From quantum dynamics of spin 1 particle in Coulomb field to jet geometric-physical objects

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Abstract. The aim of this paper is to relate particular second order ordinary differential equations, associated with quantum mechanics of spin 1 particle in Coulomb field, to certain natural jet geometrical objects, such as a nonlinear connection, a distinguished (d-) torsion or a geometrical Yang-Mills stress-like construction. In its critical points the Yang-Mills entity has the value 1/4. This is intimately connected with the turning points of the quantity P_x^2, which is meaningful both in the context of classical mechanics and quantum mechanics.

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1 Jet geometrical objects produced by a homogeneous linear ODE of second order

Starting from a given homogeneous linear ODE of superior order (generally of order n), in the monograph [1] it was constructed a natural collection of jet geometrical objects which geometrically characterize the initial ODE. More precisely, if we have the initial second order homogeneous linear ODE

$$\frac{d^2\Phi}{dr^2} + a_1(r)\frac{d\Phi}{dr} + a_2(r)\Phi = 0, \tag{1.1}$$

via the canonical ODEs system

$$\begin{cases} \dfrac{dx^1}{dr} = x^2 := X^{(1)}_{(1)}(x^1, x^2) \\ \dfrac{dx^2}{dr} = -a_2(r)x^1 - a_1(r)x^2 := X^{(2)}_{(1)}(x^1, x^2), \end{cases}$$

where $x^1 = \Phi$ and $x^2 = d\Phi/dr$, we can associate it with the following geometrical objects on the 1-jet space $J^1([0,\infty), \mathbb{R}^2)$, whose coordinates are $(r, x^1, x^2, y^1_1 := dx^1/dr, y^2_1 := dx^2/dr)$ (for more details, please see [1, p. 175] or [4]):

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1. a *jet nonlinear connection* $\Gamma = \left(0, N^{(i)}_{(1)j}\right)$, where
$$N_{(1)} = \left(N^{(i)}_{(1)j}\right)_{i,j=\overline{1,2}} = \frac{1}{2}\begin{pmatrix} 0 & -1-a_2(r) \\ 1+a_2(r) & 0 \end{pmatrix};$$
2. a *jet torsion d-tensor*
$$R^{(1)}_{(1)12} = -R^{(2)}_{(1)11} = \frac{1}{2}\frac{da_2}{dr};$$
3. a *jet energetic distinguished* 2*-form*:
$$\mathbb{E} = \mathbb{E}^{(1)}_{(i)j}\delta y^i_1 \wedge dx^i,$$
where $\delta y^i_1 = dy^i_1 + N^{(i)}_{(1)m}dx^m$ and $\mathbb{E}^{(1)}_{(i)j} = -N^{(i)}_{(1)j}$.
4. a *jet geometrical Yang-Mills stress-like construction (helicoidal energy, in terms of the terminology from Udrişte [9])*
$$\mathcal{EYM}(r) = \frac{[1+a_2(r)]^2}{4}.$$

Remark 1.1. Using the Jacobian matrix
$$J(X_{(1)}) = \begin{pmatrix} \dfrac{\partial X^{(1)}_{(1)}}{\partial x^1} & \dfrac{\partial X^{(1)}_{(1)}}{\partial x^2} \\ \dfrac{\partial X^{(2)}_{(1)}}{\partial x^1} & \dfrac{\partial X^{(2)}_{(1)}}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2(r) & -a_1(r) \end{pmatrix},$$
the following matrix relations hold true:

- the *nonlinear connection matrix* is given by
$$N_{(1)} = -\frac{1}{2}\left[J(X_{(1)}) - {}^{T}J(X_{(1)})\right];$$
- the *torsion d-tensor matrix* is given by
$$R_{(1)1} := \begin{pmatrix} R^{(1)}_{(1)11} = 0 & R^{(1)}_{(1)12} = \dfrac{1}{2}\dfrac{da_2}{dr} \\ R^{(2)}_{(1)11} = -R^{(1)}_{(1)12} & R^{(2)}_{(1)12} = 0 \end{pmatrix} = -\frac{\partial N_{(1)}}{\partial r};$$
- the *energetic skew-symmetric matrix* is given by
$$\mathbb{E}^{(1)} := \left(\mathbb{E}^{(1)}_{(i)j}\right)_{i,j=\overline{1,2}} = -N_{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1+a_2(r) \\ -1-a_2(r) & 0 \end{pmatrix};$$
- the *jet geometrical Yang-Mills stress-like entity* is given by
$$\mathcal{EYM}(r) = \frac{1}{2}\,Trace\left[\mathbb{E}^{(1)}\cdot\ {}^{T}\mathbb{E}^{(1)}\right] = \frac{[1+a_2(r)]^2}{4}.$$

Remark 1.2. It is important to note that all above constructed jet geometrical objects are independent of the first coefficient $a_1(r)$, instead they depend only on the second coefficient $a_2(r)$ of the initial SODE. Seemingly, it is because the term with df/dr can be excluded from the initial ODE by a simple substitution of the form $f(r) = \varphi(r) f(\overline{r})$. So the term $\alpha_1(r)$ cannot play any substantial role in the initial ODE.

2 Spin 1 particle in Coulomb field, radial equations

Recently, a special treatment [2] of the quantum-mechanical system, spin 1 particle in the external Coulomb field, was given on the base of tetrad based formalism in the matrix Duffin - Kemmer - Petiau approach with the use spherical metric in Minkowski space

$$dS^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \; r \geq 0.$$

In [2], three classes of solutions were pointed out; with each of them a special second order differential equation for a main (radial) function is associated (for more details, see [2]).

The main function of the first type obeys the known *scalar particle radial equation in Coulomb potential* (see Tamm's paper [8]):

$$\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} + \left[\left(\frac{\epsilon}{\hbar c} + \frac{e^2}{\hbar c}\frac{1}{r} \right)^2 - \left(\frac{mc}{\hbar} \right)^2 - \frac{j(j+1)}{r^2} \right] f(r) = 0, \tag{2.1}$$

below $e^2/(c\hbar) = \alpha = 1/137$, $M = mc/\hbar$, and j is the quantum number: $j = 1, 2, 3, ...$

By introducing special units for length and for energy, such as

$$\lambda = \frac{\hbar}{mc}, \qquad x = \frac{r}{\lambda}, \qquad E = \frac{\epsilon}{mc^2}, \tag{2.2}$$

the above equation reads

$$\frac{d^2 f}{dx^2} + \frac{2}{x}\frac{df}{dx} + \left[\left(E + \frac{\alpha}{x} \right)^2 - 1 - \frac{j(j+1)}{x^2} \right] f(x) = 0, \tag{2.3}$$

where all quantities are dimensionless. Note that by means of a special substitution one can eliminate the term with the first derivative:

$$f = \frac{1}{x} F, \qquad \frac{d^2 F}{dx^2} + \left[\left(E + \frac{\alpha}{x} \right)^2 - 1 - \frac{j(j+1)}{x^2} \right] F(x) = 0, \tag{2.4}$$

which is equivalent to

$$\left[\frac{d^2}{dx^2} + P^2(x) \right] F(x) = 0, \tag{2.5}$$

where

$$\begin{aligned} P^2(x) \quad : &= \left(E + \frac{\alpha}{x} \right)^2 - 1 - \frac{j(j+1)}{x^2} = \\ &= \frac{(E^2 - 1)x^2 + 2\alpha E x - [j(j+1) - \alpha^2]}{x^2}. \end{aligned} \tag{2.6}$$

The sign of the entity $P_x^2 := P^2$ has substantial physical sense:

- in the region when $P^2 > 0$, classical motion is possible;
- in the region when $P^2 < 0$, classical motion is impossible;
- if $P^2(x) = 0$, we have turning points.

From physical point of view, the turning points are important characteristics of P^2:

$$P^2(x) = 0 \implies x_{1,2} = +\frac{E\alpha}{1-E^2} \pm \sqrt{\left(\frac{E\alpha}{1-E^2}\right)^2 - \frac{j(j+1)-\alpha^2}{1-E^2}}.$$

These two (different) real values exist and are positive, when the following inequalities are true:

$$0 < E < 1,$$

$$\left(\frac{E\alpha}{1-E^2}\right)^2 - \frac{j(j+1)-\alpha^2}{1-E^2} > 0 \implies E^2 > \frac{j(j+1)-\alpha^2}{j(j+1)},$$

$$E^2 > 1 - \frac{\alpha^2}{j(j+1)},\ \alpha = \frac{1}{137},\ j = 1, 2, ... \tag{2.7}$$

If

$$E^2 < 1 - \frac{\alpha^2}{j(j+1)},\ \alpha = \frac{1}{137},\ j = 1, 2, ... \tag{2.8}$$

we will have two complex conjugate roots.

If $E > 1$, we always will have one negative and one positive roots:

$$x_{1,2} = -\frac{E\alpha}{E^2-1} \pm \sqrt{\left(\frac{E\alpha}{1-E^2}\right)^2 + \frac{j(j+1)-\alpha^2}{E^2-1}}. \tag{2.9}$$

For this first case we have the following conclusions:

1. When E is subunitary, then the equation of second degree has two complex roots. In that case there aren't turning points, see figure 1. But in the neighborhood of 1, we obtain two turning points. The numerical analysis is based on the following values: $j = 1, \alpha = \frac{1}{137}$ that implies $E = \sqrt{\frac{37537.5}{37538}}$. In this case the turning points are: 160.5017447 and 935.4939099.
2. When E is greater than 1, then there are two turning points (one positive and one negative) which are approximately symmetric from the origin, see figure 2.
3. When $E = 1$ then there is just one positive turning point, see figure 3.
4. When $E = 0$ then there exist no turning points, all roots of equation $P^2(x) = 0$ are complex. The graphic is the same like in the case of E subunitary.

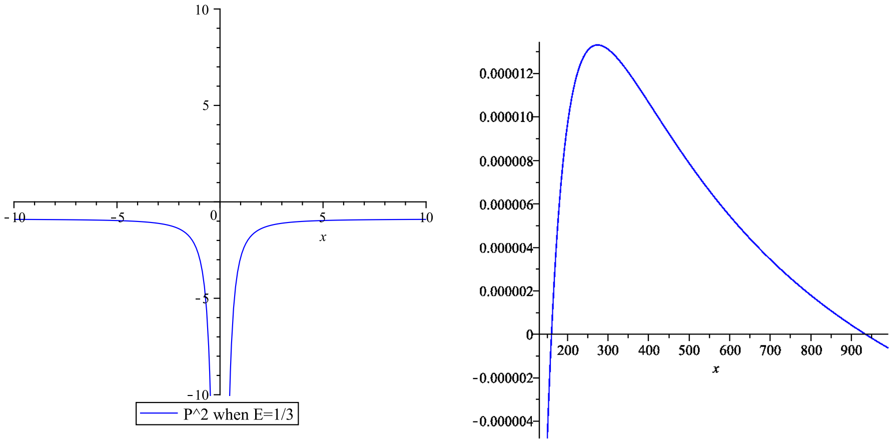

Figure 1:

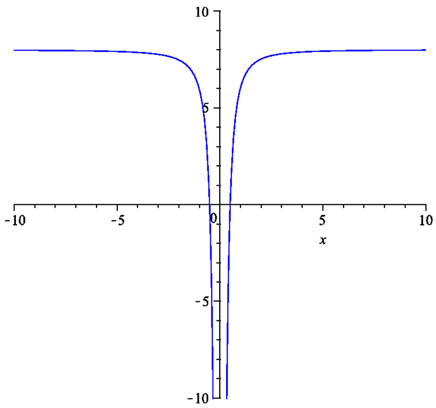

Figure 2:

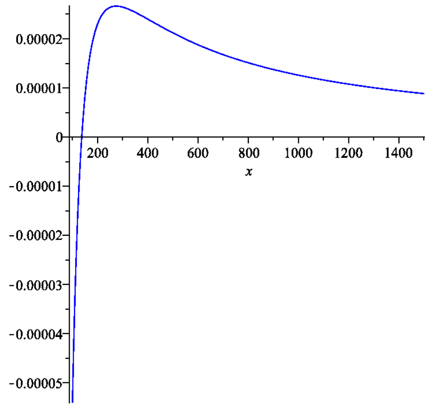

Figure 3:

The main function of the second type is (it can be related to the *confluent Heun equation [2]*)

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\left(3 - \frac{E}{E+\alpha/x}\right)\frac{df}{dx} + \\ + \left(E^2 - \frac{\alpha^2}{x^2} - 3 + 2\frac{E}{E+\alpha/x} - \frac{j(j+1)}{x^2}\right) f = 0. \tag{2.10}$$

The equation can be presented in the form

$$\frac{d^2}{dx^2} f(x) + \left(\frac{3}{x} - \frac{E}{Ex+\alpha}\right)\frac{d}{dx} f + P^2(x) f(x) = 0,$$

where (the notation $\mu = j(j+1) + \alpha^2$ is used)

$$P^2(x) = \frac{x^3 E(E^2-1) + x^2(E^2-3)\alpha - xE\mu - \alpha\mu}{x^2(Ex+\alpha)}, \tag{2.11}$$

$$x \to \pm 0 \qquad P^2(x) \sim -\tfrac{\mu}{x^2} < 0 \sim -\infty,$$
$$x \to \pm\infty \qquad P^2(x) \sim (E^2 - 1),$$
$$x \to -\tfrac{\alpha}{E} \pm 0 \quad P^2(x) \sim \tfrac{-2\alpha}{Ex+\alpha} \sim \mp\infty.$$

We conclude that the graphics of $P^2(x)$ will have three asymptotes (one horizontal and two vertical). The turning points will be determined by the cubic equation

$$x^3 E(E^2-1) + x^2(E^2-3)\alpha - xE\mu - \alpha\mu = 0. \tag{2.12}$$

Remark 2.1. From physical ground, when $0 < E < 1$, we may expect possibilities to get two positive and one negative roots; this situation corresponds to possible finite classical motion of a particle and can lead to bound states in quantum-mechanical description.

With the brief notations

$$a = -E(1 - E^2),\ b = -(3 - E^2)\alpha,\ c = -E\mu,\ d = -\alpha\mu, \tag{2.13}$$

the above equation reads

$$ax^3 + bx^2 + cx + d = 0. \tag{2.14}$$

Some of the properties of the roots can be understood from the classical identity

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3),$$

which implies the Viéte relations:

$$b = -a\,(x_1 + x_2 + x_3) \qquad \Longrightarrow \qquad x_1 + x_2 + x_3 = -\frac{(E^2 - 3)\alpha}{E(E^2 - 1)},$$

$$c = a(x_1x_2 + x_1x_3 + x_2x_3) \qquad \Longrightarrow \qquad x_1x_2 + x_1x_3 + x_2x_3 = \frac{\mu}{(1 - E^2)},$$

$$d = -ax_1x_2x_3 \qquad \Longrightarrow \qquad x_1x_2x_3 = +\frac{\alpha\mu}{E(E^2 - 1)}. \tag{2.15}$$

The third equation in (2.15) says that when $0 < E < 1$, the following inequality holds true

$$x_1x_2x_3 < 0; \tag{2.16}$$

if all roots are real, in turn it means that we could have two positive (let they be x_1, x_2) and one negative x_3 real valued roots. In this case, using the first and second equations in (2.15)

$$-\frac{(E^2 - 3)\alpha}{E(E^2 - 1)} - x_3 = (x_1 + x_2) > 0, \qquad x_1x_2 + (x_1 + x_2)x_3 = \frac{\mu}{(1 - E^2)}, \tag{2.17}$$

one expresses x_3 through x_1x_2:

$$x_1x_2 + x_3\left[-\frac{(E^2 - 3)\alpha}{E(E^2 - 1)} - x_3\right] = \frac{\mu}{(1 - E^2)},$$

so that

$$x_3 = -\frac{(3 - E^2)}{(1 - E^2)} \cdot \frac{\alpha}{2E} \pm \sqrt{\left[\frac{(3 - E^2)}{(1 - E^2)} \cdot \frac{\alpha}{2E}\right]^2 + x_1x_2 - \frac{\mu}{1 - E^2}}.$$

Allowing the condition $x_3 < 0$, we conclude that the first root is good one, that is we have

$$x_3 = -\frac{(3 - E^2)}{(1 - E^2)} \cdot \frac{\alpha}{2E} + \sqrt{\left[\frac{(3 - E^2)}{(1 - E^2)} \cdot \frac{\alpha}{2E}\right]^2 + x_1x_2 - \frac{\mu}{1 - E^2}} \tag{2.18}$$

if we additionally assume

$$x_1 x_2 - \frac{\mu}{1-E^2} > 0. \tag{2.19}$$

After changing the variable

$$x = y - \frac{b}{3a} = y - \frac{3-E^2}{1-E^2} \cdot \frac{\alpha}{3E}; \tag{2.20}$$

we get the reduced cubic equation

$$y^3 + py + q = 0, \tag{2.21}$$

where

$$p = -\frac{b^2}{3a^2} + \frac{c}{a} = -\frac{(3-E^2)^2}{(1-E^2)^2} \cdot \frac{\alpha^2}{3E^2} + \frac{\mu}{1-E^2},$$

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = \frac{(3-E^2)^3}{(1-E^2)^3} \cdot \frac{2\alpha^3}{27E^3} + \frac{2\alpha\mu E}{3(1-E^2)^2} > 0. \tag{2.22}$$

The discriminant of equation (2.21) is

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2,$$

or

$$D = \left(\frac{-b^2+3ac}{9a^2}\right)^3 + \left(\frac{2b^3-9abc+27a^2d}{54a^3}\right)^2. \tag{2.23}$$

We must assume D be negative in order to have real value for our three roots. This will be reached when

$$p < 0, \qquad \left(-\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2 \tag{2.24}$$

With the notation

$$\rho = \left(-\frac{p}{3}\right)^{3/2} \qquad \Longrightarrow \qquad \rho^2 > \left(\frac{q}{2}\right)^2, \tag{2.25}$$

and allowing the condition (2.22), the relation (2.24) is equivalent to

$$\left[\frac{(3-E^2)^2}{(1-E^2)^2} \cdot \frac{\alpha^2}{9E^2} - \frac{\mu}{3(1-E^2)}\right]^3 > \left[\frac{(3-E^2)^3}{(1-E^2)^3} \cdot \frac{\alpha^3}{27E^3} + \frac{\alpha\mu E}{3(1-E^2)^2}\right]^2. \tag{2.26}$$

With notation $1-E^2 = A$, the last inequality can be led to the form,

$$(4\alpha^2\mu^2 - 32\alpha^4\mu) + A^2(8\mu^3 - 48\alpha^2\mu^2 - 24\alpha^4\mu) >$$

$$> A^3(4\mu^3 + 4\alpha^4\mu - 8\alpha^2\mu^2) + A(4\mu^3 + 48\alpha^4\mu - 36\alpha^2\mu^2).$$

By using

$$\alpha = \frac{1}{137}, \qquad \mu = \alpha^2 + j(j+1) >> \alpha >> \alpha^2, \qquad (j = 1, 2, 3, ...),$$

the last inequality can be approximated by

$$4\alpha^2\mu^2 + A^2 8\mu^3 >> A^3 4\mu^3 + A 4\mu^3 \implies \frac{\alpha^2}{\mu} + 2A^2 > A(A^2+1).$$

It follows that

$$\frac{\alpha^2}{j(j+1)} > A(A-1)^2. \tag{2.27}$$

We remind that the first necessary condition $p < 0$ reads in the variable A as follows

$$-\frac{(2+A)^2}{A^2}\,\frac{\alpha^2}{3(1-A)} + \frac{\mu}{A} < 0 \implies \frac{\alpha^2}{\mu} < \frac{3A(1-A)}{(2+A)^2}.$$

Consequently, its approximate form is

$$\frac{\alpha^2}{j(j+1)} > \frac{3A(1-A)}{(2+A)^2}. \tag{2.28}$$

To analytically describe the roots, we should use the notations

$$\rho = \sqrt{-\frac{p^3}{27}}, \qquad \cos\phi = -\frac{q}{2\rho}.$$

In this case, the roots are given by

$$B_1 = 2\rho^{1/3}\cos\frac{\phi}{3} = 2\sqrt{-\frac{p}{3}}\cos\frac{\phi}{3},$$

$$B_2 = 2\rho^{1/3}\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right),$$

$$B_3 = 2\rho^{1/3}\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi}{3} - \frac{2\pi}{3}\right).$$

In idea that the roots B_1, B_2, B_3 had to be real, we must require

$$p < 0, \qquad \frac{q^2}{4\rho^2} < 1;$$

these conditions coincide with the above conditions (2.24) and (2.25).

For the second case we have the following conclusions:

1. When the energy is subunitary, then there is one negative turning point, see figure 5. The computation is made for $E = \frac{1}{3}$ and the final form of the function P^2 is

$$P^2(x) = -\frac{1}{168921}\cdot\frac{20570824\cdot x^3 + 1463982\cdot x^2 + 46285587\cdot x + 1013553}{x^2(137\cdot x + 3)};$$

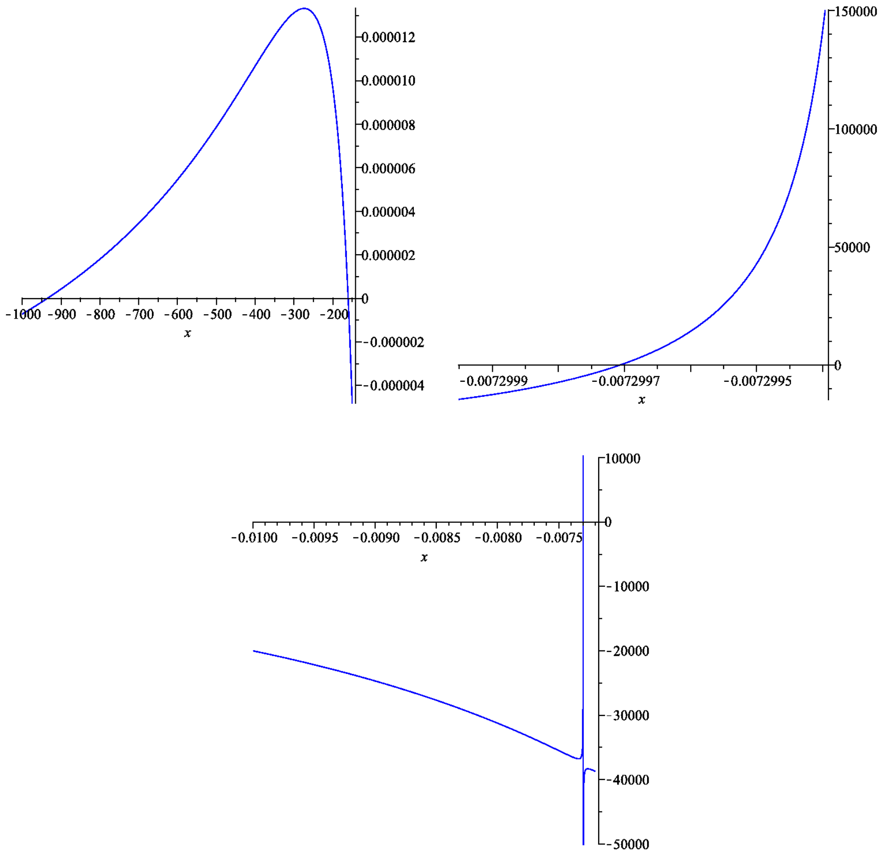

Figure 4:

the zeros of the equation $P^2(x) = 0$ are one negative and the other two complex conjugated:

$$x_1 = -0.02190831764, \qquad x_{2,3} = -0.02462978264 \pm 1.499457942 \cdot i.$$

But in the region $0 < E < 1$, like for $E = 0.9999998$, we can obtain three negative turning points as well:

$$-0.007299660466, -137.5144626, -36358.84321$$

The graphic for this case see in figure 4 The computation is made also for a smaller subunit value of E, like for $E = \frac{1}{3000}$. In this case we obtain the simplified form of $P^2(x)$:

$$P^2(x) = -\frac{23142174428647x^3 + 1520288943693000x^2}{168921000000x^2(137x + 3000)} - \\ -\frac{46285587 \cdot 10^6 x + 1013553 \cdot 10^9}{168921000000x^2(137x + 3000)},$$

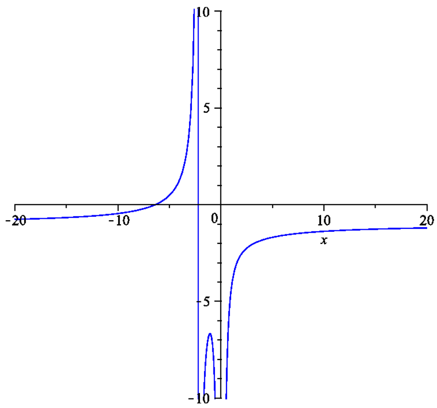

Figure 5:

with the roots:

$$x_1 = -6.567313556, \qquad x_{2,3} = -0.01014998 \pm 0.8165705745 \cdot i.$$

2. When E is greater than 1, then there are three turning points: two negative and one positive. The numerical analysis is made for a value of $E = \sqrt{\frac{37539}{37538}}$. The turning points are:

$$661.4857055 - 1.0 \cdot 10^{(-7)} \cdot i, \quad -113.4930041 - 5.562177830 \cdot 10^{(-7)} \cdot i,$$
$$-0.00729950 + 6.562177830 \cdot 10^{(-7)} \cdot i.$$

Because the imaginary parts are very small (and moreover, the corresponding roots are not complex conjugate), we must neglect these parts. The graphics for this case are presented in the figure 6.

Remark 2.2. Taking into account the previous different analytical and numerical results, we infer that, apparently, the analytical case presented as a possible physical situation in the Remark 2.1 cannot be reached ever. This would be because in the analytical case we have (in the situation $0 < E < 1$) for $P^2(x)$ three real roots (two positive and one negative), while, because of the Rolle's array associated with $P^2(x)$, in the numerical case we have three negative real roots or a negative real root and two complex conjugated roots. Although this seems to be a contradiction, however, in fact we have the same alternative situations. This is because the function $P^2(x)$ is invariant under the simultaneous transformation of the energy E and the coordinate x, given by

$$(E, x) - - - - > (-E, -x),$$

which is specific for Coulomb interaction. Moreover, taking into account that we are in a relativistic case, the sign of the energy parameter is the conventional one (plus or minus).

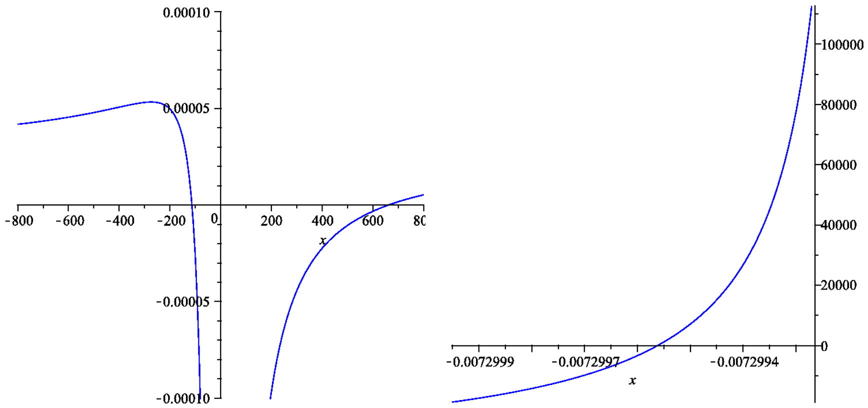

Figure 6:

Remark 2.3. Note that with the help of substitution

$$f(x) = \varphi(x)F(x), \qquad \varphi = \sqrt{\frac{Ex+\alpha}{x^3}}$$

we can eliminate the term with the first derivative from equation

$$\frac{d^2}{dx^2}f(x) + \left(\frac{3}{x} - \frac{E}{Ex+\alpha}\right)\frac{d}{dx}f + P^2(x)f(x) = 0.$$

Consequently, we will obtain then the similar (but not identical) form

$$\frac{d^2}{dx^2}f(x) + \left[\frac{\varphi''}{\varphi} + \frac{\varphi'}{\varphi}\left(\frac{3}{x} - \frac{E}{Ex+\alpha}\right) + P^2(x)\right]f(x) = 0,$$

which will modify explicitly the form of P^2 as $P^2 \rightsquigarrow P^{'2}$. Obviously, the factor $\varphi(x)$ is quite trivial and cannot substantially influence the problem of the initial ODE.

The main function of the third type is verified by the *more complicated equation*

$$\begin{aligned} &\frac{d^2f(x)}{dx^2} + \frac{1}{x}\left[6 + \frac{\alpha}{x\left(E+\alpha/x\right)}\right]\frac{df(x)}{dx} + \\ &+\left[E^2 - 1 + \frac{2E^2\alpha}{Ex+\alpha} - \frac{\alpha\nu^2}{x^4\left(Ex+\alpha\right)} - \right. \\ &\left. -\frac{1}{2}\frac{\alpha\left(-15+4\nu^2-2\alpha^2\right)}{x^2\left(Ex+\alpha\right)} - \frac{E\left(-5+2\nu^2-3\alpha^2\right)}{r\left(Ex+\alpha\right)}\right]f(x) = 0. \end{aligned} \tag{2.29}$$

Obviously, the equation (2.29) can be rewritten in the general form (2.5), corresponding to

$$P^2(x) = E^2 - 1 + \frac{2E^2\alpha}{Ex+\alpha} - \frac{\alpha\nu^2}{x^4\,(Ex+\alpha)} - \\ -\frac{1}{2}\frac{\alpha\left(-15+4\nu^2-2\alpha^2\right)}{x^2\,(Ex+\alpha)} - \frac{E\left(-5+2\nu^2-3\alpha^2\right)}{r\,(Ex+\alpha)}. \tag{2.30}$$

The behavior near singular point is given by

$$x \to 0\,, \qquad P^2(x) \to -\frac{\nu^2}{x^4}; \tag{2.31}$$

$$x \to \infty, \qquad P^2(x) \to E^2 - 1; \tag{2.32}$$

$$x \to -\frac{\alpha}{E}\,, \qquad P^2(x) \to \frac{1}{Ex+\alpha}\times$$

$$\times\left[2\alpha E^2 - \alpha\nu^2\frac{E^4}{\alpha^4} - \frac{1}{2}\frac{E^2}{\alpha}(-15+4\nu^2-2\alpha^2) + \frac{E^2}{\alpha}(-5+2\nu^2-3\alpha^2)\right] =$$

$$= \frac{1}{Ex+\alpha}\left[2\alpha E^2 - \nu^2\frac{E^4}{\alpha^3} + \frac{E^2}{\alpha}(+\frac{5}{2}-2\alpha^2)\right] = \frac{5\alpha^2 - E^2 j(j+1)}{Ex+\alpha}\frac{E^2}{2\alpha^3}. \tag{2.33}$$

We conclude that its graphics will have three (one horizontal and two vertical) asymptotes. The expression for $P^2(x)$ may be rewritten as

$$P^2(x) = \frac{1}{x^4(Ex+\alpha)}\cdot\Big[\,E(E^2-1)x^5 + \alpha(3E^2-1)x^4 +$$

$$+E[5-j(j+1)+3\alpha^2]x^3 + \alpha\left[\frac{15}{2}+\alpha^2-j(j+1)\right]x^2 - \frac{\alpha j(j+1)}{2}\Big]. \tag{2.34}$$

If $E=0$, we have the much simpler variant:

$$P^2(x) = \frac{1}{x^4}\left\{-x^4 + \left[\frac{15}{2}+\alpha^2-j(j+1)\right]x^2 - \frac{j(j+1)}{2}\right\}. \tag{2.35}$$

If $E=1$, we also have a much simpler variant:

$$P^2(x) = \frac{1}{x^4(x+\alpha)}\Big\{\,2\alpha x^4 +$$

$$+[5-j(j+1)+3\alpha^2]x^3 + \alpha\left[\frac{15}{2}+\alpha^2-j(j+1)\right]x^2 - \frac{\alpha j(j+1)}{2}\Big\}. \tag{2.36}$$

Let us turn to (2.36) with the notations

$$a = E(E^2-1), \qquad b = \alpha(3E^2-1),$$

$$c = E \cdot [5 - j(j+1) + 3\alpha^2],$$

$$d = \alpha \left[\frac{15}{2} + \alpha^2 - j(j+1) \right], \qquad f = -\frac{\alpha j(j+1)}{2}. \tag{2.37}$$

It reads

$$P^2(x) = \frac{ax^5 + bx^4 + cx^3 + dx^2 + f}{x^4(Ex+\alpha)}. \tag{2.38}$$

An equation determining possible turning points is

$$ax^5 + bx^4 + cx^3 + dx^2 + f = a(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) \tag{2.39}$$

In particular, we derive that

$$x_1 x_2 x_3 x_4 x_5 = -\frac{f}{a} = -\frac{\alpha j(j+1)}{2E(1-E^2)} \tag{2.40}$$

When $0 < E < 1$, then there exist only four possible variants

$$+,\ +,\ +,\ +,\ - \tag{2.41}$$
$$+,\ +,\ -,\ -,\ - \tag{2.42}$$
$$+,\ +,\ +i, -i,\ - \tag{2.43}$$
$$+i, -i\ +i, -i\ - \tag{2.44}$$

When $E > 1$, then there exist four possible variants

$$+,\ +,\ +,\ +,\ + \tag{2.45}$$
$$+,\ +,\ -,\ -,\ + \tag{2.46}$$
$$+,\ +,\ +i, -i,\ + \tag{2.47}$$
$$+i, -i\ +i, -i\ + \tag{2.48}$$

For the third case we have the following conclusions:

1. When $\frac{1}{675} \le E < 1$ then there are three real valued turning points: two positive and one negative; and also we have two complex conjugate roots. The computation is made for $E = \frac{1}{3}$. The behavior of the graphics is the same also for the value $\sqrt{\frac{37537,5}{37538}}$. The roots of the numerator of P^2 are:

$$x_1 = 0.0965757069792165, \qquad x_2 = 2.25862816372310,$$

$$x_3 = -2.24122563678588,$$

$$x_{4,5} = -0.652007957903348 \pm 0.913600827087149 \cdot i.$$

The graphic of the P^2 function in presented below, but for a good accuracy we display a zooming area, see figure7:

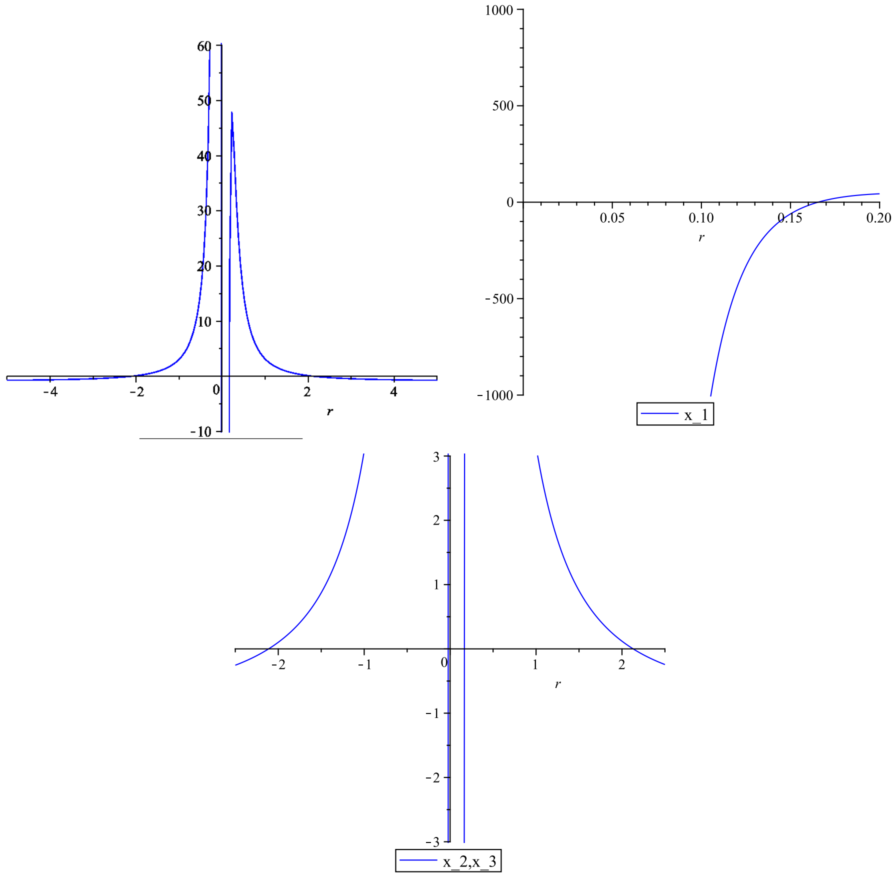

Figure 7:

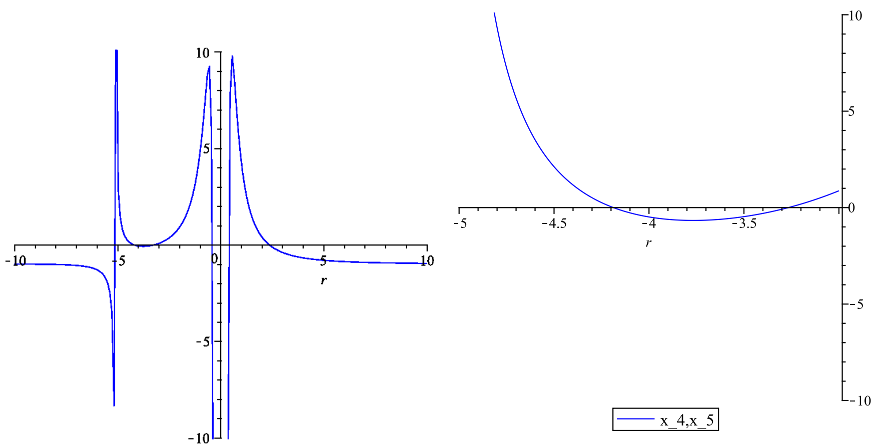

Figure 8:

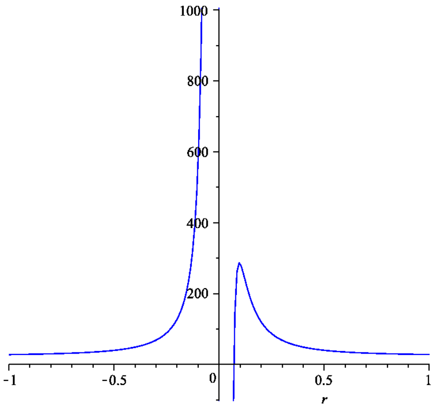

Figure 9:

2. When $0 < E < \frac{1}{675}$ then there are five real valued turning points: two positive and the remaining ones are negative. The computation is made for $E = \frac{1}{700}$, see figure 8:

$$x_1 = 0.187269398191001, \qquad x_2 = 2.48105347521760,$$

$$x_3 = -0.191801644086821, \ x_4 = -3.56288445168428, x_5 = -4.02310497363248$$

3. When the energy is greater than 1 than there is one real-valued turning point. The computation is made for $E = 5$, see figure 9

$$x_1 = 0.0423972349255576,$$

$$x_{2,3} = -0.000898895225515174 \pm 0.433012949098657 \cdot i,$$
$$x_{4,5} = -0.0225503305097697 \pm 0.0374753724335642 \cdot i$$

4. When $E = 1$, then for respective fourth order polynomial there are two real-valued turning points: one positive and one negative; these are

$$x_1 = 0.0704143, \qquad x_2 = -288.2495945,$$

$$x_{3,4} = -0.04088438 \pm 0.06390341202 \cdot i$$

For a good accuracy we represent the zooming area around the negative root, see 10

5. When $E = 0$ then there are four real-valued turning points symmetric with respect to the origin, see figure 11:

$$x_{1,2} = \pm 0.189467962, x_{2,3} = \pm 2.638968580$$

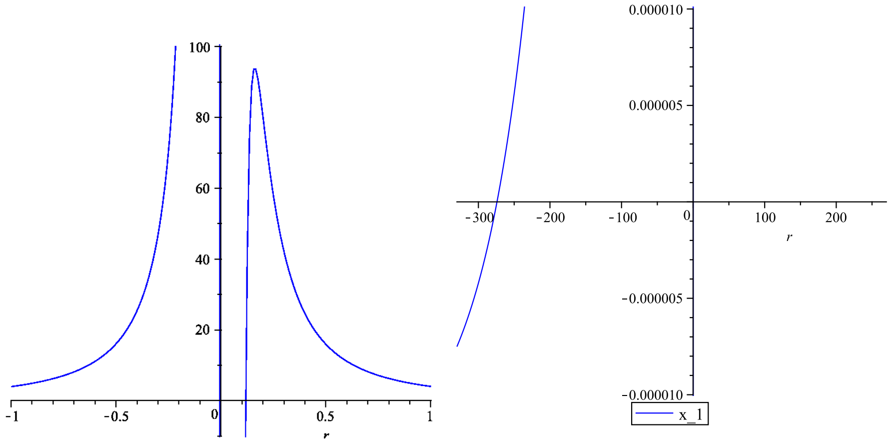

Figure 10:

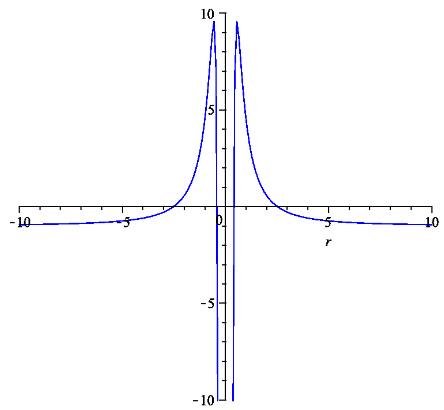

Figure 11:

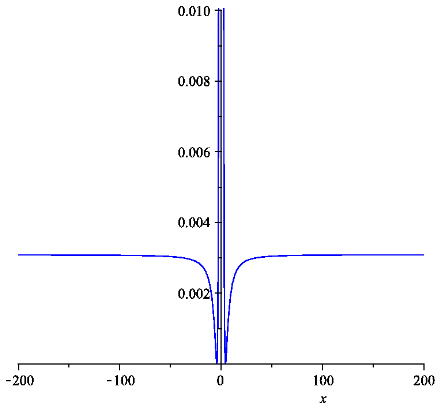

Figure 12:

3 Jet d-torsions and the Yang-Mills-like entity

Obviously, the equations (2.1), (2.10) and (2.29) are particular cases of the second order homogenous linear differential equation (1.1). Consequently, we can apply to them the discussed jet geometrical objects (the corresponding d-torsions and geometrical Yang-Mills stress-like construction).

3.1 A first type main function

In the equation (2.1) we have

$$a_1(x) = \frac{2}{x}, \qquad a_2(x) = \left(E + \frac{\alpha}{x}\right)^2 - 1 - \frac{j(j+1)}{x^2}.$$

It follows that the corresponding jet distinguished torsion has the expression

$$R^{(1)}_{(1)12} = -R^{(2)}_{(1)11} = -\frac{\alpha E}{x^2} + \frac{j(j+1) - \alpha^2}{x^3},$$

and the corresponding geometrical Yang-Mills stress-like entity is

$$\mathcal{EYM}(x) = \frac{1}{4}\left[1 + \left(E + \frac{\alpha}{x}\right)^2 - 1 - \frac{j(j+1)}{x^2}\right]^2.$$

Note that in the case when $E = \frac{1}{3}$ then the energy Yang Mills has the graph similar to the one from the figure 12. Here the horizontal asymptote is $y = \frac{1}{36}$.

In the case when $E = 3$ then the energy Yang Mills has the graph similar to the one from the figure 13. Here the horizontal asymptote is $y = \frac{81}{4}$. When $E = 10000$ then the graph is represented in the figure 14

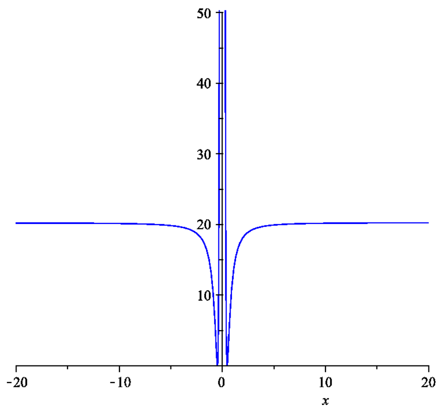

Figure 13:

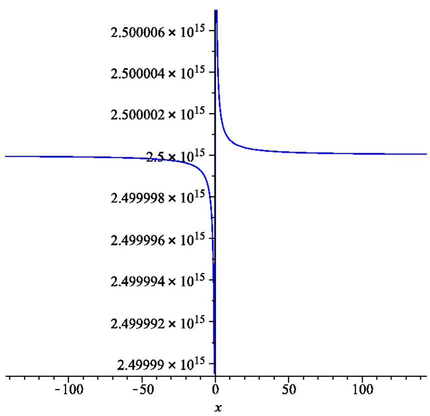

Figure 14:

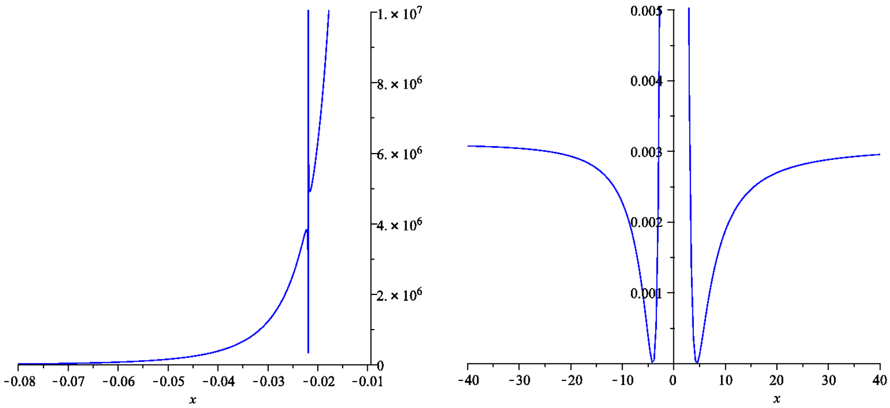

Figure 15:

3.2 A second type main function

In the equation (2.10) we find

$$a_1(x) = \frac{1}{x}\left(3 - \frac{E}{E+\alpha/x}\right), a_2(x) = E^2 - \frac{\alpha^2}{x^2} - 3 + 2\frac{E}{E+\alpha/x} - \frac{2\nu^2}{x^2}.$$

It follows that the corresponding jet distinguished torsion is

$$R^{(1)}_{(1)12} = -R^{(2)}_{(1)11} = \frac{\alpha^2 + 2\nu^2}{x^3} + \frac{\alpha E}{(Ex+\alpha)^2},$$

and the corresponding geometrical Yang-Mills stress-like entity has the form

$$\mathcal{EYM}(x) = \frac{1}{4}\left[1 + E^2 - \frac{\alpha^2}{x^2} - 3 + 2\frac{E}{E+\alpha/x} - \frac{2\nu^2}{x^2}\right]^2.$$

The behavior of $\mathcal{EYM}(x)$ for $E = \frac{1}{3}$ is given in the figure 15
The graph for $\mathcal{EYM}(x)$ when $E = 0.9999998$ is presented in the figure 16
The graph for $\mathcal{EYM}(x)$, when $E = 5$ is presented in the figure 17

3.3 A third type main function

In the Proca equation (2.29) we get

$$a_1(x) = \frac{1}{x}\left[6 + \frac{\alpha}{x\,(E+\alpha/x)}\right],$$

$$a_2(x) = \left[E^2 - 1 + \frac{2E^2\alpha}{Ex+\alpha} - \frac{\alpha\nu^2}{x^4\,(Ex+\alpha)} - \right.$$
$$\left. -\frac{1}{2}\frac{\alpha\left(-15+4\nu^2-2\alpha^2\right)}{x^2\,(Ex+\alpha)} - \frac{E\left(-5+2\nu^2-3\alpha^2\right)}{x\,(Ex+\alpha)}\right].$$

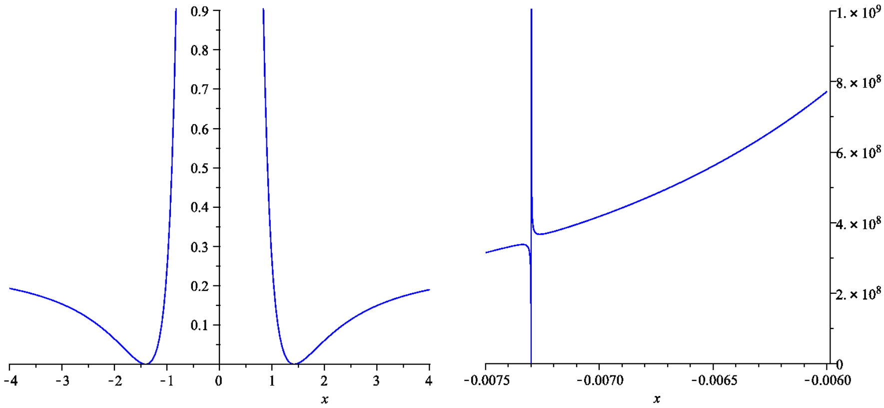

Figure 16:

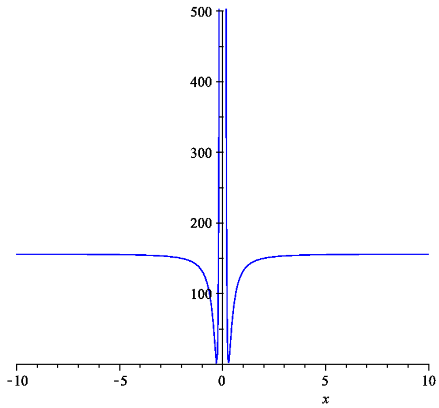

Figure 17:

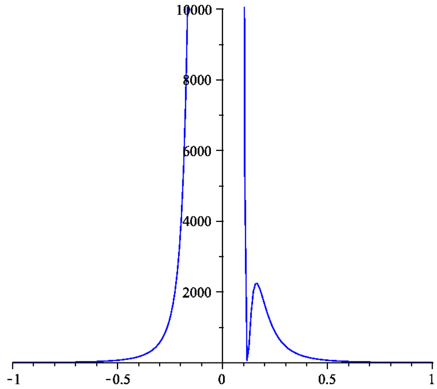

Figure 18:

It follows that the corresponding jet distinguished torsion is given by the formula

$$R^{(1)}_{(1)12} = -R^{(2)}_{(1)11} = -\frac{\epsilon^3\alpha}{(Ex+\alpha)^2} + \frac{\alpha\nu^2}{2}\frac{\left(5Ex^4+4\alpha x^3\right)}{x^8\,(Ex+\alpha)^2} +$$
$$+\frac{\alpha\left(-15+4\nu^2-2\alpha^2\right)}{4}\frac{\left(3Ex^2+2\alpha x\right)}{x^4\,(Ex+\alpha)^2} + \frac{E\left(-5+2\nu^2-3\alpha^2\right)}{2}\frac{(2Ex+\alpha)}{x^2\,(Ex+\alpha)^2}.$$

and the corresponding geometrical Yang-Mills stress-like entity becomes

$$\mathcal{EYM}(x) = \frac{1}{4}\left[1+E^2-1+\frac{2E^2\alpha}{Ex+\alpha} - \frac{\alpha\nu^2}{x^4\,(Ex+\alpha)} - \right.$$
$$\left. -\frac{1}{2}\frac{\alpha\left(-15+4\nu^2-2\alpha^2\right)}{x^2\,(Ex+\alpha)} - \frac{E\left(-5+2\nu^2-3\alpha^2\right)}{x\,(Ex+\alpha)}\right]^2.$$

When E is very closed to 1, like $E=\sqrt{\frac{37537.5}{1/37538}}$, then we have the graph of $\mathcal{EYM}(x)$ from the figure 18 When $E=\frac{1}{700}$, then we have the graph of $\mathcal{EYM}(x)$ from the figure 19 When $E=5$, then we have the graph of $\mathcal{EYM}(x)$ from the figure 20 When $E=1$, then we have the graph of $\mathcal{EYM}(x)$ from the figure 21 When $E=0$, then we have the graph of $\mathcal{EYM}(x)$ from the figure 22

4 Comments and conclusions related to the entities $\mathcal{EYM}(x)$ and $P^2(x)$

In all above three cases, we have the **"geometric-physical relation"**

$$\mathcal{EYM}(x) = \frac{1}{4}\left(1+P_x^2\right)^2 \Leftrightarrow P^2(x) = 2\sqrt{\mathcal{EYM}(x)}-1, \quad x\in\mathbb{R}, \tag{4.1}$$

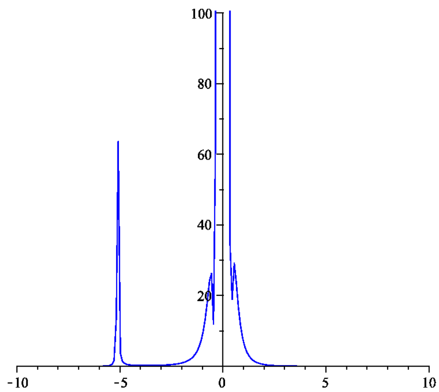

Figure 19:

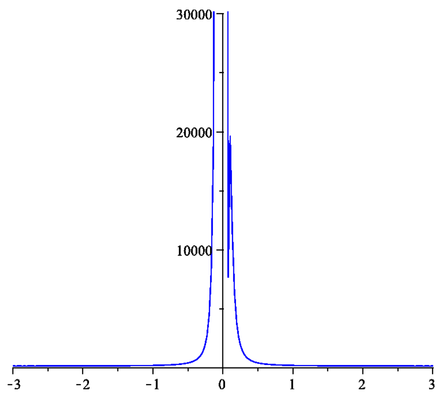

Figure 20:

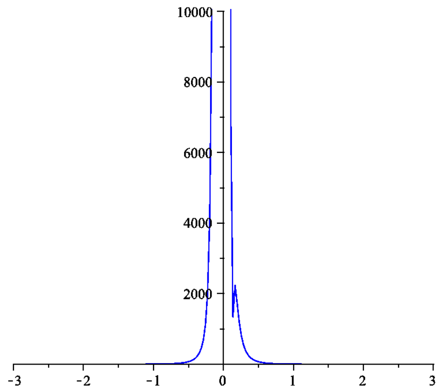

Figure 21:

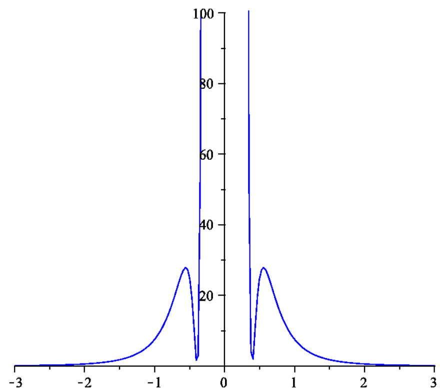

Figure 22:

where the quantity P_x^2 is meaningful both in the context of classical mechanics and quantum mechanics.

It is important to note the following geometric-physical facts that could lead to possible physical interpretations:

1. for $0 < \mathcal{EYM} < 1/4$ there do not exist any positive root P^2 satisfying the equation (4.1), so the motion is impossible;
2. $\mathcal{EYM} = 1/4$ - we have critical (turning) points;
3. for $\mathcal{EYM} > 1/4$ there exists at least a positive root P^2 satisfying the equation (4.1), so the motion is possible.

Moreover, the following relations between asymptotes are true:

- if $x = x_0$ is a vertical asymptote for P^2, then it is also a vertical asymptote for $\mathcal{EYM}$;
- if $y = y_0$ is a horizontal asymptote for P^2, then $y = (1/4)\,(1+y_0)^2$ is a horizontal asymptote for $\mathcal{EYM}$.

Quantum dynamics of a spin 1 particle in natural way has led us to three classes of ODEs, related to some corresponding functions $P^2(x)$:

$$P_I^2(x) = \frac{a_2x^2 + a_1x + a_0}{x^2}, P_{II}^2(x) = \frac{a_3x^3 + a_2x^2 + a_1x + a_0}{x^2(Ax+B)},$$

$$P_{III}^2(x) = \frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{x^4(Ax+B)}.$$

The singular point determined by equation $Ax + B \equiv Ex + \alpha = 0$ is not physical, but is very substantial mathematically. For instance, in the case of spin 1/2 Dirac particle such a singularity arises as well, and it can be eliminated by means of special trick in the solving procedure – see for instance [10].

Note that the type I is referred to class of hypergeometric differential equations, the type II is associated with the Heun equation with four singular points, and the type III is a more complicated class of ODEs. Obviously, the above series I, II, III can be continued from a mathematical point of view.

In accordance with to possibilities I, II, III, there arise three sorts of geometrical structures within the following relation:

$$\mathcal{EYM}(x) = \frac{1}{4}\left[1 + P^2(x)\right]^2, \quad \forall\, x \in \mathbb{R}. \tag{4.2}$$

All three functions $P^2(x)$ behave at infinity ($x \to \pm\infty$) the same manner:

$$P^2(x \to \pm\infty) \sim E^2 - 1.$$

This leads to

$$2\sqrt{\mathcal{EYM}(x \to \pm\infty)} \sim E^2.$$

Thus, the quantity $2\sqrt{\mathcal{EYM}}$ effectively behaves at infinity as the entity P^2 for a massless field. Indeed, this is because, in usual units, we have that:

$$E^2 - 1 \qquad \text{reads} \qquad \left(\frac{\epsilon}{\hbar c}\right)^2 - \left(\frac{mc}{\hbar}\right)^2.$$

From a physical point of view, the value $E^2 = 1$ is critical. Namely, we may expect bound states in quantum-mechanical background when $| E |< 1$, and when $| E |> 1$ we may expect unbound state of scattering particle with spin 1 on the Coulomb potential field.

Similarly, the critical value $E = 0$ is very interesting from mathematical point of view, but hardly to understand from physical standpoint.

All three types of ODEs and the corresponding functions $P^2(x)$ are invariant under the simultaneous transformation of the energy E and the coordinate x:

$$(E, x) - - - - - > (-E, -x).$$

Obviously, this is not an occasional fact and it is specific for Coulomb interaction. Moreover, in the relativistic case (which is taken under consideration in this paper), the sign of the energy parameter is the conventional one (plus or minus).

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