On $\phi$-pseudo symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection

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Abstract. The object of the present paper is to study $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection and obtain a necessary and sufficient condition of a $\phi$-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection to be $\phi$-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection.

Key words: $\phi$-pseudo symmetric, $\phi$-pseudo Ricci symmetric, Kenmotsu manifold, Einstein manifold, quarter-symmetric metric connection.

1 Introduction

In [48] Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing $\xi$ is a constant, say $c$. He proved that they could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $\mathbb{R} \times f \mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [29] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors.

As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [34] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type $(0, 0), (\alpha, 0)$ and $(0, \beta)$ are called the cosympletic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds respectively, $\alpha, \beta$ being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively.

The study of Riemann symmetric manifolds began with the work of Cartan [6]. A Riemannian manifold $(M^n, g)$ is said to be locally symmetric due to Cartan [6] if
its curvature tensor $R$ satisfies the relation $\nabla R = 0$, where $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [55], semisymmetric manifold by Szabó [47], pseudosymmetric manifold in the sense of Deszcz [20], pseudosymmetric manifold in the sense of Chaki [7].

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [55], semisymmetric manifold by Szabó [47], pseudosymmetric manifold in the sense of Deszcz [20], pseudosymmetric manifold in the sense of Chaki [7].

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be pseudosymmetric in the sense of Chaki [7] if it satisfies the relation

\[
+ A(U) R(X, Y, Z, W),
\]

i.e.,

\[
(\nabla W R)(X, Y)Z = 2 A(W) R(X, Y)Z + A(X) R(W, Y)Z
+ A(Y) R(X, W)Z + A(Z) R(X, Y)W
+ g(R(X, Y)Z, W)\rho,
\]

for any vector field $X, Y, Z, U$ and $W$, where $R$ is the Riemannian curvature tensor of the manifold, $A$ is a non-zero 1-form such that $g(X, \rho) = A(X)$ for every vector field $X$. Such an $n$-dimensional manifold is denoted by $(PS)_n$.

Every recurrent manifold is pseudosymmetric in the sense of Chaki [7] but not conversely. Also the pseudosymmetry in the sense of Chaki is not equivalent to that in the sense of Deszcz [20]. However, pseudosymmetry by Chaki will be the pseudosymmetry by Deszcz if and only if the non-zero 1-form associated with $(PS)_n$ is closed. Pseudosymmetric manifolds in the sense of Chaki have been studied by Chaki and Chaki [9], Chaki and De [10], De [12], De and Biswas [14], De, Murathan and Özgür [17], Özen and Altay ([36], [37]), Tarafder ([51], [52]), Tarafder and De [53] and others.

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor $S$ of type $(0,2)$ satisfies $\nabla S = 0$, where $\nabla$ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifold [38], Ricci semisymmetric manifold [47], pseudo Ricci symmetric manifold by Deszcz [21], pseudo Ricci symmetric manifold by Chaki [8].

A non-flat Riemannian manifold $(M^n, g)$ is said to be pseudo Ricci symmetric [8] if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

\[
(\nabla_X S)(Y, Z) = 2 A(X) S(Y, Z) + A(Y) S(X, Z) + A(Z) S(Y, X)
\]

for any vector field $X, Y, Z$, where $A$ is a nowhere vanishing 1-form and $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. Such an $n$-dimensional manifold is denoted by $(PRS)_n$. The pseudo Ricci symmetric manifolds have been also studied by Arslan et. al [3], Chaki and Saha [11], De and Mazumder [16], De, Murathan and Özgür [17], Özen [35] and many others.

The relation (1.3) can be written as

\[
(\nabla_X Q)(Y) = 2 A(X) Q(Y) + A(Y) Q(X) + S(Y, X)\rho,
\]
where $\rho$ is the vector field associated to the 1-form $A$ such that $A(X) = g(X, \rho)$ and $Q$ is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all $X, Y$.

As a weaker version of local symmetry, the notion of locally $\phi$-symmetric Sasakian manifolds was introduced by Takahashi [49]. Generalizing the notion of locally $\phi$-symmetric Sasakian manifolds, De, Shaikh and Biswas [18] introduced the notion of $\phi$-recurrent Sasakian manifolds. In this connection De [13] introduced and studied $\phi$-symmetric Kenmotsu manifolds and in [19] De, Yildiz and Yaliniz introduced and studied $\phi$-recurrent Kenmotsu manifolds. In this connection it may be mentioned that Shaikh and Hui studied locally $\phi$-symmetric $\beta$-Kenmotsu manifolds [43] and extended generalized $\phi$-recurrent $\beta$-Kenmotsu Manifolds [44], respectively. Also in [39] Prakash studied concircularly $\phi$-recurrent Kenmotsu Manifolds. In [46] Shukla and Shukla studied $\phi$-Ricci symmetric Kenmotsu manifolds. Recently the present author [26] studied $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric Kenmotsu manifolds.

**Definition 1.1.** A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n > 3)$ is said to be $\phi$-pseudo symmetric [26] if the curvature tensor $R$ satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 2A(W)R(X, Y)Z + A(X)R(W, Y)Z + A(Y)R(X, W)Z + g(R(X, Y)Z, W)\rho$$

for any vector field $X$, $Y$, $Z$ and $W$, where $A$ is a non-zero 1-form. In particular, if $A = 0$ then the manifold is said to be $\phi$-symmetric [13].

**Definition 1.2.** A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n > 3)$ is said to be $\phi$-pseudo Ricci symmetric [26] if the Ricci operator $Q$ satisfies

$$\phi^2((\nabla_X Q)(Y)) = 2A(X)QY + A(Y)QX + S(Y, X)\rho$$

for any vector field $X$, $Y$, where $A$ is a non-zero 1-form.

In particular, if $A = 0$, then (1.6) turns into the notion of $\phi$-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [46].

In [22] Friedmann and Schouten introduced the notion of semisymmetric linear connection on a differentiable manifold. Then in 1932 Hayden [24] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semisymmetric metric connection on a Riemannian manifold has been given by Yano in 1970 [56]. In 1975, Golab introduced the idea of a quarter symmetric linear connection in differentiable manifolds.

A linear connection $\nabla$ in an $n$-dimensional differentiable manifold $M$ is said to be a quarter symmetric connection [23] if its torsion tensor $\tau$ of the connection $\nabla$ is of the form

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

where $\eta$ is a 1-form and $\phi$ is a tensor of type $(1,1)$. In particular, if $\phi X = X$ then the quarter symmetric connection reduces to the semisymmetric connection. Thus the
notion of quarter symmetric connection generalizes the notion of the semisymmetric connection. Again if the quarter symmetric connection $\nabla$ satisfies the condition

\[(\nabla_X g)(Y, Z) = 0\]

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold $M$, then $\nabla$ is said to be a quarter symmetric metric connection. Quarter symmetric metric connection have been studied by many authors in several ways to a different extent such as [1], [2], [4], [25], [27], [28], [30], [31], [32], [33], [41], [42], [45], [50], [54]. Recently Prakasha [40] studied $\phi$-symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection.

Motivated by the above studies the present paper deals with the study of $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of $\phi$-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection and obtain a necessary and sufficient condition of a $\phi$-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection to be $\phi$-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection. In section 4, we have studied $\phi$-pseudo Ricci symmetric symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection.

## 2 Preliminaries

A smooth manifold $(M^n, g)$ ($n = 2m + 1 > 3$) is said to be an almost contact metric manifold [5] if it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$, an 1-form $\eta$ and a Riemannian metric $g$ which satisfy

\[
\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi,
\]

\[
g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]

for all vector fields $X, Y$ on $M$.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following condition holds [29]:

\[
\nabla_X \xi = X - \eta(X)\xi,
\]

\[
(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X,
\]

where $\nabla$ denotes the Riemannian connection of $g$.

In a Kenmotsu manifold, the following relations hold [29]:

\[(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y),\]

\[R(X, Y)\xi = \eta(X)Y - \eta(Y)X,\]
(2.8) \[ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \]
(2.9) \[ \eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \]
(2.10) \[ S(X, \xi) = -(n-1)\eta(X), \]
(2.11) \[ S(\xi, \xi) = -(n-1), \quad \text{i.e.,} \quad Q\xi = -(n-1)\xi, \]
(2.12) \[ S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \]

(2.13) \[ (\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W \]
for any vector field \( X, Y, Z \) on \( M \) and \( R \) is the Riemannian curvature tensor and \( S \) is the Ricci tensor of type \((0,2)\) such that \( g(QX, Y) = S(X, Y) \).

Let \( M \) be an \( n \)-dimensional Kenmotsu manifold and \( \nabla \) be the Levi-Civita connection on \( M \). A quarter symmetric metric connection \( \nabla \) in a Kenmotsu manifold is defined by \((23), [40]) \[
(2.14) \quad \nabla_X Y = \nabla_X Y + H(X, Y),
\]
where \( H \) is a tensor of type \((1,1)\) such that \[
(2.15) \quad H(X, Y) = \frac{1}{2} [\tau(X, Y) + \tau'(X, Y) + \tau'(Y, X)]
\]
and \[
(2.16) \quad g(\tau'(X, Y), Z) = g(\tau(Z, X), Y).
\]

From (1.7) and (2.16), we get \[
(2.17) \quad \tau'(X, Y) = g(\phi Y, X)\xi - \eta(X)\phi Y.
\]

Using (1.7) and (2.17) in (2.15), we obtain \[
(2.18) \quad H(X, Y) = -\eta(X)\phi Y.
\]

Hence a quarter symmetric metric connection \( \nabla \) in a Kenmotsu manifold is given by \[
(2.19) \quad \nabla_X Y = \nabla_X Y - \eta(X)\phi Y.
\]

If \( R \) and \( \bar{R} \) are respectively the curvature tensor of Levi-Civita connection \( \nabla \) and the quarter symmetric metric connection \( \nabla \) in a Kenmotsu manifold then we have \([40]) \[
(2.20) \quad \bar{R}(X, Y)Z = R(X, Y)Z - 2d\eta(X, Y)\phi Z + [\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)]\xi + [\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z).
\]
From (2.20) we have

\[
S(Y, Z) = S(Y, Z) - 2d\eta(\phi Z, Y) + g(\phi Y, Z) + \psi\eta(Y)\eta(Z),
\]

where \( S \) and \( S \) are respectively the Ricci tensor of a Kenmotsu manifold with respect to the quarter symmetric metric connection and Levi-Civita connection and \( \psi = \text{tr.} \omega \), where \( \omega(X, Y) = g(\phi X, Y) \). From (2.21) it follows that the Ricci tensor with respect to quarter symmetric metric connection is not symmetric.

Also from (2.21), we have

\[
\tau = r + 2(n - 1),
\]

where \( \tau \) and \( r \) are the scalar curvatures with respect to quarter symmetric metric connection and Levi-Civita connection respectively.

From (2.1), (2.2), (2.5), (2.13), (2.19) and (2.20), we get

\[
(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W + [\eta(Y)g(\phi W, X) - \eta(X)g(\phi W, Y)]\xi - \eta(W)[\eta(X)Y - \eta(Y)X] + \eta(X)\phi Y - \eta(Y)\phi X.
\]

Again from (2.19) and (2.20), we have

\[
\]

**Definition 2.1.** A Kenmotsu manifold \( M \) is said to be \( \eta \)-Einstein if its Ricci tensor \( S \) of type \((0,2)\) is of the form

\[
S = ag + b\eta \otimes \eta,
\]

where \( a, b \) are smooth functions on \( M \).

**3 \( \phi \)-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection**

**Definition 3.1.** A Kenmotsu manifold \( M^n(\phi, \xi, \eta, g) \) \((n = 2m + 1 > 3)\) is said to be \( \phi \)-pseudo symmetric with respect to quarter symmetric metric connection if the curvature tensor \( R \) with respect to quarter symmetric metric connection satisfies

\[
\phi^2((\nabla_W R)(X, Y)Z) = 2A(W)R(X, Y)Z + A(X)R(W, Y)Z + A(Y)R(X, W)Z + A(Z)R(X, Y)W + g(R(X, Y)Z, W)\rho,
\]

for any vector field \( X, Y, Z \) and \( W \), where \( A \) is a non-zero 1-form.

In particular, if \( A = 0 \) then the manifold is said to be \( \phi \)-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection [40].
On $\phi$-pseudo symmetric Kenmotsu manifolds

We now consider a Kenmotsu manifold $M^n(\phi, \xi, h, g)$ ($n = 2m + 1 > 3$), which is $\phi$-pseudo symmetric with respect to quarter symmetric metric connection. Then by virtue of (2.1), it follows from (3.1) that

$$
\tag{3.2}
(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z) \xi
= 2A(W)R(X, Y)Z + A(X)R(W, Y)Z + A(Y)R(X, W)Z
+ A(Z)R(X, Y)W + g(R(X, Y)Z, W)\rho
$$

from which it follows that

$$
\tag{3.3}
-g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U)
= 2A(W)g(R(X, Y)Z, U) + A(X)g(R(W, Y)Z, U) + A(Y)g(R(X, W)Z, U)
+ A(Z)g(R(X, Y)W, U) + g(R(X, Y)Z, W)A(U).
$$

Taking an orthonormal frame field and then contracting (3.3) over $X$ and $U$ and then using (2.1) and (2.2), we get

$$
\tag{3.4}
-g((\nabla_W S)(Y, Z), U) + \eta((\nabla_W S)(Y, Z))\xi
= 2A(W)\overline{S}(Y, Z) + A(Y)\overline{S}(W, Z) + A(Z)\overline{S}(Y, W)
+ A(\overline{R}(W, Y)Z) + A(\overline{R}(W, Z)Y).
$$

Using (2.8), (2.23) and (2.24), we have

$$
\tag{3.5}
g((\nabla_W R)(\xi, Y)Z, \xi) = -g((\nabla_W R)(\xi, Y)\xi, Z)
= g(\phi W, Y)\eta(Z) + g(\phi Y, Z)\eta(W)
+ [g(Y, Z) - \eta(Y)\eta(Z)]\eta(W).
$$

By virtue of (3.5) it follows from (3.4) that

$$
\tag{3.6}
(\nabla_W S)(Y, Z) = -2A(W)\overline{S}(Y, Z) - A(Y)\overline{S}(W, Z) - A(Z)\overline{S}(Y, W)
- A(\overline{R}(W, Y)Z) - A(\overline{R}(W, Z)Y) - g(\phi W, Y)\eta(Z)
- g(\phi Y, Z)\eta(W) - [g(Y, Z) - \eta(Y)\eta(Z)]\eta(W).
$$

This leads to the following:

**Theorem 3.1.** A $\phi$-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is pseudo Ricci symmetric with respect to quarter symmetric metric connection if and only if

$$
A(\overline{R}(W, Y)Z) + A(\overline{R}(W, Z)Y) + g(\phi W, Y)\eta(Z)
+ g(\phi Y, Z)\eta(W) + [g(Y, Z) - \eta(Y)\eta(Z)]\eta(W) = 0.
$$

Setting $Z = \xi$ in (3.4) and using (3.5), we get

$$
\tag{3.7}
-(\nabla_W S)(Y, \xi) + g(\phi W, Y)
= 2A(W)\overline{S}(Y, \xi) + A(Y)\overline{S}(W, \xi) + A(\xi)\overline{S}(Y, W)
+ A(\overline{R}(W, Y)\xi) + A(\overline{R}(W, \xi)Y).$$
By virtue of (2.7), (2.8), we have from (2.20) that
\[ \mathcal{R}(X,Y)\xi = \eta(X)[Y - \phi Y] - \eta(Y)[X - \phi X], \]
(3.8)
\[ \mathcal{R}(X,\xi)Y = [g(X,Y) - g(\phi X,Y)]\xi - \eta(Y)[X - \phi X]. \]
(3.9)

Also in view of (2.10) we get from (2.21) that
\[ \mathcal{S}(Y,\xi) = \psi - (n - 1)\eta(Y). \]
(3.10)

We know that
\[ \nabla W \mathcal{S}(Y,\xi) = \psi - (n - 1)\eta(Y). \]
(3.11)

Using (2.4), (2.10), (2.19) and (2.21) in (3.11) we get
\[ \nabla W \mathcal{S}(Y,\xi) = -S(Y,W) + 2d\eta(\phi Y,W) - g(\phi Y,W) - [\psi - (n - 1)]\eta(Y). \]
(3.12)

In view of (2.3), (2.21) and (3.8)-(3.10), we have from (3.12) that
\[ [1 - A(\xi)]S(Y,W) = [\psi - (n - 1) - A(\xi)]g(Y,W) \]
\[ + 2A(\xi)g(\phi Y,W) + [\psi - (n - 1)]2A(W)\eta(Y) \]
\[ + A(Y)\eta(W) - [1 - A(\xi)]\psi\eta(Y)\eta(W) \]
\[ + [A(Y) - A(\phi Y)]\eta(W) \]
\[ - 2[A(W) - A(\phi W)]\eta(Y). \]
(3.13)

Contracting (3.13) over \( Y \) and \( W \), we obtain
\[ [1 - A(\xi)]r = (n - 1)(\psi - n) + 2(3\psi - 2n + 1)A(\xi). \]
(3.14)

This leads to the following:

**Theorem 3.2.** In a \( \phi \)-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection the Ricci tensor and the scalar curvature are respectively given by (3.13) and (3.14).

In particular, if \( A = 0 \) then (3.13) reduces to
\[ S(Y,W) = [\psi - (n - 1)]g(Y,W) - \psi\eta(Y)\eta(W), \]
which implies that the manifold under consideration is \( \eta \)-Einstein.

This leads to the following:

**Corollary 3.3.** A \( \phi \)-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is an \( \eta \)-Einstein manifold.

Using (2.24) in (3.2), we get
\[ (\nabla W \mathcal{R})(X,Y)Z = -g((\nabla W \mathcal{R})(X,Y)\xi,\xi)\xi - 2A(W)\mathcal{R}(X,Y)Z \]
\[ - A(X)\mathcal{R}(W,Y)Z - A(Y)\mathcal{R}(X,W)Z \]
\[ - A(Z)\mathcal{R}(X,Y)W - g(\mathcal{R}(X,Y)Z,\rho). \]
(3.15)
A Kenmotsu manifold is symmetric metric connection if and only if the relation

\[ \text{Theorem 3.5.} \]

From (3.16) and (3.17), we can state the following:

\[ \text{(3.16)} \]
\[ (\nabla W R)(X, Y)Z = [R(X, Y, W, Z) + g(X, Z)g(Y, W) - g(X, W)g(Y, Z)] + \eta(X)g(\phi W, Y) - \eta(Y)g(\phi W, X) + (g(Y, Z) + g(\phi Y, Z))\eta(W)\eta(Y) - \{g(X, Z) + g(\phi X, Z)\}\eta(W)\eta(Y) + 2A(W)[R(X, Y)Z - 2d\eta(X, Y)\phi Z] + \{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\} + \{\eta(Y)\phi X - \eta(X)\phi Y\} + \{2A(W)\} + \{2d\eta(X, Y)\phi Z\} + \{\eta(Y)\phi X - \eta(X)\phi Y\} \}

for arbitrary vector fields \( X, Y, Z \) and \( W \).

This leads to the following:

\[ \text{Theorem 3.4.} A \text{ Kenmotsu manifold is } \phi \text{-pseudo symmetric with respect to quarter symmetric metric connection if and only if the relation (3.16) holds.} \]

Let us take a \( \phi \text{-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection. Then the relation (1.5) holds. By virtue of (2.1), (2.13) and the relation } g((\nabla W R)(X, Y)Z, U) = -g((\nabla W R)(X, Y)U, Z) \text{ it follows from (1.5) that} \]

\[ \text{(3.17)} \]
\[ (\nabla W R)(X, Y)Z = [R(X, Y, W, Z) + g(X, Z)g(Y, W) - g(X, W)g(Y, Z)] + \eta(X)g(\phi W, Y) - \eta(Y)g(\phi W, X) + (g(Y, Z) + g(\phi Y, Z))\eta(W)\eta(Y) - \{g(X, Z) + g(\phi X, Z)\}\eta(W)\eta(Y) + 2A(W)[R(X, Y)Z - 2d\eta(X, Y)\phi Z] + \{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\} + \{\eta(Y)\phi X - \eta(X)\phi Y\} + \{2A(W)\} + \{2d\eta(X, Y)\phi Z\} + \{\eta(Y)\phi X - \eta(X)\phi Y\} \]

From (3.16) and (3.17), we can state the following:

\[ \text{Theorem 3.5.} A \phi \text{-pseudo symmetric Kenmotsu manifold is invariant under quarter
A Kenmotsu manifold is \( S \) symmetric metric connection if and only if the relation
\[
\left[ \eta(X)g(\phi W, Y) - \eta(Y)g(\phi W, X) + \{g(Y, Z) + g(\phi Y, Z)\} \eta(W) \eta(X) \right.
\]
\[
- \left\{ g(X, Z) + g(\phi X, Z) \right\} \eta(W) \eta(Y) \eta(X) \xi + 2A(W)[2d\eta(X, Y)\phi Z
\]
\[
- \left\{ \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) \right\} \xi - \{ \eta(Y)\phi X - \eta(X)\phi Y \} \eta(Z) \]
\[
+ A(X)[2d\eta(W, Y)\phi Z - \{\eta(W)g(\phi Y, Z) - \eta(Y)g(\phi W, Z)\} \xi
\]
\[
- \left\{ \eta(Y)\phi W - \eta(W)\phi Y \right\} \eta(Z) \] \[+ A(Y)[2d\eta(X, W)\phi Z
\]
\[
- \left\{ \eta(Y)g(\phi W, Z) - \eta(W)g(\phi X, Z) \right\} \xi - \{ \eta(W)\phi X - \eta(X)\phi W \} \eta(Z) \]
\[
+ A(Z)[2d\eta(X, Y)\phi W - \{\eta(X)g(\phi Y, W) - \eta(Y)g(\phi X, W)\} \xi
\]
\[
- \left\{ \eta(Y)\phi X - \eta(X)\phi Y \right\} \eta(W) \] \[+ \{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\} \eta(W)
\]
\[
- \left\{ \eta(Y)g(\phi X, W) - \eta(X)g(\phi Y, W) \right\} \eta(Z) \] \[\rho = 0.
\]
holds for arbitrary vector fields \( X, Y, Z \) and \( W \).

4 \( \phi \)-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection

**Definition 4.1.** A Kenmotsu manifold \( M^n(\phi, \xi, \eta, g) \) \( (n = 2m + 1 > 3) \) is said to be \( \phi \)-pseudo Ricci symmetric with respect to quarter symmetric metric connection if the Ricci operator \( Q \) satisfies
\[
\phi^2((\nabla_X Q)(Y)) = 2A(X)QY + A(Y)QX + S(Y, X)\rho.
\]
for any vector field \( X, Y \), where \( A \) is a non-zero 1-form.

In particular, if \( A = 0 \), then (4.1) turns into the notion of \( \phi \)-Ricci symmetric Kenmotsu manifold with respect to quarter symmetric metric connection.

Let us take a Kenmotsu manifold \( M^n(\phi, \xi, \eta, g) \) \( (n = 2m + 1 > 3) \), which is \( \phi \)-pseudo Ricci symmetric with respect to quarter symmetric metric connection. Then by virtue of (2.1) it follows from (4.1) that
\[
-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 2A(X)QY + A(Y)QX + S(Y, X)\rho
\]
from which it follows that
\[
-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z)
\]
\[
\]
Putting \( Y = \xi \) in (4.2) and using (2.4), (2.10), (2.19), (2.21) and (3.10), we get
\[
1 - A(\xi)S(X, Z) = [\psi - (n - 1) - 2A(\xi)]g(X, Z)
\]
\[
+ A(\xi)g(\phi X, Z) + [(\psi + 2)A(\xi) - \psi\eta(X)\eta(Z)
\]
\[
+ [\psi - (n - 1)] [2A(X)\eta(Z) + A(Z)\eta(X)].
\]
This leads to the following:
Theorem 4.1. In a $\phi$-pseudo Ricci symmetric Kenmotsu manifold with quarter symmetric metric connection the Ricci tensor is of the form (4.3).

In particular, if $A = 0$ then from (4.3), we get

$$S(X, Z) = \{\psi - (n - 1)\}g(X, Z) - \psi\eta(X)\eta(Z),$$

which implies that the manifold under consideration is $\eta$-Einstein. This leads the following:

Corollary 4.2. A $\phi$-Ricci symmetric Kenmotsu manifold with quarter symmetric metric connection is an $\eta$-Einstein manifold.

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References


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