Some notes concerning Norden-Walker 8-manifolds

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Abstract. The main purpose of the present paper is to study almost Norden structures on 8-dimensional Walker manifolds. We discuss the integrability and Kähler(holomorphic) conditions for these structures. Nonexistence of (non-Kähler)quasi-Kähler structures on almost Norden-Walker 8-manifolds is proved.

Key words: Walker 8-manifolds; Proper almost complex structure; Norden metrics; Holomorphic metrics.

1 Introduction

Let $M_{2n}$ be a pseudo-Riemannian manifold with neutral metric, i.e., with pseudo-Riemannian metric $g$ of signature $(n, n)$. We denote by $\mathcal{H}_p^q(M_{2n})$ the set of all tensor fields of type $(p, q)$ on $M_{2n}$. Manifolds, tensor fields and connections are always assumed to be differentiable and of class $C^\infty$.

Let $(M_{2n}, \varphi)$ be an almost complex manifold with almost complex structure $\varphi$. This structure is said to be integrable if the matrix $\varphi = (\varphi^i_j)$ is reduced to constant form in a certain holonomic natural frame in a neighborhood $U_x$ of every point $x \in M_{2n}$. In order that an almost complex structure $\varphi$ be integrable, it is necessary and sufficient that it be possible to introduce a torsion-free affine connection $\nabla$ with respect to which the structure tensor $\varphi$ is covariantly constant, i.e., $\nabla \varphi = 0$. It is also known that the integrability of $\varphi$ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \mathcal{H}_{1,2}(M_{2n})$. If $\varphi$ is integrable, then $\varphi$ is a complex structure and, moreover, $M_{2n}$ is a C-holomorphic manifold $X_n(C)$ whose transition functions are holomorphic mappings.

A metric $g$ is a Norden metric [2] if

$$ g(\varphi X, Y) = g(X, \varphi Y), $$

for any $X, Y \in \mathcal{H}_1^0(M_{2n})$. Metrics of this type have also been studied under the names: pure and B-metrics (see [1], [2], [4], [12], [17], [19]). If $(M_{2n}, \varphi, g)$ is an almost complex manifold with Norden metric $g$, we say that $(M_{2n}, \varphi, g)$ is an almost Norden manifold. If $\varphi$ is integrable, we say that $(M_{2n}, \varphi, g)$ is a Norden manifold.

In the present paper, we shall focus our attention to the Norden manifolds in dimension eight. Using the Walker metric we constructive a new Norden-Walker metrics together with so called proper almost complex structures. Note that an indefinite Kähler-Einstein metric on an eight-dimensional Walker manifolds has been recently investigated in [8]. Many authors have also been studied recently on Norden-Walker manifolds (see [13], [14], [15], [16]).

1.1 Holomorphic (almost holomorphic) tensor fields

Let $\tilde{t}$ be a complex tensor field on $X_n(C)$. The real model of such a tensor field is a tensor field $t$ on $M_{2n}$ of the same order such that the action of the structure affinor $\varphi$ on $t$ does not depend on which vector or covector argument of $t$ $\varphi$ acts. Such tensor fields are said to be pure with respect to $\varphi$. They were studied by many authors (see, e.g., [4], [9], [10], [17]-[19], [21]). In particular, being applied to a $(0, q)$-tensor field $\omega$, the purity means that for any $X_1, ..., X_q \in \mathcal{I}_{10}(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, ..., X_q) = \omega(X_1, \varphi X_2, ..., X_q) = ... = \omega(X_1, X_2, ..., \varphi X_q).$$

We define an operator

$$\Phi_{\varphi} : \mathcal{I}_{0}^q(M_{2n}) \to \mathcal{I}_{0}^{q+1}(M_{2n})$$

applied to the pure tensor field $\omega$ by (see [21])

$$(\Phi_{\varphi}\omega)(X, Y_1, Y_2, ..., Y_q) = (\varphi X)(\omega(Y_1, Y_2, ..., Y_q)) - X(\omega(\varphi Y_1, Y_2, ..., Y_q))$$

$$+ \omega((L_{Y_1}\varphi)X, Y_2, ..., Y_q) + ... + \omega(Y_1, Y_2, ..., (L_{Y_q}\varphi)X),$$

where $L_Y$ denotes the Lie differentiation with respect to $Y$.

When $\varphi$ is a complex structure on $M_{2n}$ and the tensor field $\Phi_{\varphi}\omega$ vanishes, the complex tensor field $\hat{\omega}$ on $X_n(C)$ is said to be holomorphic (see [4], [17], [21]). Thus a holomorphic tensor field $\hat{\omega}$ on $X_n(C)$ is realized on $M_{2n}$ in the form of a pure tensor field $\omega$, such that

$$(\Phi_{\varphi}\omega)(X, Y_1, Y_2, ..., Y_q) = 0,$$

for any $X, Y_1, ..., Y_q \in \mathcal{I}_{10}(M_{2n})$. Therefore such a tensor field $\omega$ on $M_{2n}$ is also called holomorphic tensor field. When $\varphi$ is an almost complex structure on $M_{2n}$, a tensor field $\omega$ satisfying $\Phi_{\varphi}\omega = 0$ is said to be almost holomorphic.

1.2 Holomorphic Norden(Kähler-Norden) metrics

In a Norden manifold a Norden metric $g$ is called a holomorphic if

$$(\Phi_{\varphi}g)(X, Y, Z) = 0,$$
for any $X, Y, Z \in \mathcal{V}_0^1(M_{2n})$, where

\begin{equation}
(\Phi_{\varphi}g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_\varphi Y)X, Z) + g(Y, (L_\varphi Z)X).
\end{equation}

By setting $X = \partial_k, Y = \partial_i, Z = \partial_j$ in the equation (1.1), we see that the components $(\Phi_{\varphi}g)_{kij}$ of $\Phi_{\varphi}g$ with respect to a local coordinate system $x^1, ..., x^n$ may be expressed as follows:

$$(\Phi_{\varphi}g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi^m_k - \partial_k \varphi^m_i) + g_{im} \partial_j \varphi^m_k.$$

If $(M_{2n}, \varphi, g)$ is a Norden manifold with holomorphic Norden metric $g$, we say that $(M_{2n}, \varphi, g)$ is a holomorphic Norden manifold.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

**Theorem 1.1.** ([3] For paracomplex version see [11]) For an almost complex manifold with Norden metric $g$, the condition $\Phi_{\varphi}g = 0$ is equivalent to $\nabla \varphi = 0$, where $\nabla$ is the Levi-Civita connection of $g$.

Kähler-Norden manifold can be defined as a triple $(M_{2n}, \varphi, g)$ which consists of a manifold $M_{2n}$ endowed with an almost complex structure $\varphi$ and a pseudo-Riemannian metric $g$ such that $\nabla \varphi = 0$, where $\nabla$ is the Levi-Civita connection of $g$ and the metric $g$ is assumed to be Nordenian. Therefore, there exist a one-to-one correspondence between Kähler-Norden manifolds and Norden manifolds with a holomorphic metric. Recall that in such a manifold, the Riemannian curvature tensor is pure and holomorphic, also the curvature scalar is locally holomorphic function (see [3], [12]).

**Remark 1.1.** We know that the integrability of the almost complex structure $\varphi$ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla \varphi = 0$ holds. Since the Levi-Civita connection $\nabla$ of $g$ is a torsion-free affine connection, we have: If $\Phi_{\varphi}g = 0$, then $\varphi$ is integrable. Thus, almost Norden manifold with conditions $\Phi_{\varphi}g = 0$ and $N_{\varphi} \neq 0$, i.e. almost holomorphic Norden manifolds does not exist.

## 2 Norden-Walker metrics

### 2.1 Walker metric $g$

A neutral metric $g$ on a 8-manifold $M_8$ is said to be Walker metric if there exists a 4-dimensional null distribution $D$ on $M_8$, which is parallel with respect to $g$. From Walker theorem [20], there is a system of coordinates $(x^1, ..., x^8)$ with respect to which $g$ takes the local canonical form

\begin{equation}
g = (g_{ij}) = \begin{pmatrix}
0 & I_4 \\
I_4 & B
\end{pmatrix},
\end{equation}
where $I_4$ is the unit $4 \times 4$ matrix and $B$ is a $4 \times 4$ symmetric matrix whose entries are functions of the coordinates $(x^1, ..., x^8)$. Note that $g$ is of neutral signature $(+++-)$, and that the parallel null 4-plane $D$ is spanned locally by $\{\partial_1, \partial_2, \partial_3, \partial_4\}$, where $\partial_i = \frac{\partial}{\partial x^i}, (i = 1, ..., 8)$.

In this paper, we consider the specific Walker metrics on $M_8$ with $B$ of the form

$$B = \begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & b & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix},$$

where $a, b$ are smooth functions of the coordinates $(x^1, ..., x^8)$.

### 2.2 Almost Norden-Walker 8-manifolds

We can construct various almost complex structures $\varphi$ on a Walker 8-manifold $M_8$ with the metric $g$ as in (2.1), (2.2) so that $(M_8, \varphi, g)$ is almost Nordenian. The following $\varphi$ is one of the simplest examples of such an almost complex structure:

$$\begin{align*}
\varphi\partial_1 &= \partial_3, & \varphi\partial_2 &= \partial_4, & \varphi\partial_3 &= -\partial_1, & \varphi\partial_4 &= -\partial_2, \\
\varphi\partial_5 &= \frac{1}{2}(a + b)\partial_3 - \partial_7, & \varphi\partial_6 &= -\partial_8, \\
\varphi\partial_7 &= -\frac{1}{2}(a + b)\partial_1 + \partial_5, & \varphi\partial_8 &= \partial_6.
\end{align*}$$

In conformity with the terminology of Matsushita (see, [6]-[8]) we call $\varphi$ the proper almost complex structure. The proper almost complex structure $\varphi$ has the local components

$$\varphi = (\varphi^i_j) = \begin{pmatrix}
    0 & 0 & -1 & 0 & 0 & 0 & -\frac{1}{2}(a + b) & 0 \\
    0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & \frac{1}{2}(a + b) & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix},$$

with respect to the natural frame $\{\partial_i\}, i = 1, ..., 8$.

**Remark 2.1.** From (2.3) we see that in the case $a = -b$, $\varphi$ is integrable.

### 2.3 Integrability of the structure $\varphi$

We consider the general case for integrability.

The proper almost complex structure $\varphi$ on almost Norden-Walker manifolds is integrable if and only if

$$\begin{align*}
(N_\varphi)^i_j = & \varphi^m_j \partial_m \varphi^i_k - \varphi^m_k \partial_m \varphi^i_j - \varphi^m_i \partial_j \varphi^m_k + \varphi^m_i \partial_k \varphi^m_j = 0.
\end{align*}$$
Since \( N^i_{jk} = -N^i_{kj} \), we need only consider \( N^i_{jk} (j < k) \). By explicit calculation, the nonzero components of the Nijenhuis tensor are as follows:

\[
\begin{align*}
N^1_{15} &= N^3_{17} = N^3_{37} = -N^3_{35} = \frac{1}{2} (a_1 + b_1), \\
N^2_{25} &= N^1_{17} = -N^3_{15} = -N^1_{35} = -\frac{1}{2} (a_2 + b_2), \\
N^1_{17} &= -N^1_{35} = -N^3_{15} = -N^1_{37} = -\frac{1}{2} (a_3 + b_3), \\
N^1_{57} &= -\frac{1}{4} (a + b)(a_3 + b_3), \\
N^1_{27} &= N^3_{45} = N^3_{25} = N^3_{27} = \frac{1}{2} (a_4 + b_4), \\
N^1_{56} &= -N^1_{78} = N^3_{58} = -N^3_{67} = -\frac{1}{2} (a_6 + b_6), \\
N^1_{58} &= -N^1_{67} = -N^3_{58} = N^3_{78} = -\frac{1}{2} (a_8 + b_8).
\end{align*}
\]

From (2.5) we have

**Theorem 2.1.** The proper almost complex structure \( \varphi \) on almost Norden-Walker manifolds is integrable if and only if the following PDEs hold:

\[
\begin{align*}
a_1 + b_1 &= 0, \\
a_2 + b_2 &= 0, \\
a_3 + b_3 &= 0, \\
a_4 + b_4 &= 0, \\
a_6 + b_6 &= 0, \\
a_8 + b_8 &= 0.
\end{align*}
\]

**Corollary 2.2.** The proper almost complex structure \( \varphi \) on almost Norden-Walker manifolds is integrable if and only if

\[
a = -b + \xi,
\]

where \( \xi \) is any function of \( x^5 \) and \( x^7 \) alone.

### 3 Holomorphic Norden-Walker (Kähler-Norden-Walker) metrics on \((M_8, \varphi, g)\)

Let \((M_8, \varphi, g)\) be an almost Norden-Walker manifold. If

\[
(\Phi_{\varphi} g)_{kij} = \varphi^m_k \partial_m g_{ij} - \varphi^m_i \partial_k g_{mj} + g_{mj} (\partial_i \varphi^m_k - \partial_k \varphi^m_i) + g_{im} \partial_j \varphi^m_k = 0,
\]

then by virtue of Theorem 1 \( \varphi \) is integrable and the triple \((M_8, \varphi, g)\) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold. Taking account of Remark 1, we see that almost Kähler-Norden-Walker manifold with condition \( \Phi_{\varphi} g = 0 \) and \( N_{\varphi} \neq 0 \) does not exist.

We will write (2.1) and (2.2) in (3.1). Since \( (\Phi_{\varphi} g)_{ijk} = (\Phi_{\varphi} g)_{ikj} \), we need only consider \( (\Phi_{\varphi} g)_{ijk} (j < k) \). By explicit calculation, the nonzero components of the
The triple \((\Phi, \varphi, g)\) tensor are as follows:

\[
\begin{align*}
(\Phi, \varphi, g)_{155} &= a_3, \quad (\Phi, \varphi, g)_{157} = \frac{1}{2}(b_1 - a_1), \quad (\Phi, \varphi, g)_{177} = b_3, \\
(\Phi, \varphi, g)_{255} &= a_4, \quad (\Phi, \varphi, g)_{257} = \frac{1}{2}(b_2 - a_2), \quad (\Phi, \varphi, g)_{277} = b_4, \\
(\Phi, \varphi, g)_{355} &= -a_1, \quad (\Phi, \varphi, g)_{357} = \frac{1}{2}(b_3 - a_3), \quad (\Phi, \varphi, g)_{377} = -b_1, \\
(\Phi, \varphi, g)_{455} &= -a_2, \quad (\Phi, \varphi, g)_{457} = \frac{1}{2}(b_4 - a_4), \quad (\Phi, \varphi, g)_{477} = -b_2, \\
(\Phi, \varphi, g)_{517} &= -(\Phi, \varphi, g)_{275} = \frac{1}{2}(a_2 + b_2), \quad (\Phi, \varphi, g)_{527} = -(\Phi, \varphi, g)_{175} = \frac{1}{2}(a_1 + b_1), \\
(\Phi, \varphi, g)_{537} &= -(\Phi, \varphi, g)_{375} = \frac{1}{2}(a_3 + b_3), \quad (\Phi, \varphi, g)_{547} = -(\Phi, \varphi, g)_{745} = \frac{1}{2}(a_4 + b_4), \\
(\Phi, \varphi, g)_{555} &= \frac{1}{2}(a + b)a_3 - a_7, \quad (\Phi, \varphi, g)_{557} = -b_5, \\
(\Phi, \varphi, g)_{567} &= -(\Phi, \varphi, g)_{257} = \frac{1}{2}(a_6 + b_6), \quad (\Phi, \varphi, g)_{577} = \frac{1}{2}(a + b)b_3 + a_7, \\
(\Phi, \varphi, g)_{578} &= -(\Phi, \varphi, g)_{575} = \frac{1}{2}(a + b)b_3 + a_7, \quad (\Phi, \varphi, g)_{655} = -a_8, \\
(\Phi, \varphi, g)_{657} &= \frac{1}{2}(b_6 - a_6), \quad (\Phi, \varphi, g)_{677} = -b_8, \quad (\Phi, \varphi, g)_{755} = -\frac{1}{2}(a + b)a_3 - b_5, \\
(\Phi, \varphi, g)_{757} &= -a_7, \quad (\Phi, \varphi, g)_{777} = -\frac{1}{2}(a + b)b_1 + b_5, \\
(\Phi, \varphi, g)_{855} &= a_6, \quad (\Phi, \varphi, g)_{857} = \frac{1}{2}(b_4 - a_8), \quad (\Phi, \varphi, g)_{877} = b_6.
\end{align*}
\]

From (3.2) we have

**Theorem 3.1.** The triple \((M_8, \varphi, g)\) is Kähler-Norden-Walker if and only if the following PDEs hold:

\[
\begin{align*}
& a_1 = a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = 0, \\
& b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_8 = 0.
\end{align*}
\]

**Corollary 3.2.** \((M_8, \varphi, g)\) is Kähler-Norden-Walker if and only if the matrix \(B\) in (2.1) has components

\[
B = \begin{pmatrix}
a(x^5) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b(x^7) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

4 Nonexistence of (non-Kähler) quasi-Kähler-Norden-Walker structures on \((M_8, \varphi, g)\)

The basis class of almost complex manifolds with Norden metric is the class of the quasi-Kähler manifolds. An almost Norden manifold \((M_2n, \varphi, g)\) is called a quasi-Kähler [2], [5], if

\[X, \varphi g((\nabla X) \phi)Y, Z) = 0,\]

where \(\sigma\) is the cyclic sum by three arguments.

By setting

\[L_X \varphi X = L_X (\varphi X) - \varphi (L_X X) = \nabla_X (\phi X) - \nabla \varphi X Y - \varphi (\nabla_Y X) + \varphi (\nabla_Y Y)\]
in (1.1), we see that \((\Phi_{\varphi}g)(X, Y, Z)\) may be expressed as
\[
(\Phi_{\varphi}g)(X, Y, Z) = -g((\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi)Z, X) + g((\nabla_Z \varphi)X, Y).
\]

If we add \((\Phi_{\varphi}g)(X, Y, Z)\) and \((\Phi_{\varphi}g)(Z, Y, X)\), then by virtue of
\[
g(Z, (\nabla_Y \varphi)X, (\nabla_X \varphi)Z) = g((\nabla_Y \varphi)Z, X),
\]
we find
\[
(\Phi_{\varphi}g)(X, Y, Z) + (\Phi_{\varphi}g)(Z, Y, X) = 2g((\nabla_Y \varphi)Z, X).
\]

Since \((\Phi_{\varphi}g)(X, Y, Z) = (\Phi_{\varphi}g)(X, Z, Y)\), from the last equation we have
\[
(\Phi_{\varphi}g)(X, Y, Z) + (\Phi_{\varphi}g)(Y, Z, X) + (\Phi_{\varphi}g)(Z, X, Y) = \sigma_{X,Y,Z}g((\nabla_X \varphi)Y, Z).
\]

Thus we have

**Theorem 4.1.** Let \((M_{2n}, \varphi, g)\) be an almost Norden manifold. Then the Norden metric \(g\) is a quasi-Kähler-Norden if and only if
\[
(4.1) \quad (\Phi_{\varphi}g)(X, Y, Z) + (\Phi_{\varphi}g)(Y, Z, X) + (\Phi_{\varphi}g)(Z, X, Y) = 0
\]
for any \(X, Y, Z \in \mathfrak{g}^1_0(M_{2n})\).

From (4.1) we easily see that a Kähler-Norden manifold is a quasi-Kähler-Norden. Conversely, quasi-Kähler-Norden manifold is a non-Kähler-Norden, in general. In particular, let \((M_8, \varphi, g)\) be an almost Norden-Walker 8-manifold. Using (3.2) and (4.1) we have
\[
(4.2) \quad
\begin{align*}
(\Phi_{\varphi}g)_{155} + (\Phi_{\varphi}g)_{551} + (\Phi_{\varphi}g)_{515} &= a_3 = 0, \\
(\Phi_{\varphi}g)_{157} + (\Phi_{\varphi}g)_{571} + (\Phi_{\varphi}g)_{715} &= \frac{1}{2}(b_1 - a_1) = 0, \\
(\Phi_{\varphi}g)_{177} + (\Phi_{\varphi}g)_{771} + (\Phi_{\varphi}g)_{717} &= b_3 = 0, \\
(\Phi_{\varphi}g)_{255} + (\Phi_{\varphi}g)_{552} + (\Phi_{\varphi}g)_{525} &= a_4 = 0, \\
(\Phi_{\varphi}g)_{257} + (\Phi_{\varphi}g)_{572} + (\Phi_{\varphi}g)_{725} &= \frac{1}{2}(b_2 - a_2) = 0, \\
(\Phi_{\varphi}g)_{277} + (\Phi_{\varphi}g)_{772} + (\Phi_{\varphi}g)_{727} &= b_4 = 0,
\end{align*}
\]
Let a Walker 8-manifold \((M, g)\) be an almost Norden-Walker manifold. Then there does not exist a (non-K"ahler) quasi-K"ahler structure on this manifold.

\[
\begin{align*}
(\Phi \varphi g)_{555} + (\Phi \varphi g)_{553} + (\Phi \varphi g)_{535} &= -a_1 = 0, \\
(\Phi \varphi g)_{557} + (\Phi \varphi g)_{573} + (\Phi \varphi g)_{735} &= \frac{1}{2}(b_3 - a_3) = 0, \\
(\Phi \varphi g)_{377} + (\Phi \varphi g)_{773} + (\Phi \varphi g)_{737} &= -b_1 = 0, \\
(\Phi \varphi g)_{455} + (\Phi \varphi g)_{554} + (\Phi \varphi g)_{545} &= -a_2 = 0, \\
(\Phi \varphi g)_{457} + (\Phi \varphi g)_{574} + (\Phi \varphi g)_{745} &= \frac{1}{2}(b_4 - a_4) = 0, \\
(\Phi \varphi g)_{477} + (\Phi \varphi g)_{774} + (\Phi \varphi g)_{747} &= -b_2 = 0, \\
(\Phi \varphi g)_{557} + (\Phi \varphi g)_{575} + (\Phi \varphi g)_{755} &= -\frac{1}{2}(a + b)a_1 - 3b_5 = 0, \\
(\Phi \varphi g)_{567} + (\Phi \varphi g)_{675} + (\Phi \varphi g)_{756} &= \frac{1}{2}(b_6 - a_6) = 0, \\
(\Phi \varphi g)_{577} + (\Phi \varphi g)_{775} + (\Phi \varphi g)_{757} &= \frac{1}{2}(a + b)b_3 - a_7 = 0, \\
(\Phi \varphi g)_{578} + (\Phi \varphi g)_{785} + (\Phi \varphi g)_{857} &= \frac{1}{2}(b_8 - a_8) = 0, \\
(\Phi \varphi g)_{655} + (\Phi \varphi g)_{556} + (\Phi \varphi g)_{565} &= -a_8 = 0, \\
(\Phi \varphi g)_{677} + (\Phi \varphi g)_{776} + (\Phi \varphi g)_{767} &= -b_8 = 0, \\
(\Phi \varphi g)_{855} + (\Phi \varphi g)_{558} + (\Phi \varphi g)_{585} &= a_6 = 0, \\
(\Phi \varphi g)_{877} + (\Phi \varphi g)_{778} + (\Phi \varphi g)_{787} &= b_6 = 0, \\
(\Phi \varphi g)_{555} &= \frac{1}{2}(a + b)a_3 - a_7 = 0, \\
(\Phi \varphi g)_{777} &= -\frac{1}{2}(a + b)b_1 + b_5 = 0.
\end{align*}
\]

From (3.2) and (4.2) we see that the triple \((M_8, \varphi, g)\) is quasi-K"ahler-Norden-Walker if and only if the PDEs in the form (3.3) holds. On the other hand, the equation (3.3) is a K"ahler condition of almost Norden-Walker manifolds. Thus we have

**Theorem 4.2.** Let \((M_8, \varphi, g)\) be an almost Norden-Walker Manifold. Then there does not exist a (non-K"ahler) quasi-K"ahler structure on this manifold.

5 Conclusions

A Walker \(n\)-manifold is a semi-Riemannian manifold which admits a field of parallel null \(r\) planes with \(r \leq \frac{n}{2}\). In this article, we study the almost Norden structures of a Walker 8-manifold \((M, g)\) which admits a field of parallel null 4-planes. The metric \(g\) is necessarily of neutral signature \((+++--\ldots)\). In [8], the authors consider Goldberg’s conjecture but for the metrics with neutral signature. They initially display examples of almost Kahler-Einstein neutral structures on \(R^8\) such that the almost complex structure is not integrable. Then, they obtain the structures of the same type on the torus \(T^8\). Therefore, it is proved that the neutral version of Goldberg’s conjecture fails. For such restricted Walker 8-manifolds, we study almost Norden structures on 8-dimensional Walker manifolds. We discuss the integrability and K"ahler(holomorphic) conditions for these structures. Also, nonexistence of (non-K"ahler) quasi-K"ahler structures on almost Norden-Walker 8-manifolds is proved.
Some notes concerning Norden-Walker 8-manifolds

References


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