Ordinary and fractional mathematical models on language competition and bilingualism

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Abstract. A general view on mathematical models of language competition and bilingualism, in which ordinary and fractional differential equations were used, is given. Equilibria and stability are investigated for models of fractional differential systems with two components and three components. Also applications of the fractional case is given in the sense of numerical simulation.

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1 Introduction

Many of living languages in the world encounter risk of extinction. It is easy enough to see that this extinction deeply affect cultural diversity. To deal with the language extinction and preserve the diversity are a significant issue to be focused on, both practically and theoretically. In this respect, we zoom in on examining language competition and bilingualism in a mathematical point of view. We firstly present a general view on existing mathematical models on language competition and bilingualism. Then, we investigate the dynamics of fractional models in this context.

2 Abrams-Strogatz model

A pioneering study in modelling language competition is Abrams-Strogatz model [1], which is constructed as a first order differential rate equation

$$\frac{dx}{dt} = yP_{YX}(x, s_x) - xP_{XY}(y, s_y), \tag{2.1}$$

where x is the fraction of the population speaking language X, and y is of language Y (provided that $x + y = 1$), with the following assumptions.

1. The population consists of two monolingual groups competing with each other for speakers.

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2. Conversion from one language to the other is based on the attractiveness of the competing language.
3. Attractiveness of a language increases with both its number of speakers and its perceived status.
4. No one will adopt a language that has no speakers or no status.
5. Bilingual population is not considered.

s_x and s_y represent the measures of the relative status of X and Y, respectively, and both are in the interval $[0,1]$. The conversion from Y to X per unit time is represented by $P_{YX}(x,s_x)$ and the conversion from X to Y is given by $P_{XY}(y,s_y)$ likewise. As a result of an assumption that $P_{YX}(x,s_x) = cs_x x^a$ and $P_{XY}(y,s_y) = c(1-s_x)(1-x)^a$, where the parameter a was unexpectedly found with $a = 1.31 \pm 0.25$ (mean $\pm$ standard deviation) and gives the relation between the attractiveness of a language and the number of its speakers, the equation (2.1) takes the following form.

$$\frac{dx}{dt} = c(1-x)s_x x^a - cx(1-s_x)(1-x)^a. \tag{2.2}$$

The parameter c represents some social factors concerning the languages, such as rate of interactions effecting the competition between the languages, cultural or political motives affecting or inducing learning the second language, etc.

Since bilingual population is ignored in the Abrams-Strogatz model, in [2] it was modified by introducing a parameter b to represent bilingual population B, provided that $x + y + b = 1$. The modified version of the model is

$$\begin{aligned} \frac{dx}{dt} &= c\left[(1-x)(1-k)s_x(1-y)^a - x(1-s_x)(1-x)^a\right] \\ \frac{dy}{dt} &= c\left[(1-y)(1-k)(1-s_x)(1-x)^a - ys_x(1-y)^a\right], \end{aligned}$$

where the parameter k $(0 \le k \le 1)$ reflects the similarity of two languages.

3 Baggs-Freedman model

Another leading mathematical model within this scope is Baggs-Freedman model [3], which consists of a system of two autonomous ordinary differential equations as follows.

$$\begin{cases} x'(t) = (B_1 - D_1)x(t) - L_1 x^2(t) - \frac{\alpha x(t)y(t)}{1+x(t)} + P_1 B_2 y(t) \\ y'(t) = (P_2 B_2 - D_2)y(t) - L_2 y^2(t) + \frac{\alpha x(t)y(t)}{1+x(t)}, \end{cases} \tag{3.1}$$

where $B_i > D_i$ $(i = 1,2)$, $0 < P_1 < 1$ and $P_2 = 1 - P_1$. While P_1 represents the rate of the children of y, which enter the population as unilinguals, B_1 and $D_1 + L_1 x$ respectively are the specific birth rate and death rate of x, and similarly B_1 and $D_1 + L_1 x$ are of y. Carrying capacities of the environment of x and y is denoted by K_1 and K_2, respectively, where $K_1 = \frac{B_1 - D_1}{L_1}$, $K_2 = \frac{B_2 - D_2}{L_2}$. The parameter α denotes the conversion rate from unilingual to bilingual.

In [4], Baggs and Freedman developed a general model for the interaction of two unilingual components and one bilingual component of a population and they investigated conditions under which all three components persist and conditions under which one dominated unilingual component will become extinct.

4 Fractional models

In recent years, however, it has turned out that many phenomena in different fields can be described very successfully by the models using fractional order differential equations [5]. The fractional calculus was reasonably developed by 19th century. It was realized, only in the past few decades that these derivatives are better models to study physical phenomenon in transient state [6]. Also, fractional operators are a very natural tool to model memory-dependent phenomena [7].

Despite a few other definitions of fractional derivative, we use Caputo's definition. Because the initial conditions for fractional differential equations with Caputo derivatives are in the same form as for integer-order differential equations [6].

The definition of fractional derivative of Caputo-type is as follows.

Definition 4.1. The Caputo-type fractional derivative of order $q > 0$ for a function $f : (0, \infty) \to R$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau,$$

where $n = \lceil q \rceil$ and $\lceil . \rceil$ is the ceiling function.

Here and elsewhere Γ denotes the gamma function and $0 < q < 1$. Notice that $q = 1$ corresponds the classical first order derivative.

4.1 Unilinguals as one compartment

This model mainly based on the assumption that conversion from unilingual to bilingual does not exist, which differs from system (3.1). The population is considered as two groups; the first group consists of the unilingual speakers of either dominant language or minority language in the population, and the second group is bilinguals. That is, two unilingual groups are considered as one compartment. Imposing fractional q-order instead of the first order classical derivative, we have the following system.

$$\left\{ \begin{array}{c} D^q x(t) = (B_1 - D_1)x(t) - L_1 x^2(t) + P_1 B_2 y(t) \\ D^q y(t) = (P_2 B_2 - D_2)y(t) - L_2 y^2(t). \end{array} \right. \tag{4.1}$$

Before stability analysis, we should note the following important criterion for the stability of fractional differential equations.

If all the eigenvalues λ of the Jacobian matrix evaluated at equilibrium point satisfy the inequality $|\arg(\lambda)| > \frac{q\pi}{2}$, then the equilibrium point is asymptotically stable [8], [9].

The equilibrium points of system (4.1) with nonnegative components are $E_0(0,0)$, $E_1(K_1,0)$ and $E^*\left(\frac{1}{2}K_1+\frac{1}{2}\sqrt{K_1^2+\frac{4P_1B_2(P_2B_2-D_2)}{L_1L_2}}, \frac{P_2B_2-D_2}{L_2}\right)$.

Theorem 4.1. *$E_0(0,0)$ is unstable.*

Proof. The general Jacobian matrix $J(x,y)$ for system (4.1) is

$$J(x,y)=\begin{bmatrix} B_1-D_1-2L_1x & P_1B_2 \\ 0 & P_2B_2-D_2-2L_2y \end{bmatrix}. \tag{4.2}$$

The Jacobian matrix at E_0 is

$$J(E_0)=J(0,0)=\begin{bmatrix} B_1-D_1 & P_1B_2 \\ 0 & P_2B_2-D_2 \end{bmatrix}.$$

Solving the characteristic equation $\det(J(E_0)-\lambda I)=0$, we obtain the equation $(B_1-D_1-\lambda)(P_2B_2-D_2-\lambda)=0$, which has the roots $\lambda_1=B_1-D_1$ and $\lambda_2=P_2B_2-D_2$. Since we assume that $B_1-D_1>0$, E_0 is unstable. □

Theorem 4.2. *$E_1(K_1,0)$ is asymptotically stable, if $P_2B_2-D_2<0$.*

Proof. Following the same procedure as in the investigation of stability for E_0, we get the Jacobian matrix at E_1 as

$$J(E_1)=\begin{bmatrix} -(B_1-D_1) & P_1B_2 \\ 0 & P_2B_2-D_2 \end{bmatrix},$$

and the eigenvalues of the characteristic equation $\det(J(E_1)-\lambda I)=0$ as $\lambda_1=-(B_1-D_1)$ and $\lambda_2=P_2B_2-D_2$. Since $\lambda_1=-(B_1-D_1)<0$ and $\lambda_2=P_2B_2-D_2<0$, the equilibrium point E_1 is asymptotically stable. □

Theorem 4.3. *Let E^* exists and is unique. If $P_2B_2-D_2>0$, then E^* is asymptotically stable.*

Proof. The Jacobian matrix at E^* is

$$J(E^*)=\begin{bmatrix} -L_1\sqrt{K_1^2+\frac{4P_1B_2(P_2B_2-D_2)}{L_1L_2}} & P_1B_2 \\ 0 & -(P_2B_2-D_2) \end{bmatrix},$$

and the roots of the characteristic equation $\det(J(E^*)-\lambda I)=0$ are $\lambda_1=-L_1\times\sqrt{K_1^2+\frac{4P_1B_2(P_2B_2-D_2)}{L_1L_2}}<0$ and $\lambda_2=-(P_2B_2-D_2)<0$, since $P_2B_2-D_2>0$. Thus, E^* is asymptotically stable. □

4.2 Unilinguals as two compartments

In the previous subsection, we consider the unilinguals of dominant language and of minority language as one group. Now, we investigate the interaction between three components, that is, unilinguals of dominant language, bilinguals and one unilinguals of minority language. The situation is modeled by an autonomous system which consists of three differential equations of fractional order with the following assumptions.

1. Birth and death process are continuous.
2. Children of unilingual parents enter the population as unilinguals.
3. Children of bilingual parents may enter the population as bilinguals or unilinguals.
4. Emigration from the environment and immigration to the environment are not considered.
5. Conversion from bilingual to unilingual exists.
6. Conversion from dominant unilingual to bilingual doesn't exist.

(2.1) is a standard assumption for human populations with intermediate to large numbers of individuals. (2), (3) and (5) are also assumptions of Baggs-Freedman models. The model with emigration is studied in [4], however the model with immigration is not appropriate for this kind of models.

The assumption (6) is crucial in our model, because we assume that the need for the usage of minority languages decreases, particularly in last few decades, due to some social, political and economical factors.

The concentration of dominant unilinguals at time $t \geq 0$ is represented by $x_1(t)$; while the concentration of bilinguals and unilinguals of minority language are represented by $x_2(t)$ and $x_3(t)$, respectively. Thus, with all these assumptions and considerations, our model is

$$\begin{aligned}
D^q x_1 &= (B_1 - D_1)x_1 - L_1 x_1^2 + \left(P_1 - \frac{P_3 x_3}{x_1 + x_2 + x_3}\right) B_2 x_2 \\
D^q x_2 &= (P_2 B_2 - D_2)x_2 - L_2 x_2^2 + \frac{x_2 x_3}{1 + x_2 + x_3}\left(\alpha + \frac{\beta x_1}{1 + x_1}\right) \qquad (4.3) \\
D^q x_3 &= (B_3 - D_3)x_3 - L_3 x_3^2 - \frac{x_2 x_3}{1 + x_2 + x_3}\left(\alpha + \frac{\beta x_1}{1 + x_1}\right) \\
&\quad + \frac{P_3 x_3}{x_1 + x_2 + x_3} B_2 x_2.
\end{aligned}$$

In this system, B_i and $D_i + L_i x_i$ are the specific birth rate and death rate of x_i $(i = 1, 2, 3)$ with the assumption $B_i > D_i$. P_2, $P_1 - \frac{P_3 x_3}{x_1+x_2+x_3}$ and $\frac{P_3 x_3}{x_1+x_2+x_3}$ is the probability that a child born to bilingual parents will enter the population as a member of population x_1, x_2, and x_3, respectively, where $P_1 + P_2 = 1$, $P_3 \leq P_1$. The conversion rate of monolingual component x_3 to the bilingual component x_2, per unit time, is $\frac{1}{1+x_2+x_3}(\alpha + \frac{\beta x_1}{1+x_1})$.

The equilibrium points of system (6) are $E_0(0,0,0)$, $E_1(K_1,0,0)$, $E_3(0,0,K_3)$, $\tilde{E}(K_1, 0, K_3)$, $\hat{E}(\hat{x}_1, \hat{x}_2, 0)$ and possibly $\bar{E}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $K_i = \frac{B_i - D_i}{L_i}$ is the carrying capacity of x_i $(i = 1, 2, 3)$. Nonzero components of $\hat{E}$ are found as $\hat{x}_1 = \frac{1}{2}K_1 + \frac{1}{2}\sqrt{K_1^2 + \frac{4P_1 B_2 (P_2 B_2 - D_2)}{L_1 L_2}}$ and $\hat{x}_2 = \frac{P_2 B_2 - D_2}{L_2}$.

The Jacobian matrix $J(\tilde{E})$ is evaluated as

$$J(\tilde{E}) = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 \\ 0 & \tilde{a}_{22} & 0 \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{bmatrix},$$

where

$$\begin{aligned}
\tilde{a}_{11} &= D_1 - B_1 \\
\tilde{a}_{12} &= P_1 B_2 - \frac{P_3 B_2 K_3}{(K_1 + K_3)} \\
\tilde{a}_{22} &= P_2 B_2 - D_2 + \frac{K_3}{1 + K_1}\left(\alpha + \frac{\beta K_1}{1 + K_1}\right) \\
\tilde{a}_{32} &= -\frac{K_3}{1 + K_1}\left(\alpha + \frac{\beta K_1}{1 + K_1}\right) + \frac{P_3 B_2 K_3}{(K_1 + K_3)} \\
\tilde{a}_{33} &= D_3 - B_3
\end{aligned}$$

and the eigenvalues of $J(\tilde{E})$ are $\tilde{\lambda}_1 = D_1 - B_1$, $\tilde{\lambda}_2 = P_2 B_2 - D_2 + \frac{K_3}{1+K_1}\left(\alpha + \frac{\beta K_1}{1+K_1}\right)$ and $\tilde{\lambda}_3 = D_3 - B_3$. The Jacobian matrix of $\hat{E}$ is

$$J(\hat{E}) = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} \\ 0 & \hat{a}_{22} & \hat{a}_{23} \\ 0 & 0 & \hat{a}_{33} \end{bmatrix},$$

where

$$\begin{aligned}
\hat{a}_{11} &= -L_1 \sqrt{K_1^2 + \frac{4 P_1 B_2 (P_2 B_2 - D_2)}{L_1 L_2}} \\
\hat{a}_{12} &= P_1 B_2 \\
\hat{a}_{13} &= \frac{-P_3 B_2 \left(\frac{P_2 B_2 - D_2}{L_2}\right)}{\hat{x}_1 + \hat{x}_2} \\
\hat{a}_{22} &= -(P_2 B_2 - D_2) \\
\hat{a}_{23} &= \left(\alpha + \frac{\beta \hat{x}_1}{1 + \hat{x}_1}\right) \frac{\hat{x}_2}{1 + \hat{x}_2} \\
\hat{a}_{33} &= B_3 - D_3 - \left(\alpha + \frac{\beta \hat{x}_1}{1 + \hat{x}_1}\right) \frac{\hat{x}_2}{1 + \hat{x}_2} + \frac{P_3 B_2 \hat{x}_2}{\hat{x}_1 + \hat{x}_2},
\end{aligned}$$

and the eigenvalues of $J(\hat{E})$ are $\hat{\lambda}_1 = -L_1 \sqrt{K_1^2 + \frac{4 P_1 B_2 (P_2 B_2 - D_2)}{L_1 L_2}}$, $\hat{\lambda}_2 = -(P_2 B_2 - D_2)$ and $\hat{\lambda}_3 = B_3 - D_3 - \left(\alpha + \frac{\beta \hat{x}_1}{1+\hat{x}_1}\right) \frac{\hat{x}_2}{1+\hat{x}_2} + \frac{P_3 B_2 \hat{x}_2}{\hat{x}_1 + \hat{x}_2}$.

The existence of the positive equilibrium point $\bar{E}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ with condition $\bar{x}_i > 0$ $(i = 1, 2, 3)$ is stated in the following corollary [3], [11].

Corollary 4.4. *If $\tilde{\lambda}_2 > 0$ and $\hat{\lambda}_3 > 0$, then the positive equilibrium $\bar{E}$ exists.*

Now, we discuss the stability of $\bar{E}$. The characteristic equation $\det(J(\bar{E}) - \lambda I) = 0$ leads to the equation

$$P(\lambda) = \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0,$$

where

$$\begin{aligned}
A_1 &= -(a_{11}+a_{22}+a_{33}),\\
A_2 &= -a_{12}a_{21}+a_{11}a_{22}-a_{12}a_{31}-a_{23}a_{32}+a_{11}a_{33}+a_{22}a_{33},\\
A_3 &= a_{13}a_{31}a_{22}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33},\\
a_{11} &= B_1-D_1-2L_1\bar{x}_1,\\
a_{12} &= P_1B_2-\frac{P_3B_2\bar{x}_3(\bar{x}_1+\bar{x}_3)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2},\\
a_{13} &= -\frac{P_3B_2\bar{x}_2(\bar{x}_1+\bar{x}_2)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2},\\
a_{21} &= \frac{P_3\bar{x}_2\bar{x}_3}{(1+\bar{x}_2+\bar{x}_3)(1+\bar{x}_1)^2},\\
a_{22} &= P_2B_2-D_2-2L_2\bar{x}_2+\frac{\bar{x}_3(1+\bar{x}_3)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2}\left(\alpha+\frac{\beta\bar{x}_1}{1+\bar{x}_1}\right),\\
a_{23} &= \frac{\bar{x}_2(1+\bar{x}_2)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2}\left(\alpha+\frac{\beta\bar{x}_1}{1+\bar{x}_1}\right),\\
a_{31} &= \frac{-\beta\bar{x}_2\bar{x}_3}{(1+\bar{x}_2+\bar{x}_3)(1+\bar{x}_1)^2}-\frac{P_3B_2\bar{x}_2\bar{x}_3}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2},\\
a_{32} &= -\left(\alpha+\frac{\beta\bar{x}_1}{1+\bar{x}_1}\right)\frac{\bar{x}_3(1+\bar{x}_3)}{(1+\bar{x}_2+\bar{x}_3)^2}+\frac{P_3B_2\bar{x}_3(\bar{x}_1+\bar{x}_3)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2},\\
a_{33} &= B_3-D_3-2L_3\bar{x}_3-\frac{\bar{x}_2(1+\bar{x}_2)}{(1+\bar{x}_2+\bar{x}_3)^2}\left(\alpha+\frac{\beta\bar{x}_1}{1+\bar{x}_1}\right)+\frac{P_3B_2\bar{x}_2(\bar{x}_1+\bar{x}_2)}{(\bar{x}_1+\bar{x}_2+\bar{x}_3)^2}.
\end{aligned}$$

Corollary 4.5. *Let $D(P)$ denote the discriminant of the polynomial $P(\lambda)=\lambda^3+A_1\lambda^2+A_2\lambda+A_3$. If one of the following conditions holds, then the positive equilibrium $\bar{E}$ is asymptotically stable [8], [9], [11].*

i) $D(P)>0,\ A_1>0,\ A_3>0$ *and* $A_1A_2>A_3$.
ii) $D(P)<0,\ A_1\geq 0,\ A_2\geq 0,\ A_3>0$ *and* $q<\frac{2}{3}$.
iii) $D(P)<0,\ A_1<0,\ A_2<0$ *and* $q>\frac{2}{3}$.

5 Numerical simulations

Substituting $B_1=0.14$, $B_2=0.2$, $D_1=D_2=0.02$, $P_1=0.6$, $P_2=0.4$, $L_1=L_2=0.002$, and $q=0.9$ in system (4.1), we obtain

$$\begin{aligned}
D^{0.9}x &= 0.12x-0.002x^2+0.12y\\
D^{0.9}y &= 0.06y-0.002y^2,
\end{aligned} \tag{5.1}$$

and the positive equilibrium point is $E^*(81.9615,\ 30)$. Letting the initial conditions

$$x(0)=60,\ y(0)=15, \tag{5.2}$$

and using predictor-corrector method given in [12], which is a numerical method to solve fractional initial value problems, we have the solution of the system (5.1)-(5.2) as shown in Figure-1.

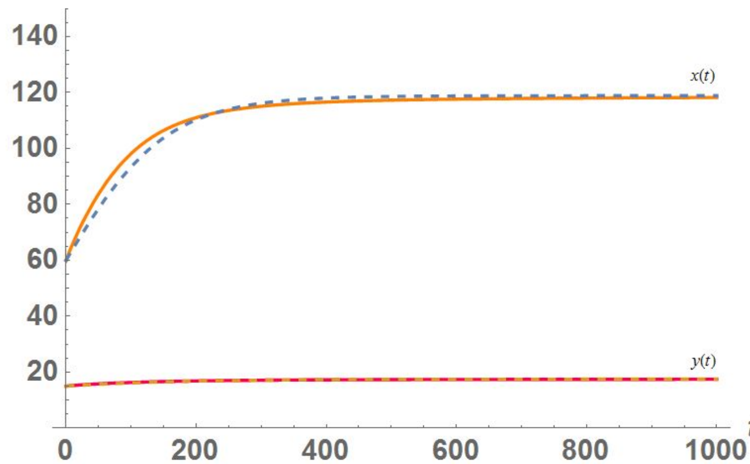

Figure 1: Solutions of the system (5.1)-(5.2). $q = 0.9$ (solid); $q = 1$ (dashed).

By substituting the following values,

$$\begin{aligned} B_1 &= 0.017, D_1 = 0.006, L_1 = 0.0001, P_1 = 0.3, \\ B_2 &= 0.020, D_2 = 0.007, L_2 = 0.0004, P_2 = 0.7, \\ B_3 &= 0.022, D_3 = 0.007, L_3 = 0.0001, P_3 = 0.2, \\ \alpha &= \beta = 0.005, \end{aligned}$$

and $q = 0.9$ in system (4.3), we obtain

$$\begin{aligned} D^{0.9}x_1 &= 0.011x_1 - 0.0001x_1^2 + \left(0.3 - \frac{0.2x_3}{x_1 + x_2 + x_3}\right) 0.02x_2 \\ D^{0.9}x_2 &= 0.007x_2 - 0.0004x_2^2 + \frac{x_2 x_3}{1 + x_2 + x_3}\left(0.005 + \frac{0.005x_1}{1 + x_1}\right) \\ D^{0.9}x_3 &= 0.015x_3 - 0.0001x_3^2 - \frac{x_2 x_3}{1 + x_2 + x_3}\left(0.005 + \frac{0.005x_1}{1 + x_1}\right) + \frac{0.004x_2 x_3}{x_1 + x_2 + x_3}. \end{aligned} \tag{5.3}$$

and the positive equilibrium point $\bar{E}(121.35,\ 23.85,\ 8.50)$, where $x_i = x_i(t)$. Let the initial conditions be

$$x_1(0) = 60,\ x_2(0) = 15, x_3(0) = 0.5. \tag{5.4}$$

Similarly using predictor-corrector method, we have the solutions of the system (5.3)-(5.4) in Figure-2.

6 Conclusions

In this paper, we have analyzed some kinds of mathematical models on bilingualism and language competition models. We have particularly focused on the fractional

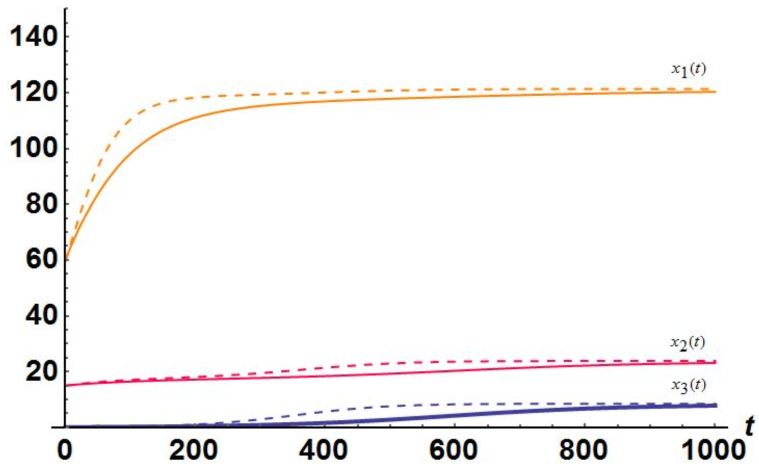

Figure 2: Solutions of the system (5.3)-(5.4). $q = 0.9$ (solid); $q = 1$ (dashed).

models by means of stability analyzes of equilibrium points of two fractional systems and supported our analyzes by numerical simulations. We have observed that fractional models of this type may be as stable as their classical counterparts, and even their approach to the equilibrium point may be faster than the classical case.

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