A classification of homothetical hypersurfaces in Euclidean spaces via Allen determinants and its applications

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Abstract. The present authors, in [3], classified the composite functions of the form $f = F(h_1(x_1) \times ... \times h_n(x_n))$ via the Allen determinants used to calculate the Allen's elasticity of substitution of production functions in microeconomics. In this paper, we adapt this classification to the homothetical hypersurfaces in the Euclidean spaces. An application for the homothetical hypersurfaces is also given.

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1 Introduction

Let $\mathcal{A}$ be a real $n \times n$ symmetric matrix and $\mathbf{a} \in \mathbb{R}^n$ column vector, $\mathbf{a} \neq 0$. The real $(n+1) \times (n+1)$ symmetric matrix given by

$$\mathcal{A}^B = \begin{pmatrix} 0 & \mathbf{a}^\intercal \\ \mathbf{a} & \mathcal{A} \end{pmatrix}$$

is referred to as a *bordered matrix* [17]. Further, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f = f(x_1, ..., x_n)$, be a twice differentiable function. Denote by (f_{x_i}) and $(f_{x_i x_j})$ the column vector of first-order partial derivatives of f and the Hessian matrix of f, respectively. Thus, the *bordered Hessian matrix* of the function f is defined as

$$\mathcal{H}^B(f) = \begin{pmatrix} 0 & (f_{x_i})^\intercal \\ (f_{x_i}) & (f_{x_i x_j}) \end{pmatrix}, \tag{1.1}$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for $1 \leq i, j \leq n$.

In the theory of constrained optimization, the bordered Hessian determinantal criterion is used to test whether an objective function has an extremun at a critical point [21]. In addition, the bordered Hessian matrices are used to analyze quasi-convexity of the functions. If the signs of the bordered principal diagonal determinants

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of the bordered Hessian matrix of a function are alternate (resp. negative), then the function is quasi-concave (resp. quasi-convex), see [5,16,17,23].

Another example is the application of the bordered Hessian matrices to elasticity of substitutions of production functions in microeconomics. Explicitly, let $f = f(x_1, ..., x_n)$ be a production function. Then the *Allen's elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is defined by

$$A_{ij}(\mathbf{x}) = -\frac{x_1 f_{x_1} + x_2 f_{x_2} + ... + x_n f_{x_n}}{x_i x_j} \frac{\left(\mathcal{H}^B(f)\right)_{ij}}{\det\left(\mathcal{H}^B(f)\right)}$$

for $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$, $i, j = 1, ..., n$, $i \neq j$. Here $\left(\mathcal{H}^B(f)\right)_{ij}$ is the co-factor of the element $f_{x_i x_j}$ in the determinant of $\mathcal{H}^B(f)$ [25,29].

The bordered Hessian matrix $\mathcal{H}^B(f)$ given by (1.1) is also called *Allen's matrix* and $\det\left(\mathcal{H}^B(f)\right)$ *Allen determinant* [2,3].

On the other hand, a *homothetical hypersurface* M^n in the $(n+1)-$dimensional Euclidean space $\mathbb{R}^{n+1}$ is the graph of a function of the form:

$$f(x_1, ..., x_n) = \prod_{j=1}^{n} h_j(x_j),$$

where $h_1, ..., h_n$ are one variable functions of class C^∞. We call $h_1, ..., h_n$ by the *components* of f and denote the homothetical hypersurface M^n by a pair (M^n, f).

Homothetical hypersurfaces have been studied by many authors as focusing on minimality property [19,22,27,28,30].

In this paper, we give a classification for the homothetical hypersurfaces by using Allen's matrices. An application for the homothetical hypersurfaces is also given.

Throughout the present article, we assume that $h_1, ..., h_n : \mathbb{R} \longrightarrow \mathbb{R}$ are thrice differentiable functions and $F : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a twice differentiable function with $F'(u) \neq 0$ such that $I \subset \mathbb{R}$ is an interval of positive length.

2 Basics on hypersurfaces in Euclidean spaces

Let M^n be a hypersurface of the Euclidean space $\mathbb{R}^{n+1}$. For general references on the geometry of hypersurfaces, see [7,18,20].

The *Gauss map* $\nu : M^n \longrightarrow \mathbb{S}^{n+1}$ maps M^n to the unit hypersphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. The differential $d\nu$ of the Gauss map ν is known as *shape operator* or *Weingarten map*. Denote by $T_p M^n$ the tangent space of M^n at the point $p \in M^n$. Then, for $v, w \in T_p M^n$, the shape operator A_p at the point $p \in M^n$ is defined by

$$g(A_p(v), w) = g(d\nu(v), w),$$

where g is the induced metric tensor on M^n from the Euclidean metric on $\mathbb{R}^{n+1}$. The determinant of the shape operator A_p is called the *Gauss-Kronocker curvature.* A hypersurface having zero Gauss-Kronecker curvature is said to be *developable.* In this case the hypersurface can be flattened onto a hyperplane without distortion. We remark that cylinders and cones are examples of developable surfaces, but the spheres are not under any metric.

For a given function $f = f(x_1, ..., x_n)$, the graph of f is the non-parametric hypersurface of $\mathbb{R}^{n+1}$ defined by

$$\varphi(\mathbf{x}) = (x_1, ..., x_n, f(\mathbf{x})) \tag{2.1}$$

for $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$.

Let us put

$$\omega = \sqrt{1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2}. \tag{2.2}$$

The Gauss-Kronecker curvature of the graph hypersurface, given by (2.1), of f is

$$G = \frac{\det(\mathcal{H}(f))}{\omega^{n+2}}, \tag{2.3}$$

where $\mathcal{H}(f)$ is the Hessian matrix of f.

3 Production functions in microeconomics

In microeconomics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+, \quad f = f(x_1, x_2, ..., x_n),$$

where f is the quantity of output, n are the number of inputs and $x_1, x_2, ..., x_n$ are the inputs. The production functions satisfy the following conditions:

1. f is equivalently zero in absence of an input.
2. $\frac{\partial f}{\partial x_i} > 0$, for $i = 1, ..., n$, which means that the production function is strictly increasing with respect to any factor of production.
3. $\frac{\partial^2 f}{\partial x_i^2} < 0$, for $i = 1, ..., n$, i.e., the production has decrasing efficiency with respect to any factor of production.
4. $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, which implies that the production has non-decreasing global efficiency, see [10,11,26,32].

A production function $f(x_1, x_2, ..., x_n)$ is said to be *homogeneous of degree* p or $p-$*homogenous* if

$$f(tx_1, tx_2, ..., tx_n) = t^p f(x_1, x_2, ..., x_n) \tag{3.1}$$

holds for each $t \in \mathbb{R}_+$ for which (3.1) is defined. A homogeneous function of degree one is called *linearly homogeneous.* If $h > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $h < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale [8].

Important properties of homogeneous production functions in microeconomics were interpreted in terms of the geometry of their graph hypersurfaces by [6,13,14,33].

A homothetic function is a production function of the form:

$$f(x_1, ..., x_n) = F(h(x_1, ..., x_n)),$$

where $h(x_1, ..., x_n)$ is homogeneous function of arbitrary given degree and F is a monotonically increasing function. Homothetic functions are production functions whose marginal technical rate of substitution is homogeneous of degree zero [11,14,24].

In 1928, C. W. Cobb and P. H. Douglas introduced [15] a famous two-factor production function

$$Y = bL^k C^{1-k},$$

where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function*. In its generalized form the Cobb-Douglas production function may be expressed as

$$f(\mathbf{x}) = \gamma \prod_{j=1}^{n} x_j^{\alpha_j},$$

where $\gamma, \alpha_1, ..., \alpha_n$ are positive constants.

A homothetic production function of form:

$$f(\mathbf{x}) = F\left(\prod_{j=1}^{n} x_j^{\alpha_j}\right)$$

is called a *homothetic generalized Cobb-Douglas production function* [10].

4 An application for composite functions

Next completely classifies the composite functions in the form:

$$f = F\left(\prod_{j=1}^{n} h_j(x_j)\right) \tag{4.1}$$

which have singular Allen's matrices [3].

Theorem 4.1. *Let f be a composite function given by* (4.1). *Then the Allen's matrix $\mathcal{H}^B(f)$ of f is singular if and only if f is one the following:*

(i)

$$f(\mathbf{x}) = F\left(\gamma e^{\alpha_1 x_1 + \alpha_2 x_2} \prod_{j=3}^{n} h_j(x_j)\right),$$

where $\gamma, \alpha_1, \alpha_2$ are nonzero constants;

(ii)

$$f(\mathbf{x}) = F\left(\gamma \prod_{j=1}^{n} (x_j + \beta_j)^{\alpha_j}\right),$$

where γ, α_i are nonzero constants satisfying $\alpha_1 + ... + \alpha_n = 0$ and β_i some constants.

Now, let F be a monotonically increasing function and $g(\mathbf{x}) = \prod_{j=1}^{n} (x_j + \beta_j)^{\alpha_j}$, for nonzero constants γ, α_i with $\alpha_1 + ... + \alpha_n = 0$ and some constants β_i . Because of $\sum_{i=1}^{n} \alpha_i = 0$, $g(\mathbf{x})$ is a homogeneous function of degree zero, which implies $F(g(\mathbf{x}))$ is a homothetic function. Thus we have the following result:

Corollary 4.2. *Let $F(u)$ be a twice differentiable, positive and monotonically increasing function and let f be a composite function given by*

$$f = F\left(\prod_{j=1}^{n} h_j(x_j)\right),$$

where $h_1, ..., h_n$ are thrice differentiable functions such that $h_i(x_i) \neq e^{\alpha_i x_i}$, $x_i > 0$, for all $i \in \{1, ..., n\}$. Then the Allen's matrix $\mathcal{H}^B(f)$ of f is singular if and only if, up to suitable translations of $x_1, ..., x_n$, f is a homothetic production function given by

$$f(\mathbf{x}) = F\left(\prod_{j=1}^{n} x_j^{\alpha_j}\right),$$

where α_i are nonzero constants with $\alpha_1 + ... + \alpha_n = 0$.

5 Classification of homothetical hypersurfaces

A homothetical hypersurface (M^n, f) in $\mathbb{R}^{n+1}$ is parametrized by

$$\varphi(\mathbf{x}) = \left(x_1, ..., x_n, \prod_{j=1}^{n} h_j(x_j)\right)$$

for $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$. The geometric representation of the generalized Cobb-Douglas production which is also a kind of homothetical hypersurface is given by

$$\varphi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^{n+1},\ \varphi(\mathbf{x}) = \left(x_1, ..., x_n, \gamma \prod_{j=1}^{n} x_j^{\alpha_j}\right).$$

Such a hypersurface is called the *Cobb-Douglas hypersurface* [31]. Thus, Theorem 4.1 can be adapted to the homothetical hypersurfaces in $\mathbb{R}^{n+1}$ as follows:

Theorem 5.1. *Let (M^n, f) be a homothetical hypersurface in $\mathbb{R}^{n+1}$. Then Allen's matrix $\mathcal{H}^B(f)$ of f is singular if and only if (M^n, f) is parametrized by one of the following*

(a)

$$\varphi(\mathbf{x}) = \left(x_1, ..., x_n, \gamma e^{\alpha_1 x_1 + \alpha_2 x_2} \prod_{j=3}^{n} h_j(x_j)\right),$$

where $\gamma, \alpha_1, \alpha_2$ are nonzero constants and $h_3, ..., h_n$ are functions of class C^∞;

(b) up to suitable translations of $x_1, ..., x_n$,

$$\varphi(\mathbf{x}) = \left(x_1, ..., x_n, \gamma \prod_{j=1}^{n} x_j^{\alpha_j} \right),$$

where γ, α_j *are nonzero constants satisfying* $\alpha_1 + ... + \alpha_n = 0$.

Remark 5.1. We have that for a generalized Cobb-douglas production function the values of $\alpha_1, ..., \alpha_n$ are positive constants. Hence, in reality the homothetical hypersurface (M^n, f) given by the statement (b) of Theorem 5.1 is not a Cobb-Douglas hypersurface, while it is correct in mathematical perspective

As a consequence of Theorem 5.1, we have the following:

Corollary 5.2. *Let* (M^n, f) *be a homothetical hypersurface in* $\mathbb{R}^{n+1}$ *such that* f *is a non-vanishing function on an open domain* $D \subset \mathbb{R}^n$, $n \geq 2$, *and* $h_j(x_j) \neq \gamma_j e^{\alpha_j x_j}$ *for all* $j \in \{1, ..., n\}$. *If the Allen's matrix* $\mathcal{H}^B(f)$ *of* f *is singular, then* (M^n, f) *is always non-developable.*

Proof. Let (M^n, f) be a homothetical hypersurface in $\mathbb{R}^{n+1}$. Assume that the Allen's matrix $\mathcal{H}^B(f)$ of f is singular. Thus, by Theorem 5.1, (M^n, f) is parametrized by

$$\varphi(\mathbf{x}) = \left(x_1, ..., x_n, \gamma \prod_{j=1}^{n} x_j^{\alpha_j} \right)$$

for $\mathbf{x} = (x_1, ..., x_n) \in D$ and nonzero constants γ, α_j with $\alpha_1 + ... + \alpha_n = 0$. From (2.2), we have

$$\omega = \left(1 + f^2(\mathbf{x}) \left(\sum_{j=1}^{n} \left(\frac{\alpha_j}{xj} \right)^2 \right) \right)^{1/2}, \tag{5.1}$$

where $f(\mathbf{x}) = \gamma x_1^{\alpha_1} ... x_n^{\alpha_n}$.

On the other hand, the determinant of the Hessian matrix of the function f is

$$\det(\mathcal{H}(f)) = f^n(\mathbf{x}) \begin{vmatrix} \frac{\alpha_1(\alpha_1 - 1)}{x_1^2} & \frac{\alpha_1}{x_1}\frac{\alpha_2}{x_2} & \cdots & \frac{\alpha_1}{x_1}\frac{\alpha_n}{x_n} \\ \frac{\alpha_1}{x_1}\frac{\alpha_2}{x_2} & \frac{\alpha_2(\alpha_2 - 1)}{x_2^2} & \cdots & \frac{\alpha_2}{x_2}\frac{\alpha_n}{x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\alpha_1}{x_1}\frac{\alpha_n}{x_n} & \frac{\alpha_2}{x_2}\frac{\alpha_n}{x_n} & \cdots & \frac{\alpha_n(\alpha_n - 1)}{x_n^2} \end{vmatrix}. \tag{5.2}$$

After calculating for the determinant from the formula (5.2), we deduce

$$\det(\mathcal{H}(f)) = (-1)^n (f^n(\mathbf{x})) \prod_{k=1}^{n} \left(\frac{\alpha_k}{x_k^2} \right). \tag{5.3}$$

By substituting (5.1) and (5.3) in (2.3), we obtain

$$G(\mathbf{x}) = \frac{(-1)^n f^n(\mathbf{x})}{\omega^{n+2}} \prod_{k=1}^{n} \left(\frac{\alpha_k}{x_k^2} \right). \tag{5.4}$$

Under the hypothesis of the theorem, (5.4) completes the proof. □

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