On the properties of certain fuzzy sub-implicative ideals in $BCI-$algebras

Muhammad Zulfiqar and Muhammad Shabir

Abstract. In this paper, we introduce the concept of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal in $BCI-$algebra and investigate some of their related properties.

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1 Introduction

The concept of sub-implicative ideal in $BCI-$algebra was initiated by Liu and Meng [13] and investigated some of their properties. The fundamental concept of fuzzy set given by Zadeh in his classic paper [22], of 1965 provides a natural framework for generalizing some of the basic notions of algebra. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [20], where he introduced the fuzzy subgroup of a group. Since then the literature of various algebraic structures has been fuzzified. Jun studied fuzzy sub-implicative ideals of $BCI-$algebras in [8]. Liu et al. [14] discussed FSI-ideals and FSC-ideals of $BCI-$algebras. In [17], Hedayati studied connections between generalized fuzzy ideals and sub-implicative ideals in $BCI-$algebras.

In 1971, Rosenfeld formulated the elements of theory of fuzzy groups [20]. A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [4] by using the combined notions of belongingness and quasi-coincidence of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [19]. Murali [18] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfelds fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [2-3] introduced the concept of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [9-12, 23-24]). In [6], Davvaz studied $(\in, \in \vee q)$-fuzzy subnear-rings and ideals. In [9-11], Jun defined the notion of (α, β)-fuzzy subalgebras/ideals in $BCK/BCI-$algebras where α, β are any of $\{\in, q, \in \vee q, \in \wedge q\}$with $\alpha \neq \in \wedge q$. The

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concept of (α, β)-fuzzy positive implicative ideal in BCK-algebras was introduced by Zulfiqar in [24]. In [15], Ma et al. defined the notion of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy filters of BL-algebras. Zhan and Jun studied $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals in $BCI-$algebras [23].

In the present paper, the purpose of this paper is to define the concept of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal in $BCI-$algebra and some related properties are investigated.

2 Preliminaries

Throughout this paper X always denotes a $BCI-$algebra. We further remind some basic aspects which will be used throughout the paper.

A $BCI-$algebra X ([1]) is a general algebra $(X, *, 0)$ of type $(2, 0)$, satisfying the following conditions:

$(BCI-1)$ $((x * y) * (x * z)) * (z * y) = 0$

$(BCI-2)$ $(x * (x * y)) * y = 0$

$(BCI-3)$ $x * x = 0$

$(BCI-4)$ $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$.

We can define a partial order $\leqslant$ on X by $x \leqslant y$ if and only if $x * y = 0$.

Proposition 2.1. [1, 16, 17] *In any $BCI-$algebra X, the following are true:*

(i) $(x * y) * z = (x * z) * y$

(ii) $(x * z) * (y * z) \leqslant x * y$

(iii) $(x * y) * (x * z) \leqslant z * y$

(iv) $x * 0 = x$

(v) $x * (x * (x * y)) = x * y$

for all $x, y, z \in X$.

Definition 2.1. [13] A non-empty subset I of a $BCI-$algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$and $y \in I$ imply $x \in I$, for all $x, y \in X$.

Definition 2.2. [13] A non-empty subset I of a $BCI-$algebra X is called a sub-implicative ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in I$,

(I3) $((x * (x * y)) * (y * x)) * z \in I$ and $z \in I \Rightarrow y * (y * x) \in I$,

for all $x, y, z \in X$.

We now recollect several fuzzy logic concepts. Recall that the real unit interval $[0, 1]$ with the totally ordered relation $\leqslant$ is a complete lattice, with $\wedge$ = min and $\vee$ = max, 0 and 1 being the least element and the greatest element, respectively.

A fuzzy set λ of a universe X is a function from X into the unit closed interval [0, 1], that is $\lambda : X \to [0,1]$. For a fuzzy set λ of a $BCI-$algebra X and $t \in (0,1]$, the crisp set

$$\lambda_t = \{x \in X | \lambda(x) \geqslant t\}$$

is called the level subset of λ [5].

Definition 2.3. [16]. A fuzzy set λ of a $BCI-$algebra X is called a fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where:

(F1) $\lambda(0) \geqslant \lambda(x)$,

(F2) $\lambda(x) \geqslant \lambda(x * y) \wedge \lambda(y)$,

for all $x, y \in X$.

Definition 2.4. [8]. A fuzzy set λ of a $BCI-$algebra X is called a fuzzy sub-implicative ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\lambda(0) \geqslant \lambda(x)$,

(F3) $\lambda(y * (y * x)) \geqslant \lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z)$,

for all $x, y, z \in X$.

Theorem 2.2. [8] *Every fuzzy sub-implicative ideal of a BCI−algebra X is a fuzzy ideal of X.*

Theorem 2.3. [8] *A fuzzy set λ of a BCI−algebra X is a fuzzy sub-implicative ideal of X if and only if, for every $t \in (0,1]$, λ_t is either empty or a sub-implicative ideal of X.*

A fuzzy set λ of a $BCI-$algebra X having the form

$$\lambda(y) = \begin{cases} t \in (0,1], & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t [9].

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [19] gave meaning to the symbol $x_t \alpha t$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geqslant t$ (resp. $\lambda(x) > t + 1$). By $x_t \in \vee q \lambda (x_t \in \wedge q \lambda)$, we mean that $x_t \in \lambda$or $x_t q \lambda$ ($x_t \in \lambda$ and $x_t q \lambda$). For all$t_1, t_2 \in [0,1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$ and $t_1 \vee t_2$, respectively.

In what follows let α and β denote any one of $\in, q, \in \vee q, \in \wedge q$ and $\alpha \neq \in \wedge q$, unless otherwise specified. To say that $x_t \bar{\alpha} \lambda$, means that $x_t \alpha \lambda$ does not hold.

3 $(\bar{\alpha}, \bar{\beta})$-fuzzy ideals

Throughout this paper X will denote a $BCI-$algebra and $\bar{\alpha}$, $\bar{\beta}$ are any one of $\bar{\in}$, $\bar{q}$, $\bar{\in} \vee \bar{q}$, $\bar{\in} \wedge \bar{q}$ unless otherwise specified.

Definition 3.1. A fuzzy set λ of a $BCI-$algebra X is called an $(\bar{\alpha},\ \bar{\beta})$-fuzzy subalgebra of X, where $\bar{\alpha} \neq \bar{\in} \wedge\ \bar{q}$, if it satisfies the condition

$$(x * y)_{t_1 \wedge t_2} \bar{\alpha}\, \lambda \Rightarrow x_{t_1} \bar{\beta} \lambda \text{ or } y_{t_2} \bar{\beta} \lambda,$$

for all $t_1, t_2 \in (0,1]$and $x, y \in X$. Let λ be a fuzzy set of a $BCI-$algebra X such that$\lambda(x) \geqslant 0.5$ for all $x \in X$. Let $x \in X$ and $t \in (0,1]$ be such that

$$x_t \bar{\in} \wedge \bar{q} \lambda.$$

Then $\lambda(x) < t$ and $\lambda(x) + t \leqslant 1$, and it follows that

$$2\lambda(x) = \lambda(x) + \lambda(x) < \lambda(x) + t \leqslant 1,$$

whence $\lambda(x) < 0.5$. This means that

$$\{x_t | x_t \bar{\in} \wedge \bar{q} \lambda\} = \phi,$$

and therefore, the case $\bar{\alpha} = \bar{\in}\ \wedge\ \bar{q}$ in the above definition is omitted.

Definition 3.2. A fuzzy set λ of a $BCI-$algebra X is called an $(\bar{\alpha},\ \bar{\beta})$-fuzzy ideal of X, where $\bar{\alpha} \neq\ \bar{\in}\ \wedge\ \bar{q}$, if it satisfies the conditions (A) and (B), where

(A) $0_t \bar{\alpha}\, \lambda \Rightarrow x_t \bar{\beta}\, \lambda$,

(B) $x_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (x * y)_{t_1} \bar{\beta} \lambda$ or $y_{t_2} \bar{\beta} \lambda$,

for all $t, t_1, t_2 \in (0,1]$and $x, y \in X$.

Theorem 3.1. *A fuzzy set λ of a $BCI-$algebra X is a fuzzy ideal of X if and only if λ is an $(\bar{\in},\ \bar{\in})$-fuzzy ideal of X.*

Proof. Suppose λ is a fuzzy ideal of X. Let $0_t \bar{\in} \lambda$ for $t \in (0,1]$. Then $\lambda(0) < t$. By Definition 2.4, we have

$$t > \lambda(0) \geqslant \lambda(x),$$

this implies that $t > \lambda(x)$, that is,$x_t \bar{\in} \lambda$. Let $x, y \in X$ and $t, r \in (0,1]$ be such that

$$x_{t \wedge r} \bar{\in} \lambda.$$

Then

$$\lambda(x) < t \wedge r.$$

Since λ is a fuzzy ideal of X. So

$$t \wedge r > \lambda(x) \geqslant \lambda\left(x * y\right) \wedge \lambda\left(y\right).$$

This implies that

$$t > \lambda\left(x * y\right) \text{ or } r > \lambda\left(y\right),$$

that is,

$$(x * y)_t \bar{\in} \lambda \text{ or } y_r \bar{\in} \lambda.$$

This shows that λ is an $(\bar{\in},\ \bar{\in})$-fuzzy ideal of X.

Conversely, assume that λ is an $(\bar{\in}, \bar{\in})$-fuzzy ideal of X. To show λ is a fuzzy ideal of X, suppose there exists $x \in X$ such that

$$\lambda(0) < \lambda(x).$$

Select $t \in (0,1]$ such that $\lambda(0) < t \leqslant \lambda(x)$. Then $0_t \bar{\in} \lambda$ but $x_t \in \lambda$, which is a contradiction. Hence

$$\lambda(0) \geqslant \lambda(x), \quad \text{for all } x \in X.$$

Now suppose there exist $x, y \in X$ such that $\lambda(x) < \lambda(x * y) \wedge \lambda(y)$. Select $t \in (0,1]$ such that

$$\lambda(x) < t \leqslant \lambda(x * y) \wedge \lambda(y).$$

Then $x_t \bar{\in} \lambda$ but $(x * y)_t \in \lambda$ and $y_t \in \lambda$, which is a contradiction. Hence

$$\lambda(x) \geqslant \lambda(x * y) \wedge \lambda(y).$$

This shows that λ is a fuzzy ideal of X. □

4 $(\bar{\alpha}, \bar{\beta})$-fuzzy sub-implicative ideals

In this section, we introduce the concept of $(\bar{\alpha}, \bar{\beta})$-fuzzy sub-implicative ideal in a $BCI-$algebra and investigate some of their properties.

Definition 4.1. A fuzzy set λ of a $BCI-$algebra X is called an $(\bar{\alpha}, \bar{\beta})$-fuzzy sub-implicative ideal of X, where $\bar{\alpha} \neq \bar{\in} \wedge \bar{q}$, if it satisfies the conditions (A) and (C), where

(A) $0_t \bar{\alpha} \lambda \Rightarrow x_t \bar{\beta} \lambda$,

(C) $(y * (y * x))_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (((x * (x * y) * (y * x)) * z)_{t_1} \bar{\beta} \lambda$ or $z_{t_2} \bar{\beta} \lambda$,

for all $t, t_1, t_2 \in (0,1]$and $x, y, z \in X$.

Theorem 4.1. *Every $(\bar{\alpha}, \bar{\beta})$-fuzzy sub-implicative ideal of a $BCI-$algebra X is an $(\bar{\alpha}, \bar{\beta})$-fuzzy ideal of X.*

Proof. Let λ be an $(\bar{\alpha}, \bar{\beta})$-fuzzy sub-implicative ideal of X. Then for all $t_1, t_2 \in (0,1]$ and $x, y, z \in X$, we have

$$(y * (y * x))_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (((x * (x * y) * (y * x)) * z)_{t_1} \bar{\beta} \lambda \text{ or } z_{t_2} \bar{\beta} \lambda.$$

Put $y = x$ in above, we get

$$(x * (x * x))_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (((x * (x * x) * (x * x)) * z)_{t_1} \bar{\beta} \lambda \text{ or } z_{t_2} \bar{\beta} \lambda.$$

This implies

$$(x * 0)_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (((x * 0) * 0) * z)_{t_1} \bar{\beta} \lambda \text{ or } z_{t_2} \bar{\beta} \lambda \quad \text{(BCI - 3)}$$

$$x_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow ((x * 0) * z)_{t_1} \bar{\beta} \lambda \text{ or } z_{t_2} \bar{\beta} \lambda \qquad \text{(by Proposition 2.1 (iv))}$$

$$x_{t_1 \wedge t_2} \bar{\alpha} \lambda \Rightarrow (x * z)_{t_1} \bar{\beta} \lambda \text{ or } z_{t_2} \bar{\beta} \lambda \qquad \text{(by Proposition 2.1 (iv))}$$

This means that λ satisfies the condition (B). Combining with (A) implies that λ is an $(\bar{\alpha}, \bar{\beta})$-fuzzy ideal of X. □

Theorem 4.2. *A fuzzy set* λ *of a* $BCI-$*algebra* X *is a fuzzy sub-implicative ideal of* X *if and only if* λ *is an* $(\bar{\in}, \bar{\in})$*-fuzzy sub-implicative ideal of* X.

Proof. Suppose λ is a fuzzy sub-implicative ideal of X. Let $0_t \bar{\in} \lambda$ for $t \in (0,1]$. Then $\lambda(0) < t$. By Definition 2.4,

$$t > \lambda(0) \geqslant \lambda(x),$$

this implies that $t > \lambda(x)$, that is, $x_t \bar{\in} \lambda$. Let $x, y \in X$ and $t, r \in (0,1]$ be such that

$$(y * (y * x))_{t \wedge r} \bar{\in} \lambda.$$

Then

$$\lambda\left(y * (y * x)\right) < t \wedge r.$$

Since λ is a fuzzy sub-implicative ideal of X. So

$$t \wedge r > \lambda\left(y * (y * x)\right) \geqslant \lambda\left(((x * (x * y)) * (y * x)) * z\right) \wedge \lambda\left(z\right).$$

This implies that

$$t > \lambda\left(((x * (x * y)) * (y * x)) * z\right) \ \text{ or } \ r > \lambda\left(z\right),$$

that is,

$$\left(((x * (x * y)) * (y * x)) * z\right)_t \bar{\in} \lambda \ \text{ or } \ z_r \bar{\in} \lambda.$$

This shows that λ is an $(\bar{\in}, \bar{\in})$-fuzzy sub-implicative ideal of X.

Conversely, assume that λ is an $(\bar{\in}, \bar{\in})$-fuzzy sub-implicative ideal of X. To show λ is a fuzzy sub-implicative ideal of X, suppose there exists $x \in X$ such that

$$\lambda(0) < \lambda(x).$$

Select $t \in (0,1]$ such that $\lambda(0) < t \leqslant \lambda(x)$. Then $0_t \bar{\in} \lambda$ but $x_t \in \lambda$, which is a contradiction. Hence

$$\lambda(0) \geqslant \lambda(x), \quad \text{for all } x \in X.$$

Now suppose there exist $x, y, z \in X$ such that

$$\lambda\left(y * (y * x)\right) < \lambda\left(((x * (x * y)) * (y * x)) * z\right) \wedge \lambda\left(z\right).$$

Select $t \in (0,1]$ such that

$$\lambda\left(y * (y * x)\right) < t \leqslant \lambda\left(((x * (x * y)) * (y * x)) * z\right) \wedge \lambda\left(z\right).$$

Then$(y * (y * x))_t \bar{\in} \lambda$ but $\left(((x * (x * y)) * (y * x)) * z\right)_t \in \lambda$ and $z_t \in \lambda$, which is a contradiction. Hence

$$\lambda\left(y * (y * x)\right) \geqslant \lambda\left(((x * (x * y)) * (y * x)) * z\right) \wedge \lambda\left(z\right).$$

This shows that λ is a fuzzy sub-implicative ideal of X. □

5 $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals

In this section, we introduce the concept of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal in a $BCI-$algebra and investigate some of their properties.

Definition 5.1. [23] Let λ be a fuzzy set of a $BCI-$algebra X. Then λ is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal of X if it satisfies the conditions (D) and (E), where

(D) $0_t \bar{\in} \lambda \Rightarrow x_t \bar{\in} \vee \bar{q}\lambda$,

(E) $x_{t\wedge r} \bar{\in} \lambda \Rightarrow (x * y)_t \bar{\in} \vee \bar{q}\lambda$ or $y_r \bar{\in} \vee \bar{q}\lambda$,,

for all $x, y \in X$ and $t, r \in (0, 1]$.

Example 5.2. Let $X = \{0, 1, 2, 3\}$ be a $BCI-$algebra with the following Cayley table [17]:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let λ be a fuzzy set in X defined by $\lambda(0) = 0.50$, $\lambda(1) = \lambda(2) = 0.45$ and$\lambda(3) = 0.32$. Simple calculations show that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal of X.

Theorem 5.1. *[17] The conditions (D) and (E) in Definition 5.1, are equivalent to the following conditions, respectively:*

(F) $\lambda(0) \vee 0.5 \geqslant \lambda(x)$,

(G) $\lambda(x) \vee 0.5 \geqslant \lambda(x * y) \wedge \lambda(y)$,

for all $x, y \in X$.

Corollary 5.2. *A fuzzy set λ of a $BCI-$algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal of X if it satisfies the conditions (F) and (G).*

6 $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideals

In this section, we introduce the concept of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal in a $BCI-$algebra and investigate some of their properties.

Definition 6.1. Let λ be a fuzzy set of a $BCI-$algebra X. Then λ is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X if it satisfies the conditions (D) and (H), where

(D) $0_t \bar{\in} \lambda \Rightarrow x_t \bar{\in} \vee \bar{q}\lambda$,

(H) $(y * (y * x))_{t\wedge r} \bar{\in} \lambda \Rightarrow (((x * (x * y)) * (y * x)) * z)_t \bar{\in} \vee \bar{q}\lambda$ or $z_r \bar{\in} \vee \bar{q}\lambda$,

for all $x, y, z \in X$ and $t, r \in (0, 1]$.

Example 6.2. Let $X = \{0, 1, 2, 3\}$ be a $BCI-$algebra with Cayley table as follows [17]:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let λ be a fuzzy set in X defined by $\lambda(0) = 0.67$,$\lambda(3) = 0.28$ and $\lambda(1) = \lambda(2) = 0.37$. Simple calculations show that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X.

Theorem 6.1. *Let λ be a fuzzy set of a $BCI-$algebra X. Then the condition (H) is equivalent to (I), where*

(I) $\lambda(y * (y * x)) \vee 0.5 \geqslant \lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z)$.,

for all $x, y, z \in X$.

Proof. (H) $\Rightarrow$ (I). Suppose there exist $x, y, z \in X$ such that

$$\lambda(y * (y * x)) \vee 0.5 < t = \lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z).$$

Then
$t \in (0.5, 1]$, $(y * (y * x))_t \bar{\in} \lambda$ and $(((x * (x * y)) * (y * x)) * z)_t \in \lambda$, $z_t \in \lambda$.
It follows that

$$(((x * (x * y)) * (y * x)) * z)_t \in \bar{q}\lambda \text{ or } z_t \bar{q}\lambda.$$

Then

$$\lambda(((x * (x * y)) * (y * x)) * z) + t \leqslant 1$$

or $\lambda(z) + t \leqslant 1$. Since $t \leqslant \lambda(((x * (x * y)) * (y * x)) * z)$ and $t \leqslant \lambda(z)$, it follows that

$$t \leqslant 0.5.$$

This is a contradiction. So

$$\lambda(y * (y * x)) \vee 0.5 \geqslant \lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z).$$

(I) $\Rightarrow$ (H). Let $x, y \in X$ and $t, r \in (0, 1]$ be such that

$$(y * (y * x))_{t \wedge r} \bar{\in} \lambda.$$

Then

$$\lambda(y * (y * x)) < t \wedge r.$$

(a) If $\lambda(y * (y * x)) \geqslant 0.5$, then by condition (I)

$$\lambda(y * (y * x)) \geqslant \lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z).$$

Thus

$$\lambda(((x * (x * y)) * (y * x)) * z) \wedge \lambda(z) < t \wedge r,$$

and consequently

$$\lambda\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)<t \text{ or } \lambda\left(z\right)<r.$$

It follows that

$$\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)_t \bar{\in} \lambda \text{ or } z_r \bar{\in} \lambda,$$

and hence

$$\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)_t \bar{\in} \vee \bar{q}\lambda \text{ or } z_r \bar{\in} \vee \bar{q}\lambda.$$

(b) If $\lambda\left(y*\left(y*x\right)\right)<0.5$, then by condition (I)

$$0.5 \geqslant \lambda\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right) \wedge \lambda\left(z\right).$$

Suppose $\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)_t \in \lambda$ and $z_r \in \lambda$. Then

$$\lambda\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right) \geqslant t \text{ and } \lambda\left(z\right) \geqslant r.$$

Thus $0.5 \geqslant t \wedge r$. Hence

$$\lambda\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right) \wedge \lambda\left(z\right) + t \wedge r \leqslant 0.5+0.5=1,$$

that is,

$$\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)_t \bar{q}\lambda \text{ or } z_r \bar{q}\lambda.$$

This implies that

$$\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right)_t \bar{\in} \vee \bar{q}\lambda \text{ or } z_r \bar{\in} \vee \bar{q}\lambda.$$

□

Corollary 6.2. *A fuzzy set λ of a $BCI-$algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X if and only if it satisfies the conditions (F) and (I).*

Theorem 6.3. *Every $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of a $BCI-$algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal of X.*

Proof. The proof follows from Theorem 4.1. □

Theorem 6.4. *A fuzzy set λ of a $BCI-$algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X if and only if for any $t \in (0.5, 1]$, $\lambda_t = \{x \in X | \lambda(x) \geqslant t\}$is a sub-implicative ideal of X.*

Proof. Let λ be an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X and $0.5 < t \leqslant 1$. If $\lambda_t \neq \phi$, then $x \in \lambda_t$. This implies that $\lambda(x) \geqslant t$. By condition (F)

$$\lambda(0) \vee 0.5 \geqslant \lambda(x) \geqslant t.$$

Thus $\lambda(0) \geqslant t$. Hence $0 \in \lambda_t$.
Let $((x*(x*y))*(y*x))*z \in \lambda_t$ and $z \in \lambda_t$. Then

$$\lambda\left(\left(\left(x*\left(x*y\right)\right)*\left(y*x\right)\right)*z\right) \geqslant t \text{ and } \lambda\left(z\right) \geqslant t.$$

By condition (I), it follows

$$\lambda (y * (y * x)) \vee 0.5 \geqslant \lambda (((x * (x * y)) * (y * x)) * z) \wedge \lambda (z) \geqslant t \wedge t = t.$$

Thus $\lambda (y * (y * x)) \geqslant t$, that is $y*(y*x) \in \lambda_t$. Therefore λ_t is a sub-implicative ideal of X.

Conversely, assume that λ is a fuzzy set of X such that $\lambda_t \, (\neq \phi)$ is a sub-implicative ideal of X for all $0.5 < t \leqslant 1$. Let $x \in X$ be such that

$$\lambda(0) \vee 0.5 < \lambda(x).$$

Select $0.5 < t \leqslant 1$ such that

$$\lambda(0) \vee 0.5 < t \leqslant \lambda(x).$$

Then $x \in \lambda_t$ but$0 \notin \lambda_t$, a contradiction. Hence

$$\lambda(0) \vee 0.5 \geqslant \lambda(x).$$

Now assume that $x, y, z \in X$ such that

$$\lambda (y * (y * x)) \vee 0.5 < \lambda (((x * (x * y)) * (y * x)) * z) \wedge \lambda (z) .$$

Select $0.5 < t \leqslant 1$ such that

$$\lambda (y * (y * x)) \vee 0.5 < t \leqslant \lambda (((x * (x * y)) * (y * x)) * z) \wedge \lambda (z) .$$

Then $((x * (x * y)) * (y * x)) * z$ and z are in λ_t but $y * (y * x) \notin \lambda_t$, a contradiction. Hence

$$\lambda (y * (y * x)) \vee 0.5 \geqslant \lambda (((x * (x * y)) * (y * x)) * z) \wedge \lambda (z) .$$

This shows that λ is an $(\bar{\in}, \ \bar{\in} \ \vee \ \bar{q})$-fuzzy sub-implicative ideal of X. □

Remark 6.3. Let λ be a fuzzy set of a $BCI-$algebra X and

$$I_t = \{t | t \in (0,1] \text{ such that } \lambda_t \text{ is a sub - implicative ideal of } X\}$$

In particular,

(1) If $I_t = (0, 1]$, then λ is a fuzzy sub-implicative ideal of X (Theorem 2.3).

(2) If $I_t = (0.5, 1]$, then λ is an $(\bar{\in}, \ \bar{\in} \ \vee \ \bar{q})$-fuzzy sub-implicative ideal of X (Theorem 6.4).

Corollary 6.5. *Every fuzzy sub-implicative ideal of a $BCI-$algebra X is an $(\bar{\in}, \ \bar{\in} \ \vee \ \bar{q})$-fuzzy sub-implicative ideal of X.*

Theorem 6.6. *Let H be a non-empty subset of a $BCI-$algebra X. Then H is a sub-implicative ideal of X if and only if the fuzzy set λ of X defined by:*

$$\lambda(x) = \begin{cases} \leqslant 0.5 & if\, x \in X - H \\ 1 & if\, x \in H, \end{cases}$$

is an $(\bar{\in}, \ \bar{\in} \ \vee \ \bar{q})$-fuzzy sub-implicative ideal of X.

Proof. Let I be a sub-implicative ideal of X. Then $0 \in I$. This implies that $\lambda(0) = 1$. Thus

$$\lambda(0) \vee 0.5 = 1 \geqslant \lambda(x).$$

It means that λ satisfies the condition (F).

Now let $x, y, z \in X$. If $((x*(x*y))*(y*x))*z$ and z are in I, then $y*(y*x) \in I$. This implies that

$$\lambda\left(y*(y*x)\right) \vee 0.5 = 1 = \lambda\left(((x*(x*y))*(y*x))*z\right) \wedge \lambda\left(z\right).$$

If one of $((x*(x*y))*(y*x))*z$ and z is not in I, then

$$\lambda\left(((x*(x*y))*(y*x))*z\right) \vee \lambda\left(z\right) \leqslant 0.5 \leqslant \lambda\left(y*(y*x)\right) \vee 0.5.$$

Thus λ satisfies the condition (G). Hence is an $(\bar{\in},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideal of X.

Conversely, assume that λ is an $(\bar{\in},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideal of X.
Let $x \in I$ Then by condition (F)

$$\lambda(0) \vee 0.5 \geqslant \lambda(x) = 1.$$

This implies that $0 \in I$. Let $x, y, z \in X$ be such that $((x*(x*y))*(y*x))*z$ and z are in I. Then by condition (G)

$$\lambda\left(y*(y*x)\right) \vee 0.5 \geqslant \lambda\left(((x*(x*y))*(y*x))*z\right) \wedge \lambda\left(z\right) = 1.$$

This implies that $\lambda\left(y*(y*x)\right) = 1$, that is $y*(y*x) \in I$.
Hence I is a sub-implicative ideal of X. □

Theorem 6.7. *Let H be a non-empty subset of a $BCI-algebra$ X. Then H is a sub-implicative ideal of X if and only if the fuzzy set λ of X defined by:*

$$\lambda(x) = \begin{cases} \leqslant 0.5 & if\, x \in X - H \\ 1 & if\, x \in H, \end{cases}$$

is an $(\bar{q},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideal of X.

Proof. Let I be a sub-implicative ideal of X. Let $t \in (0, 1]$ be such that $0_t\bar{q}\lambda$. Then

$$\lambda(0) + t \leqslant 1,$$

so $0 \notin I$. This implies that $I = \phi$. Thus, if $t > 0.5$, then

$$\lambda(x) \leqslant 0.5 < t,$$

so $x_t\bar{\in}\lambda$. If $t \leqslant 0.5$, then

$$\lambda(x) + t \leqslant 0.5 + 0.5 = 1.$$

This implies that $x_t\bar{q}\lambda$. Hence

$$x_t\bar{\in} \vee \bar{q}\lambda.$$

Now let $x, y \in X$and $t, r \in (0,1]$ be such that

$$(y * (y * x))_{t\wedge r}\bar{q}\lambda.$$

Then

$$\lambda\left(y * (y * x)\right) + t \wedge r \leqslant 1,$$

so $y * (y * x) \notin I$. This implies that either

$$((x * (x * y)) * (y * x)) * z \notin I \text{ or } z \notin I.$$

Suppose $((x * (x * y)) * (y * x)) * z \notin I$. Thus, if $t \wedge r > 0.5$, then

$$\lambda\left(((x * (x * y)) * (y * x)) * z\right) \leqslant 0.5 < t \wedge r$$

and so $\lambda\left(((x * (x * y)) * (y * x)) * z\right) \leqslant t$. This implies that

$$(((x * (x * y)) * (y * x)) * z)_t \bar{\in} \lambda.$$

If$t \wedge r < 0.5$ and $(((x * (x * y)) * (y * x)) * z)_t \in \lambda$, then

$$\lambda\left(((x * (x * y)) * (y * x)) * z\right) \geqslant t.$$

As

$$0.5 \geqslant \lambda\left(((x * (x * y)) * (y * x)) * z\right),$$

so $0.5 \geqslant t$. Thus

$$\lambda\left(((x * (x * y)) * (y * x)) * z\right) + t \leqslant 0.5 + 0.5 = 1,$$

that is

$$(((x * (x * y)) * (y * x)) * z)_t \bar{q}\lambda.$$

Hence

$$(((x * (x * y)) * (y * x)) * z)_t \bar{\in} \vee \bar{q}\lambda.$$

Similarly, if $z \notin I$, then

$$z_r \bar{\in} \vee \bar{q}\lambda.$$

This shows that λ is a $(\bar{q},\ \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X.

Conversely, assume that λ is a $(\bar{q},\ \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X. Let $x \in I$. If $0 \notin I$, then $\lambda(0) \leqslant 0.5$. Now for any $t \in (0, 0.5]$

$$\lambda(0) + t \leqslant 0.5 + 0.5 = 1,$$

this implies that $0_t\bar{q}\lambda$. Thus $x_t \bar{\in} \vee \bar{q}\lambda$. But

$$\lambda(x) = 1 > t \text{ and } \lambda(x) + t > 1$$

implies that $x_t \in \wedge q\lambda$, which is a contradiction. Hence $0 \in I$.

Now suppose $x, y, z \in X$ such that $((x * (x * y)) * (y * x)) * z$ and $z \in I$. We have to show that $y * (y * x) \in I$. On contrary assume that $y * (y * x) \notin I$. Then $\lambda\left(y * (y * x)\right) \leqslant 0.5$. Now for $t \in (0, 0.5]$, we have

$$\lambda\left(y * (y * x)\right) + t \leqslant 0.5 + 0.5 = 1,$$

this is $(y*(y*x))_t \bar{q}\lambda$. Thus

$$(((x*(x*y))*(y*x))*z)_t \bar{\in} \vee \bar{q}\lambda \text{ or } z_t \bar{\in} \vee \bar{q}\lambda.$$

But $((x*(x*y))*(y*x))*z \in I$ and $z \in I$ implies

$$\lambda\left(((x*(x*y))*(y*x))*z\right) = \lambda(z) = 1.$$

This implies that

$$(((x*(x*y))*(y*x))*z)_t \in \wedge q\lambda \text{ and } z_t \in \wedge q\lambda,$$

which is a contradiction. Hence $y*(y*x) \in I$. □

Theorem 6.8. *Let H be a non-empty subset of a $BCI-$algebra X. Then H is a sub-implicative ideal of X if and only if the fuzzy set λ of X defined by:*

$$\lambda(x) \begin{cases} \leqslant 0.5, & \text{if } x \in X \backslash H \\ = 1, & \text{if } x \in H, \end{cases}$$

is an $(\bar{\in} \vee \bar{q}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X.

Proof. The proof follows from the proof of Theorem 6.6 and Theorem 6.7. □

Theorem 6.9. *The intersection of any family of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideals of a $BCI-$algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X.*

Proof. Let $\{\lambda_i\}_i \in I$ be a family of $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideals of a $BCI-$algebra X and $x \in X$. So

$$\lambda_i(0) \vee 0.5 \geqslant \lambda_i(x)$$

for all $i \in I$. Thus

$$\left(\underset{i\in I}{\wedge} \lambda_i\right)(0) \vee 0.5 = \underset{i\in I}{\wedge} \lambda_i(0) \vee 0.5 \geqslant \underset{i\in I}{\wedge} (\lambda_i(x)) = \left(\underset{i\in I}{\wedge} \lambda_i\right)(x).$$

Thus

$$\left(\underset{i\in I}{\wedge} \lambda_i\right)(0) \vee 0.5 \geqslant \left(\underset{i\in I}{\wedge} \lambda_i\right)(x).$$

Let $x, y, z \in X$. Since eachλ_i is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy sub-implicative ideal of X. So

$$\lambda_i(y*(y*x)) \vee 0.5 \geqslant \lambda_i(((x*(x*y))*(y*x))*z) \wedge \lambda_i(z),$$

for all $i \in I$. Thus

$$\begin{aligned}\left(\underset{i\in I}{\wedge} \lambda_i\right)(y*(y*x)) \vee 0.5 &= \underset{i\in I}{\wedge} \lambda_i(y*(y*x)) \vee 0.5 \\ &\geqslant \underset{i\in I}{\wedge} \lambda_i(((x*(x*y))*(y*x))*z) \wedge \lambda_i(z) \\ &= \left(\underset{i\in I}{\wedge} \lambda_i\right)(((x*(x*y))*(y*x))*z) \wedge \left(\underset{i\in I}{\wedge} \lambda_i\right)(z)\end{aligned}$$

Thus

$$\left(\underset{i\in I}{\wedge}\lambda_i\right)(y*(y*x))\vee 0.5 \geqslant \left(\underset{i\in I}{\wedge}\lambda_i\right)(((x*(x*y))*(y*x))*z)\wedge\left(\underset{i\in I}{\wedge}\lambda_i\right)(z)\,.$$

Hence, $\underset{i\in I}{\wedge}\lambda_i$ is an $(\bar{\in},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideal of X. □

Theorem 6.10. *The union of any family of $(\bar{\in},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideals of a $BCI-$algebra X is an $(\bar{\in},\ \bar{\in}\ \vee\ \bar{q})$-fuzzy sub-implicative ideal of X.*

Proof. Straightforward. □

7 Conclusion

In the study of fuzzy algebraic system, we see that the fuzzy sub-implicative ideals with special properties always play a vital role.

The purpose of this paper is to define the concept of -fuzzy sub-implicative ideal in $BCI-$algebra and some related properties are investigated.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy $BCI-$algebras and their applications in other branches of algebra. In the future study of fuzzy $BCI-$algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of $BCI-$algebras by using this notion;

(2) To apply this notion to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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Authors' addresses:

Muhammad Zulfiqar
Department of Mathematics,
GC University, Lahore, Pakistan.
E-mail: mzulfiqarshafi@hotmail.com

Muhammad Shabir
Department of Mathematics,
Quaid-i-Azam University, Islamabad, Pakistan.
E-mail: mshabirbhatti@yahoo.co.uk