the KdV equation (cf.[KheMis]).

It is well known (e.g. [FMPZ], [MagMor], [Olv], [Sch], [ZubMag], ...) that this equation can be written in Hamiltonian form in two distinct ways. Moreover, there exists an infinite hierarchy of commuting conservation laws and Hamiltonian flows generated by a recursion operator linking both Poisson brackets. Such an equation can be viewed as a complete integrable system and has a lot of remarkable properties, including soliton solutions.

In [KisLeu], the framework of variational Lie algebroids is used to describe such an evolutionary equation.

Here we consider the KdV equation on the circle $\mathbb{S}^1$ of unit length

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u$$

where $t \in \mathbb{R}$ and $x \in \mathbb{S}^1$.

This equation can be seen as an infinite dimensional system on the Hilbert tower $\left(H^n\left(\mathbb{S}^1\right)\right)_{n\in\mathbb{N}}$ (cf. [KapMak] and [KapPos]). This system can be written in a bi-hamiltonian way relatively to the compatible shift Hilbert Poisson tensors ∂_x, of type $(1,0)$, and L_q of type $(2,1)$.

7 Conclusion

Different frameworks are used to describe evolution equations. The notion of shift Hilbert Poisson tensor of type (r,s) presented in this paper fits well with to different evolution equations of the form $u_t = \varphi(u_x^{[k]})$, where $u_x^{[k]}$ stands for the k-jet at x of a function u on the circle $\mathbb{S}^1$. The famous KdV equation $\partial_t u = -\partial_x^3 u + 6u\partial_x u$ examined in this paper is of this type and can be written in a (bi)Hamiltonian form, i.e. with a pair of such compatible tensors. This is also the case for other evolution equations, e.g. the inviscid Burgers equation $\partial_t u = -3u\partial_x$.

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