Boolean functions: time-reversal symmetry and the generalized technical condition of proper operation

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Abstract. The paper relates time-reversal symmetry that we have adapted to the Boolean functions with a strong version of the generalized technical condition of proper operation.

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1 Introduction and preliminaries

We denote in the following with $\mathbf{B}$ the Boolean algebra with two elements, i.e. the set $\{0,1\}$ endowed with the complement $'-'$, the intersection $'\cdot'$, the union $'\cup'$, and the modulo 2 sum $'\oplus'$. These laws induce laws denoted with the same symbols on the set $\mathbf{B}^n$.

The asynchronous flows [2] model the behavior of the digital devices from electronics. They are generated by Boolean functions $\Phi:\mathbf{B}^n\to\mathbf{B}^n$ that iterate their coordinates $\Phi_1,...,\Phi_n$ independently on each other. The time instants and the order in which $\Phi_1,...,\Phi_n$ are computed are not known and the generalized technical condition of proper operation [2] (the generalization of race-freedom) states particular circumstances when the flow behaves 'almost' deterministically.

Time-reversal symmetry [1] is one of the fundamental symmetries discussed in natural science. Consequently, it arises in many physically motivated dynamical systems, in particular in classical and quantum mechanics. Our aim is to relate the time-reversal symmetry, adapted to asynchronicity, with a strengthened form of generalized technical condition of proper operation, restricting our attention to Boolean functions - not to flows, for reasons of brevity. This is possible since, in discrete time, reasoning goes on recurrently (and real time equivalent constructions exist also). Even if time does not occur in the paper, we have kept the terminology of 'time-reversal symmetry', since other symmetries of the Boolean functions exist also.

Definition 1.1. We define for $\mu,\lambda\in\mathbf{B}^n$ the sets

$$\mu\boxplus\lambda=\{i|i\in\{1,...,n\},\mu_i\oplus\lambda_i=1\},$$

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$$\Phi_\mu = \{i | i \in \{1, ..., n\}, \mu_i \oplus \Phi_i(\mu) = 1\}.$$

The coordinates $i \in \Phi_\mu$ are called *excited*, or *enabled*, or *unstable*.

Definition 1.2. The points $\mu, \lambda \in \mathbf{B}^n$ define the sets $[\mu, \lambda] = \{\mu \oplus \underset{i \in A}{\Xi} \varepsilon^i | A \subset \mu \boxplus \lambda\}$, $[\mu, \lambda) = [\mu, \lambda] \smallsetminus \{\lambda\}$, $(\mu, \lambda] = [\mu, \lambda] \smallsetminus \{\mu\}$, $(\mu, \lambda) = [\mu, \lambda] \smallsetminus \{\mu, \lambda\}$ where we have denoted $\varepsilon^i = (0, ..., \underset{i}{1}, ..., 0) \in \mathbf{B}^n$, $i \in \{1, ..., n\}$ and $\underset{i}{\Xi}$ is the modulo 2 summation satisfying by definition $\underset{i \in \varnothing}{\Xi} \varepsilon^i = \mathbf{0} \in \mathbf{B}^n$.

Remark 1.3. The sets $[\mu, \lambda]$ are affine spaces and they fulfill $[\mu, \lambda] = [\lambda, \mu], [\mu, \mu] = \{\mu\}, [\mu, \lambda] = [\mu, \lambda'] \Longrightarrow \lambda = \lambda', \nu \in [\mu, \lambda] \Longrightarrow [\nu, \lambda] \subset [\mu, \lambda]$, etc.

Definition 1.4. Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and $\lambda \in \mathbf{B}^n$. We define $\Phi^\lambda : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}, \Phi_i^\lambda(\mu) = \begin{cases} \mu_i, if\ \lambda_i = 0, \\ \Phi_i(\mu), if\ \lambda_i = 1. \end{cases}$

Definition 1.5. The sets of *predecessors* and *successors* of $\mu \in \mathbf{B}^n$ are defined by $\mu_\Phi^- = \{\nu | \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, \Phi^\lambda(\nu) = \mu\}, \mu_\Phi^+ = \{\Phi^\lambda(\mu) | \lambda \in \mathbf{B}^n\}$.

Remark 1.6. Since $\Phi^{(0,...,0)}(\mu) = \mu$, we infer $\mu \in \mu_\Phi^-, \mu \in \mu_\Phi^+$.

Theorem 1.1. *We have that* $\mu_\Phi^+ = [\mu, \Phi(\mu)]$ *and* $\mu_\Phi^+ = \{\mu\} \Longleftrightarrow \Phi(\mu) = \mu$.

Proof. These follow from:

$$\mu_\Phi^+ = \{\Phi^\lambda(\mu) | \lambda \in \mathbf{B}^n\} = \{\underset{i \in \{1,...,n\}}{\Xi} \Phi_i^\lambda(\mu) \varepsilon^i | \lambda \in \mathbf{B}^n\}$$

$$= \{\underset{i \in \{1,...,n\}}{\Xi} ((1 \oplus \lambda_i)\mu_i \oplus \lambda_i \Phi_i(\mu)) \varepsilon^i | \lambda \in \mathbf{B}^n\}$$

$$= \{\underset{i \in \{1,...,n\}}{\Xi} \mu_i \varepsilon^i \oplus \underset{i \in \{1,...,n\}}{\Xi} \lambda_i (\mu_i \oplus \Phi_i(\mu)) \varepsilon^i | \lambda \in \mathbf{B}^n\}$$

$$= \{\mu \oplus \underset{i \in \{j | j \in \{1,...,n\}, \lambda_j = 1\} \cap \Phi_\mu}{\Xi} \varepsilon^i | \lambda \in \mathbf{B}^n\}$$

$$= \{\mu \oplus \underset{i \in A \cap \Phi_\mu}{\Xi} \varepsilon^i | A \subset \{1, ..., n\}\} = \{\mu \oplus \underset{i \in A}{\Xi} \varepsilon^i | A \subset \Phi_\mu\} = [\mu, \Phi(\mu)],$$

$$\mu_\Phi^+ = \{\mu\} = [\mu, \mu] = [\mu, \Phi(\mu)] \Longleftrightarrow \Phi(\mu) = \mu.$$

□

2 Time-reversal symmetry and the strong generalized technical condition of proper operation

Definition 2.1. The functions $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ are called *time-reversed symmetrical* if $\forall \mu \in \mathbf{B}^n$, we have $\mu_\Phi^- = \mu_\Psi^+$ and $\mu_\Phi^+ = \mu_\Psi^-$.

Definition 2.2. We say that Φ fulfills the *strong generalized technical condition of proper operation* (sgtcpo) if $\forall \mu \in \mathbf{B}^n$,

$$\exists \lambda \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda, \mu) \text{ or } \exists \lambda \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda, \mu], \tag{2.1}$$

$$\forall \lambda \in (\mu, \Phi(\mu)), \Phi^{-1}(\lambda) = \varnothing. \tag{2.2}$$

Theorem 2.1. *We suppose that Φ fulfills sgtcpo. Then the generalized technical condition of proper operation (gtcpo) holds:*

$$\forall\mu\in\mathbf{B}^n,\forall\lambda\in[\mu,\Phi(\mu)),\Phi(\mu)=\Phi(\lambda). \tag{2.3}$$

Proof. Let $\mu\in\mathbf{B}^n$ arbitrary, fixed. We have $\mu\in\Phi^{-1}(\Phi(\mu))$, thus $\Phi^{-1}(\Phi(\mu))\neq\varnothing$ and $\omega\in\mathbf{B}^n$ exists with the property $\Phi^{-1}(\Phi(\mu))=[\omega,\Phi(\mu))$ or $\Phi^{-1}(\Phi(\mu))=[\omega,\Phi(\mu)]$. If $\mu=\Phi(\mu)$, then (2.3) is trivially true under the form $\forall\lambda\in\varnothing=[\mu,\mu),\Phi(\mu)=\Phi(\lambda)$, thus we can suppose that $\mu\neq\Phi(\mu)$. As $\mu\in[\omega,\Phi(\mu))\Longrightarrow[\mu,\Phi(\mu))\subset[\omega,\Phi(\mu))$ and $\mu\in[\omega,\Phi(\mu)]\Longrightarrow[\mu,\Phi(\mu))\subset[\omega,\Phi(\mu)]$, we can write $\lambda\in[\mu,\Phi(\mu))\subset\Phi^{-1}(\Phi(\mu))$, thus $\Phi(\mu)=\Phi(\lambda)$. □

Theorem 2.2. *If Φ fulfills gtcpo and $\omega\in[\mu,\Phi(\mu))$, then $\Phi_\omega\subset\Phi_\mu$.*

Proof. From $\mu\oplus\underset{i\in\Phi_\mu}{\Xi}\varepsilon^i=\Phi(\mu)=\Phi(\omega)=\omega\oplus\underset{i\in\Phi_\omega}{\Xi}\varepsilon^i$ and from the existence of $A\subset\Phi_\mu$ with $\omega=\mu\oplus\underset{i\in A}{\Xi}\varepsilon^i$, we get $\underset{i\in A}{\Xi}\varepsilon^i=\underset{i\in\Phi_\mu}{\Xi}\varepsilon^i\oplus\underset{i\in\Phi_\omega}{\Xi}\varepsilon^i=\underset{i\in\Phi_\mu\Delta\Phi_\omega}{\Xi}\varepsilon^i$; $A=\Phi_\mu\Delta\Phi_\omega\subset\Phi_\mu$ implies $\Phi_\omega\subset\Phi_\mu$. □

Theorem 2.3. *If Φ fulfills sgtcpo, for any $\mu\in\mathbf{B}^n$, the following possibilities exist:*
i) $\Phi^{-1}(\mu)=\varnothing,\Phi(\mu)\neq\mu,\mu_\Phi^+\neq\{\mu\}$, and either
i.1) $\mu_\Phi^-=\{\mu\}$, or
i.2) $\exists\lambda\in\mathbf{B}^n$ such that

$$\Phi^{-1}(\Phi(\lambda))=[\lambda,\Phi(\lambda))\ \textit{or}\ \Phi^{-1}(\Phi(\lambda))=[\lambda,\Phi(\lambda)], \tag{2.4}$$

$\mu\in(\lambda,\Phi(\lambda))$ and we have:

$$\{\mu\}\neq\mu_\Phi^-=[\lambda,\mu]=\{\lambda\oplus\underset{i\in A}{\Xi}\varepsilon^i|A\subset\lambda\boxplus\mu\},$$

$$\mu_\Phi^+=[\mu,\Phi(\lambda)]=\{\lambda\oplus\underset{i\in A}{\Xi}\varepsilon^i|\lambda\boxplus\mu\subset A\subset\Phi_\lambda\};$$

ii) $\Phi^{-1}(\mu)=\{\mu\}$, $\mu_\Phi^-=\{\mu\},\Phi(\mu)=\mu,\mu_\Phi^+=\{\mu\}$;
iii) $\exists\lambda\in\mathbf{B}^n,\Phi^{-1}(\mu)=[\lambda,\mu)$, where $\lambda\neq\mu,\{\mu\}\neq\mu_\Phi^-=[\lambda,\mu],\Phi(\mu)\neq\mu,\mu_\Phi^+\neq\{\mu\}$;
iv) $\exists\lambda\in\mathbf{B}^n,\Phi^{-1}(\mu)=[\lambda,\mu]$, where $\lambda\neq\mu,\{\mu\}\neq\mu_\Phi^-=[\lambda,\mu],\Phi(\mu)=\mu,\mu_\Phi^+=\{\mu\}$.

Proof. i) When in (2.1) $\lambda=\mu$, it is possible that $\Phi^{-1}(\mu)=[\mu,\mu)=\varnothing$. In this situation $\Phi(\mu)\neq\mu$ and $\mu_\Phi^+=[\mu,\Phi(\mu)]\neq[\mu,\mu]=\{\mu\}$. In addition:
i.1) $\mu_\Phi^-=\{\mu\}$ is a possibility, otherwise
i.2) $\mu_\Phi^-\neq\{\mu\}$. Then $\nu\in\mathbf{B}^n,\nu\neq\mu$ and $\omega\in\mathbf{B}^n$ exist such that $\Phi^\omega(\nu)=\mu$. We denote with $\lambda\in\mathbf{B}^n$ the point that makes true

$$\Phi^{-1}(\Phi(\nu))=[\lambda,\Phi(\nu))\text{ or }\Phi^{-1}(\Phi(\nu))=[\lambda,\Phi(\nu)].$$

As $\nu,\lambda\in\Phi^{-1}(\Phi(\nu))$ we obviously have $\Phi(\lambda)=\Phi(\nu)$, wherefrom the truth of (2.4). But $\mu\in[\nu,\Phi(\nu)]$; since $\mu\neq\nu$ is a consequence of the initial supposition $\mu_\Phi^-\neq\{\mu\}$ and because $\mu=\Phi(\nu)$ is impossible, as this would imply $\Phi^{-1}(\mu)\neq\varnothing$, we get $\mu\in(\nu,\Phi(\nu))\subset(\lambda,\Phi(\lambda))$.

We prove now two relations:

$$\lambda \boxplus \mu \cap \Phi_\mu = \varnothing, \tag{2.5}$$

$$\lambda \boxplus \mu \cup \Phi_\mu = \Phi_\lambda. \tag{2.6}$$

We suppose against all reason the falsity of (2.5), i.e. $i \in \lambda \boxplus \mu \cap \Phi_\mu$ exists, thus $\lambda_i \oplus \mu_i = \mu_i \oplus \Phi_i(\mu) = 1$, wherefrom $\lambda_i \oplus \Phi_i(\mu) = \lambda_i \oplus \Phi_i(\lambda) = 0$, i.e. $i \notin \Phi_\lambda$. But $i \in \Phi_\mu$ and $\Phi_\mu \subset \Phi_\lambda$ from Theorem 2.2, contradiction.

We show $\lambda \boxplus \mu \cup \Phi_\mu \subset \Phi_\lambda$: $\lambda \boxplus \mu \subset \Phi_\lambda$ follows from $\mu \in (\lambda, \Phi(\lambda))$ and $\Phi_\mu \subset \Phi_\lambda$ is clear.

We prove $\Phi_\lambda \subset \lambda \boxplus \mu \cup \Phi_\mu$ and let $i \in \Phi_\lambda$, meaning that $\lambda_i \oplus \Phi_i(\lambda) = 1$. Two possibilities exist: if $\lambda_i \oplus \mu_i = 1$, then $i \in \lambda \boxplus \mu$ and the inclusion holds; otherwise, if $\lambda_i \oplus \mu_i = 0$, then $\mu \oplus \underset{j\in\Phi_\mu}{\Xi} \varepsilon^j = \Phi(\mu) = \Phi(\lambda) = \lambda \oplus \underset{j\in\Phi_\lambda}{\Xi} \varepsilon^j$ gives $\mu \oplus \lambda = \underset{j\in\Phi_\mu \Delta \Phi_\lambda}{\Xi} \varepsilon^j = \underset{j\in\Phi_\lambda \smallsetminus \Phi_\mu}{\Xi} \varepsilon^j$ and from $i \in \Phi_\lambda, i \notin \Phi_\lambda \smallsetminus \Phi_\mu$ we infer $i \in \Phi_\mu$. (2.5), (2.6) are proved.

At this moment we can compute:

$$[\mu, \Phi(\lambda)] = [\mu, \Phi(\mu)] = \{\mu \oplus \underset{i\in A}{\Xi} \varepsilon^i | A \subset \Phi_\mu\} = \{\lambda \oplus \underset{i\in\lambda\boxplus\mu}{\Xi} \varepsilon^i \oplus \underset{i\in A}{\Xi} \varepsilon^i | A \subset \Phi_\mu\}$$

$$= \{\lambda \oplus \underset{i\in\lambda\boxplus\mu\Delta A}{\Xi} \varepsilon^i | A \subset \Phi_\mu\} \overset{(2.5)}{=} \{\lambda \oplus \underset{i\in\lambda\boxplus\mu\cup A}{\Xi} \varepsilon^i | A \subset \Phi_\mu\}$$

$$= \{\lambda \oplus \underset{i\in A}{\Xi} \varepsilon^i | \lambda \boxplus \mu \subset A \subset \lambda \boxplus \mu \cup \Phi_\mu\} \overset{(2.6)}{=} \{\lambda \oplus \underset{i\in A}{\Xi} \varepsilon^i | \lambda \boxplus \mu \subset A \subset \Phi_\lambda\}.$$

We prove $[\lambda, \mu] \subset \mu_\Phi^-$. We take an arbitrary $\nu \in [\lambda, \mu] \subset [\lambda, \Phi(\lambda))$ for which $\lambda \boxplus \nu \subset \lambda \boxplus \mu$ and $\nu = \lambda \oplus \underset{i\in\lambda\boxplus\nu}{\Xi} \varepsilon^i$. We have $\Phi(\nu) = \Phi(\lambda)$ and let $\omega \in \mathbf{B}^n$. We infer, as $\lambda \boxplus \nu \cap \Phi_\nu = \varnothing, \lambda \boxplus \nu \cup \Phi_\nu = \Phi_\lambda$ (similarly with (2.5),(2.6)):

$$\Phi^\omega(\nu) = \underset{i\in\{1,\dots,n\}}{\Xi}((1 \oplus \omega_i)\nu_i \oplus \omega_i \Phi_i(\nu))\varepsilon^i = \underset{i\in\{1,\dots,n\}}{\Xi} \nu_i \varepsilon^i \oplus \underset{i\in\{1,\dots,n\}}{\Xi} \omega_i(\nu_i \oplus \Phi_i(\nu))\varepsilon^i$$

$$= \nu \oplus \underset{i\in\{j|j\in\{1,\dots,n\},\omega_j=1\}\cap\Phi_\nu}{\Xi} \varepsilon^i = \lambda \oplus \underset{i\in\lambda\boxplus\nu}{\Xi} \varepsilon^i \oplus \underset{i\in\{j|j\in\{1,\dots,n\},\omega_j=1\}\cap\Phi_\nu}{\Xi} \varepsilon^i$$

$$= \lambda \oplus \underset{i\in\lambda\boxplus\nu\Delta(\{j|j\in\{1,\dots,n\},\omega_j=1\}\cap\Phi_\nu)}{\Xi} \varepsilon^i = \lambda \oplus \underset{i\in\lambda\boxplus\nu\cup(\{j|j\in\{1,\dots,n\},\omega_j=1\}\cap\Phi_\nu)}{\Xi} \varepsilon^i.$$

We define $\omega_j = \begin{cases} 1, if\ j \in \lambda \boxplus \mu, \\ 0, else \end{cases}, j \in \{1, ..., n\}$. Then, as far as $\lambda \boxplus \nu \subset \lambda \boxplus \mu \subset \Phi_\lambda$, we get:

$$\lambda \boxplus \nu \cup (\{j|j \in \{1, ..., n\}, \omega_j = 1\} \cap \Phi_\nu) = \lambda \boxplus \nu \cup (\lambda \boxplus \mu \cap \Phi_\nu)$$

$$= (\lambda \boxplus \nu \cup \lambda \boxplus \mu) \cap (\lambda \boxplus \nu \cup \Phi_\nu) = \lambda \boxplus \mu \cap \Phi_\lambda = \lambda \boxplus \mu,$$

therefore

$$\Phi^\omega(\nu) = \lambda \oplus \underset{i\in\lambda\boxplus\mu}{\Xi} \varepsilon^i = \mu.$$

It results that $\nu \in \mu_\Phi^-$.

We prove $\mu_\Phi^- \subset [\lambda, \mu]$. We assume *ad absurdum* that there exist $\nu \notin [\lambda, \mu]$ and $\omega \in \mathbf{B}^n$, which satisfy $\Phi^\omega(\nu) = \mu$. Then $\nu_\Phi^+ = [\nu, \Phi(\nu)], \mu \neq \nu, \mu \neq \Phi(\nu)$ ($\mu = \Phi(\nu)$ gives the contradiction $\Phi^{-1}(\mu) \neq \varnothing$) imply $\mu \in (\nu, \Phi(\nu))$. But $\Phi(\lambda) \overset{gtcpo}{=} \Phi(\mu) \overset{gtcpo}{=} \Phi(\nu)$, hence $\nu \in \Phi^{-1}(\Phi(\lambda))$, therefore $(\nu, \Phi(\nu)) \subset (\lambda, \Phi(\lambda))$. We get $\nu \in [\lambda, \Phi(\lambda)) \smallsetminus [\lambda, \mu]$ and $\lambda \boxplus \nu \subset \Phi_\lambda$ fulfills $\nu = \lambda \oplus \underset{i \in \lambda \boxplus \nu}{\Xi} \varepsilon^i$, not $(\lambda \boxplus \nu \subset \lambda \boxplus \mu)$. We infer like previously:

$$\Phi^\omega(\nu) = \lambda \oplus \underset{i \in \lambda \boxplus \nu \cup (\{j | j \in \{1,...,n\}, \omega_j = 1\} \cap \Phi_\nu)}{\Xi} \varepsilon^i.$$

The equation $\Phi^\omega(\nu) = \mu$ holds only if the equation

$$\lambda \boxplus \nu \cup (\{j | j \in \{1, ..., n\}, \omega_j = 1\} \cap \Phi_\nu) = \lambda \boxplus \mu$$

holds i.e. only if $\lambda \boxplus \nu \subset \lambda \boxplus \mu$. Since we know already that not $(\lambda \boxplus \nu \subset \lambda \boxplus \mu)$, we have obtained a contradiction.

ii) We suppose that in (2.1) we have $\lambda = \mu$ and $\Phi^{-1}(\mu) = [\mu, \mu] = \{\mu\}$. $\mu \in \mu_\Phi^-$ and we suppose against all reason the existence of $\nu \in \mathbf{B}^n, \nu \neq \mu$ and $\omega \in \mathbf{B}^n$ such that $\Phi^\omega(\nu) = \mu$. As $\mu \neq \Phi(\nu)$ ($\mu = \Phi(\nu)$ gives the contradiction $\nu \in \Phi^{-1}(\mu)$), we get $\mu \in (\nu, \Phi(\nu))$ hence $\Phi^{-1}(\mu) = \varnothing$, contradiction again. We have proved that $\mu_\Phi^- = \{\mu\}$. Obviously $\mu_\Phi^+ = [\mu, \Phi(\mu)] = [\mu, \mu] = \{\mu\}$.

iii) We suppose that in (2.1) $\lambda \neq \mu$ and $\Phi^{-1}(\mu) = [\lambda, \mu)$, therefore $\Phi(\mu) \neq \mu$ and $\mu_\Phi^+ = [\mu, \Phi(\mu)] \neq [\mu, \mu] = \{\mu\}$. We must prove that $\mu_\Phi^- = [\lambda, \mu]$.

$[\lambda, \mu) \subset \mu_\Phi^-$ is obvious, together with $\mu \in \mu_\Phi^-$.

In order to prove $\mu_\Phi^- \subset [\lambda, \mu]$, we suppose against all reason the existence of $\nu \notin [\lambda, \mu]$ and $\omega \in \mathbf{B}^n$ with $\Phi^\omega(\nu) = \mu$, therefore $\mu \in [\nu, \Phi(\nu)]$. Since $\mu \neq \nu$ and $\mu \neq \Phi(\nu)$ ($\mu = \Phi(\nu)$ implies the contradiction $\nu \in \Phi^{-1}(\mu) = [\lambda, \mu)$), we have $\mu \in (\nu, \Phi(\nu))$, i.e. $\Phi^{-1}(\mu) = \varnothing$, contradiction.

iv) We have the possibility in (2.1) that $\lambda \neq \mu$ and $\Phi^{-1}(\mu) = [\lambda, \mu]$. We infer $\Phi(\mu) = \mu$ and $\mu_\Phi^+ = [\mu, \Phi(\mu)] = [\mu, \mu] = \{\mu\}$. The fact that $\mu_\Phi^- = [\lambda, \mu]$ is proved like at iii). □

Definition 2.3. Let Φ. A point $\mu \in \mathbf{B}^n$ is called

a) *isolated fixed point* if $\mu_\Phi^- = \{\mu\}, \mu_\Phi^+ = \{\mu\}$ (previous case ii));

b) *source* if $\mu_\Phi^- = \{\mu\}, \mu_\Phi^+ \neq \{\mu\}$ (previous case i.1));

c) *sink* if $\mu_\Phi^- \neq \{\mu\}, \mu_\Phi^+ = \{\mu\}$ (previous case iv));

d) *transient point* if $\mu_\Phi^- \neq \{\mu\}, \mu_\Phi^+ \neq \{\mu\}$, which can be either *synchronous*, when $\Phi^{-1}(\mu) \neq \varnothing$ (previous case iii)), or *asynchronous*, when $\Phi^{-1}(\mu) = \varnothing$ (previous case i.2)).

Corollary 2.4. *If Φ fulfills sgtcpo, for any $\mu \in \mathbf{B}^n$, a unique $\lambda \in \mathbf{B}^n$ exists such that $\mu_\Phi^- = [\lambda, \mu]$.*

Proof. This follows from Theorem 2.3. □

Theorem 2.5. *We suppose that Φ fulfills sgtcpo and we define $\Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \Psi(\mu) = \lambda$, where λ is the unique one with $\mu_\Phi^- = [\lambda, \mu]$. Then Ψ is the time-reversed symmetrical function of Φ and it fulfills sgtcpo.*

Proof. We fix an arbitrary $\mu \in \mathbf{B}^n$, thus $\lambda \in \mathbf{B}^n$ is uniquely fixed itself. We see that

$$\mu_\Psi^+ = [\mu, \Psi(\mu)] = [\mu, \lambda] = \mu_\Phi^- \tag{2.7}$$

and Theorem 2.3 shows the existence of several possibilities.

j) Case $\Phi^{-1}(\mu) = \varnothing, \Phi(\mu) \neq \mu, \mu_\Phi^+ \neq \{\mu\}$,

j.1) Case $\mu_\Phi^- = \{\mu\} = [\mu, \mu], \Psi(\mu) = \mu$.

We claim the truth of one of

$$\Phi^{-1}(\Phi(\mu)) = [\mu, \Phi(\mu)), \tag{2.8}$$

$$\Phi^{-1}(\Phi(\mu)) = [\mu, \Phi(\mu)]. \tag{2.9}$$

For any $\nu \in (\mu, \Phi(\mu))$, Theorem 2.3 i.2) shows that $\nu_\Phi^- = [\mu, \nu]$, thus $\Psi(\nu) = \mu$. If (2.8) holds, we can apply Theorem 2.3 iii) and if (2.9) holds, we can apply Theorem 2.3 iv); in both situations we get $\Phi(\mu)_\Phi^- = [\mu, \Phi(\mu)]$, hence $\Psi(\Phi(\mu)) = \mu$.

If $\Phi(\Phi(\mu)) \neq \Phi(\mu)$, we prove (2.8). $[\mu, \Phi(\mu)) \subset \Phi^{-1}(\Phi(\mu))$ is a consequence of gtcpo fulfilled by Φ, see Theorem 2.1.

$\Phi^{-1}(\Phi(\mu)) \subset [\mu, \Phi(\mu))$. We suppose against all reason that this is not true, i.e. $\nu \notin [\mu, \Phi(\mu))$ exists such that $\Phi(\nu) = \Phi(\mu)$. We have obtained the existence of $\delta \in \mathbf{B}^n$ with $\Phi^{-1}(\Phi(\mu)) = [\delta, \Phi(\mu))$ and $\mu \in (\delta, \Phi(\delta))$. From Theorem 2.3 i.2) we infer however that $\mu_\Phi^- = [\delta, \mu] \neq \{\mu\}$, contradiction. (2.8) is proved.

If $\Phi(\Phi(\mu)) = \Phi(\mu)$, relation (2.9) is proved similarly.

We have shown that $[\mu, \Phi(\mu)] \subset \Psi^{-1}(\mu)$. We state that

$$\Psi^{-1}(\mu) = [\mu, \Phi(\mu)] \tag{2.10}$$

thus the inclusion $\Psi^{-1}(\mu) \subset [\mu, \Phi(\mu)]$ must be proved. We suppose against all reason the existence of $\nu \notin [\mu, \Phi(\mu)]$ such that $\Psi(\nu) = \mu$, meaning from the definition of Ψ that $\nu_\Phi^- = [\mu, \nu]$, i.e. $\nu \in [\mu, \Phi(\mu)]$, contradiction. (2.10) is proved. We show that

$$\mu_\Psi^- = [\mu, \Phi(\mu)]. \tag{2.11}$$

The inclusion $[\mu, \Phi(\mu)] \subset \mu_\Psi^-$ is clear from (2.10), we prove $\mu_\Psi^- \subset [\mu, \Phi(\mu)]$. Let against all reason $\nu \notin [\mu, \Phi(\mu)]$ and $\omega \in \mathbf{B}^n$ with $\Psi^\omega(\nu) = \mu$, i.e. $\mu \in [\nu, \Psi(\nu)]$. From the definition of Ψ we infer $\nu_\Phi^- = [\Psi(\nu), \nu]$ and then $\delta \in \mathbf{B}^n$ exists with $\Phi^\delta(\mu) = \nu$, i.e. $\nu \in [\mu, \Phi(\mu)]$, contradiction. (2.11) is proved, hence

$$\mu_\Psi^- = [\mu, \Phi(\mu)] = \mu_\Phi^+.$$

The property

$$\forall \nu \in (\mu, \Psi(\mu)), \Psi^{-1}(\nu) = \varnothing \tag{2.12}$$

is trivially satisfied, as far as $(\mu, \Psi(\mu)) = (\mu, \mu) = \varnothing$.

j.2) Case $\exists \lambda \in \mathbf{B}^n$ such that

$$\Phi^{-1}(\Phi(\lambda)) = [\lambda, \Phi(\lambda)) \text{ or } \Phi^{-1}(\Phi(\lambda)) = [\lambda, \Phi(\lambda)],$$

$\mu \in (\lambda, \Phi(\lambda)), \mu_{\Phi}^{-} = [\lambda, \mu]$. In this case $\Psi(\mu) = \lambda$ and $\forall \nu \in (\lambda, \Phi(\lambda))$ we have similarly $\Psi(\nu) = \lambda$. If $\Phi^{-1}(\Phi(\lambda)) = [\lambda, \Phi(\lambda))$ we use Theorem 2.3 iii) and if $\Phi^{-1}(\Phi(\lambda)) = [\lambda, \Phi(\lambda)]$ we use Theorem 2.3 iv) to infer that $\Phi(\lambda)_{\Phi}^{-} = [\lambda, \Phi(\lambda)]$, hence $\Psi(\Phi(\lambda)) = \lambda$.

We prove that

$$\Psi^{-1}(\mu) = \varnothing \tag{2.13}$$

and let us suppose, against all reason, that this is false, i.e. $\nu \in \mathbf{B}^n$ exists with $\Psi(\nu) = \mu$, i.e. $\nu_{\Phi}^{-} = [\mu, \nu]$, thus $\nu \in [\mu, \Phi(\mu)]$. If $\nu = \Phi(\mu)$, then $\Psi(\nu) = \lambda \neq \mu$, contradiction. We infer that $\nu \in [\mu, \Phi(\mu))$. But $\Phi(\nu) \overset{gtcpo}{=} \Phi(\mu) \overset{gtcpo}{=} \Phi(\lambda)$, thus $\nu \in [\mu, \Phi(\lambda)) \subset (\lambda, \Phi(\lambda))$, wherefrom $\Psi(\nu) = \lambda \neq \mu$, contradiction again. (2.13) is proved.

We show that

$$\mu_{\Psi}^{-} = [\mu, \Phi(\mu)] \tag{2.14}$$

and we prove first $[\mu, \Phi(\mu)] \subset \mu_{\Psi}^{-}$. Let $\nu \in [\mu, \Phi(\mu)]$ arbitrary, thus A exists ($A = \lambda \boxplus \nu$) with $\lambda \boxplus \mu \subset A \subset \Phi_{\lambda}$, see Theorem 2.3 i.2), such that $\nu = \lambda \oplus \underset{i \in A}{\Xi} \varepsilon^i$. We have $\Psi(\nu) = \lambda$, thus the equation $\Psi^{\omega}(\nu) = \mu (= \lambda \oplus \underset{i \in \lambda \boxplus \mu}{\Xi} \varepsilon^i)$ with the unknown $\omega \in \mathbf{B}^n$ has the solution $\forall j \in \{1, ..., n\}, \omega_j = \begin{cases} 1, \text{ if } j \in A \setminus \lambda \boxplus \mu, \\ 0, \text{ otherwise.} \end{cases}$ Indeed, we have for any j :

$$\Psi_j^{\omega}(\nu) = \begin{cases} \Psi_j(\nu), \text{ if } \omega_j = 1 \\ \nu_j, \text{ otherwise} \end{cases} = \begin{cases} \Psi_j(\nu), \text{ if } j \in A \setminus \lambda \boxplus \mu \\ \nu_j, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \nu_j \oplus 1, \; if \; j \in A \setminus \lambda \boxplus \mu, \\ \nu_j, \; otherwise \end{cases} = \begin{cases} \nu_j \oplus 1, \; if \; j \in \nu \boxplus \mu, \\ \nu_j, \; otherwise \end{cases} = \mu_j.$$

We prove now $\mu_{\Psi}^{-} \subset [\mu, \Psi(\mu)]$. Let against all reason $\nu \notin [\mu, \Phi(\mu)]$ and $\omega \in \mathbf{B}^n$ such that $\Psi^{\omega}(\nu) = \mu$, meaning that $\mu \in [\nu, \Psi(\nu)]$. The definition of Ψ implies $\nu_{\Phi}^{-} = [\Psi(\nu), \nu]$, therefore $\delta \in \mathbf{B}^n$ exists with $\Phi^{\delta}(\mu) = \nu$. We have obtained the contradiction $\nu \in [\mu, \Phi(\mu)]$. (2.14) is proved, thus

$$\mu_{\Psi}^{-} = [\mu, \Phi(\mu)] = \mu_{\Phi}^{+}.$$

We claim that property (2.12) holds. Indeed, $(\mu, \Psi(\mu)) = (\mu, \lambda)$ and for any $\nu \in (\mu, \lambda)$, we get $\nu \in (\lambda, \Phi(\lambda))$. We prove that

$$\Psi^{-1}(\nu) = \varnothing. \tag{2.15}$$

Let us suppose against all reason that $\omega \in \mathbf{B}^n$ exists with $\Psi(\omega) = \nu$, thus $\omega_{\Phi}^{-} = [\nu, \omega]$, in other words $\omega \in [\nu, \Phi(\nu)]$. If $\omega = \Phi(\nu) (= \Phi(\lambda))$, then $\Psi(\omega) = \Psi(\Phi(\lambda)) = \lambda \neq \nu$, contradiction. We infer from here that $\omega \in [\nu, \Phi(\nu))$. From gtcpo we obtain $\Phi(\omega) = \Phi(\nu) = \Phi(\lambda)$, i.e. $\omega \in [\nu, \Phi(\lambda)) \subset (\lambda, \Phi(\lambda))$, therefore $\Psi(\omega) = \lambda \neq \nu$, contradiction. Statement (2.15) holds.

jj) Case $\Phi^{-1}(\mu) = \{\mu\}, \mu_{\Phi}^{-} = \{\mu\}, \Phi(\mu) = \mu, \mu_{\Phi}^{+} = \{\mu\}, \Psi(\mu) = \mu$.

We show that

$$\Psi^{-1}(\mu) = \{\mu\} \tag{2.16}$$

and let us suppose, against all reason that $\nu \in \mathbf{B}^n$ exists, $\nu \neq \mu$, such that $\Psi(\nu) = \mu$. This has its origin in $\nu_\Phi^- = [\mu, \nu]$. We have obtained the contradiction $\nu \in (\mu, \Phi(\mu)] = (\mu, \mu] = \varnothing$, showing the truth of (2.16). We prove that

$$\mu_\Psi^- = \{\mu\}. \tag{2.17}$$

Let against all reason $\nu \in \mathbf{B}^n, \nu \neq \mu$ and $\omega \in \mathbf{B}^n$ with $\Psi^\omega(\nu) = \mu$. Since $\Psi(\nu) \neq \mu$ (otherwise $\nu \in \Psi^{-1}(\mu) = \{\mu\}$, contradiction), we get $\mu \in (\nu, \Psi(\nu))$. But the definition of Ψ shows that $\nu_\Phi^- = [\Psi(\nu), \nu]$, wherefrom we get the existence of $\delta \in \mathbf{B}^n$ with $\Phi^\delta(\mu) = \nu$. We have obtained the contradiction $\mu = \Phi(\mu) = \Phi^\delta(\mu) = \nu$, showing the truth of (2.17). We infer

$$\mu_\Psi^- = \{\mu\} = \mu_\Phi^+.$$

Property (2.12) is trivially fulfilled since $(\mu, \Psi(\mu)) = (\mu, \mu) = \varnothing$.

jjj) Case $\exists\lambda \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda, \mu)$, where $\lambda \neq \mu, \mu_\Phi^- = [\lambda, \mu], \Phi(\mu) \neq \mu$ and $\Psi(\mu) = \lambda$.

Let $\nu \in (\mu, \Phi(\mu))$ arbitrary. The hypothesis of Theorem 2.3 i.2) holds, since $\Phi^{-1}(\nu) = \varnothing$ (from sgtcpo), $\nu \neq \Phi(\nu) = \Phi(\mu)$ (from gtcpo) and (2.4) is true under the form $\Phi^{-1}(\mu) = [\lambda, \mu)$. We conclude that $\nu_\Phi^- = [\mu, \nu]$, wherefrom $\Psi(\nu) = \mu$. If $\Phi^{-1}(\Phi(\mu)) = [\mu, \Phi(\mu))$, we can use Theorem 2.3 iii) and if $\Phi^{-1}(\Phi(\mu)) = [\mu, \Phi(\mu)]$, we can use Theorem 2.3 iv); we infer in both situations $\Phi(\mu)_\Phi^- = [\mu, \Phi(\mu)]$, thus $\Psi(\Phi(\mu)) = \mu$. We have proved that $(\mu, \Phi(\mu)] \subset \Psi^{-1}(\mu)$ and we prove now that

$$\Psi^{-1}(\mu) = [\Phi(\mu), \mu). \tag{2.18}$$

In order to prove the inclusion $\Psi^{-1}(\mu) \subset [\Phi(\mu), \mu)$, we suppose against all reason its falsity; then $\nu \notin [\Phi(\mu), \mu)$ exists such that $\Psi(\nu) = \mu$. We have $\nu_\Phi^- = [\mu, \nu]$, i.e. $\nu \in [\mu, \Phi(\mu)]$. The only possibility is $\nu = \mu$, but this implies the contradiction $\mu_\Phi^- = \{\mu\}$. Statement (2.18) is proved.

We have also proved that $[\Phi(\mu), \mu] \subset \mu_\Psi^-$ and we prove now the equality

$$\mu_\Psi^- = [\Phi(\mu), \mu]. \tag{2.19}$$

Let us suppose, against all reason, that $\mu_\Psi^- \subset [\Phi(\mu), \mu]$ is false; then $\nu \notin [\Phi(\mu), \mu]$ and $\omega \in \mathbf{B}^n$ exist with $\Psi^\omega(\nu) = \mu$, in other words $\mu \in [\nu, \Psi(\nu)]$. We have $\mu \neq \nu$ from the hypothesis and we have also $\mu \neq \Psi(\nu)$ (otherwise we get $\nu \in \Psi^{-1}(\mu) = (\mu, \Phi(\mu)]$, contradiction), therefore $\mu \in (\nu, \Psi(\nu))$. The definition of Ψ in ν gives $\nu_\Phi^- = [\Psi(\nu), \nu]$, and as $\mu \in (\Psi(\nu), \nu)$, we infer the existence of $\delta \in \mathbf{B}^n$ with $\Phi^\delta(\mu) = \nu$, hence $\nu \in [\mu, \Phi(\mu)]$, contradiction again. (2.19) is true, thus

$$\mu_\Psi^- = [\Phi(\mu), \mu] = \mu_\Phi^+.$$

In order to prove the truth of (2.12), we fix $\nu \in (\lambda, \mu) = (\lambda, \Phi(\lambda))$ arbitrary. The hypothesis of Theorem 2.3 i.2) is fulfilled, because $\Phi^{-1}(\nu) = \varnothing$ (from sgtcpo), $\nu \neq \Phi(\nu) = \mu$ (from gtcpo) and (2.4) takes place under the form $\Phi^{-1}(\Phi(\lambda)) = \Phi^{-1}(\mu) = [\lambda, \mu)$. We infer that $\nu_\Phi^- = [\lambda, \nu]$, thus $\Psi(\nu) = \lambda$.

We continue to keep $\nu \in (\mu, \Psi(\mu)) = (\mu, \lambda)$ arbitrary, fixed and we show that $\Psi^{-1}(\nu) = \varnothing$ holds. Let us suppose against all reason the existence of $\delta \in \mathbf{B}^n$ such that $\Psi(\delta) = \nu$, wherefrom $\delta_\Phi^- = [\nu, \delta]$ i.e. $\exists\omega \in \mathbf{B}^n$ with $\Phi^\omega(\nu) = \delta$. But in this situation $\delta \in [\nu, \Phi(\nu)] = [\nu, \mu] \subset (\lambda, \mu]$, hence $\Psi(\delta) = \lambda$. The last conclusion is in contradiction with the supposition that $\nu = \Psi(\delta) \in (\mu, \lambda)$. (2.12) is proved.

jv) $\exists \lambda \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda, \mu]$, where $\lambda \neq \mu, \mu_\Phi^- = [\lambda, \mu], \Phi(\mu) = \mu, \Psi(\mu) = \lambda$.

We prove

$$\Psi^{-1}(\mu) = \varnothing \tag{2.20}$$

and let us suppose that this is not true, i.e. $\nu \in \mathbf{B}^n$ exists with $\Psi(\nu) = \mu$, hence $\nu_\Phi^- = [\mu, \nu]$, wherefrom $\nu \in [\mu, \Phi(\mu)] = \{\mu\}$. But $\Psi(\mu) = \lambda$, contradiction showing the truth of (2.20). We prove that

$$\mu_\Psi^- = \{\mu\}. \tag{2.21}$$

If, against all reason, (2.21) is false, then $\nu \in \mathbf{B}^n, \nu \neq \mu$ and $\omega \in \mathbf{B}^n$ exist with $\Psi^\omega(\nu) = \mu$, i.e. $\mu \in [\nu, \Psi(\nu)]$. We have from the definition of Ψ in $\nu : \nu_\Phi^- = [\Psi(\nu), \nu]$, in other words $\exists \delta \in \mathbf{B}^n$ with $\Phi^\delta(\mu) = \nu$. But $\mu = \Phi(\mu) = \Phi^\delta(\mu) = \nu$ represents a contradiction. (2.21) holds and its truth implies:

$$\mu_\Psi^- = \{\mu\} = \mu_\Phi^+.$$

In order to prove the satisfaction of (2.12), we notice that $\forall \nu \in (\lambda, \mu), \nu_\Phi^- = [\lambda, \nu]$, from Theorem 2.3 i.2), thus $\Psi(\nu) = \lambda$. Let an arbitrary $\nu \in (\mu, \Psi(\mu)) = (\mu, \lambda)$. We suppose against all reason the existence of $\delta \in \mathbf{B}^n$ with $\Psi(\delta) = \nu$, thus $\delta_\Phi^- = [\nu, \delta]$ and $\delta \in [\nu, \Phi(\nu)] = [\nu, \mu] \subset [\mu, \lambda)$. But $\Psi(\delta) = \lambda$, representing a contradiction. (2.12) holds.

We finally refer to Definition 2.1 of time-reversal symmetry and Definition 2.2 of sgtcpo and we see that these properties are true in all the previous cases. □

Example 2.4. The identity function $1_{\mathbf{B}^n} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ fulfills sgtcpo and it is the time-reversed symmetrical of itself; we get $\forall \mu \in \mathbf{B}^n, \mu_{1_{\mathbf{B}^n}}^+ = \mu_{1_{\mathbf{B}^n}}^- = \{\mu\}$.

Example 2.5. Let $\omega \in \mathbf{B}^n$. We define the constant functions $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ in the following way: $\forall \mu \in \mathbf{B}^n, \Phi(\mu) = \omega$ and $\forall \mu \in \mathbf{B}^n, \Psi(\mu) = \overline{\omega} = (\overline{\omega}_1, ..., \overline{\omega}_n)$. They fulfill sgtcpo and we have:

$$\overline{\omega}_\Phi^- = \{\overline{\omega}\} = \overline{\omega}_\Psi^+,\ \overline{\omega}_\Phi^+ = \mathbf{B}^n = \overline{\omega}_\Psi^-,$$

$$\omega_\Phi^- = \mathbf{B}^n = \omega_\Psi^+,\ \omega_\Phi^+ = \{\omega\} = \omega_\Psi^-,$$

$$\forall \mu \in (\overline{\omega}, \omega), \mu_\Phi^- = [\overline{\omega}, \mu] = \mu_\Psi^+, \mu_\Phi^+ = [\mu, \omega] = \mu_\Psi^-.$$

Example 2.6. The following functions $\Phi, \Psi : \mathbf{B}^3 \longrightarrow \mathbf{B}^3$, $\forall \mu \in \mathbf{B}^3, \Phi(\mu) = (\mu_1, \overline{\mu_1} \cup \mu_1\mu_2, \overline{\mu_1} \cup \mu_1\mu_3)$ and $\forall \mu \in \mathbf{B}^3, \Psi(\mu) = (\mu_1, \mu_1\mu_2, \mu_1\mu_3)$ fulfill sgtcpo and they are time-reversed symmetrical. We have:

$$\forall \mu \in \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}, \mu_\Phi^- = \mu_\Psi^+ = \{\mu\}, \mu_\Phi^+ = \mu_\Psi^- = \{\mu\},$$

$$(0,0,0)_\Phi^+ = (0,0,0)_\Psi^- = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\} \text{ etc.}$$

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