Skew lattice structures on the financial events plane

David Carfì, Karin Cvetko-Vah

Abstract. In this paper we show that the plane of financial events (introduced recently by one of the authors) can be endowed, in a natural way, with skew lattice structures. These structures, far from being merely pure mathematical ones, have a precise financial dynamical meaning, indeed the real essence of the structures introduced in the paper is a dynamical one. Moreover this dynamical structures fulfill several meaningful properties. In the paper several theorems are proved about these structures and some applications are given.

M.S.C. 2010: 62P05, 91B02, 91B06, 06B99

Key words: Skew lattice; Green’s equivalences; financial event; compound interest.

1 Preliminaries on skew lattices

Skew lattices represent the most studied class of non-commutative lattices. The study of non-commutative variations of lattices originates in Jordan’s 1949 paper [15]. The current study of skew lattices began with the 1989 paper of Leech [13], where the fundamental structural theorems were proved. The importance of skew lattices lies in the structural role they play in the study of discriminator varieties, see Bignall and Leech [2]. A recent result of Cvetko-Vah and Leech states that if the set of idempotents $E(R)$ in a ring $R$ is closed under multiplication then the join operation can be defined so that $E(R)$ forms a skew lattice, see [12] for the details.

1.1 Basic definitions

An algebraic structure $(S, \wedge, \vee)$ is said a skew lattice if

- both operations $\wedge$ and $\vee$ are associative;
- the two operations satisfy the absorption laws

$$x \wedge (x \vee y) = x, \quad (y \vee x) \wedge x = x$$

and their corresponding dual relations.

- If one of the two operations $\wedge, \vee$ is commutative, then so is the other one, and we have a (commutative) lattice.
This implies that all the elements of a skew lattice are idempotent for both operations, in other words the two equalities $x \land x = x$ and $x \lor x = x$ hold for all elements $x \in S$.

**Definition 1.1.** A skew lattice is said to be **cancellative** if

- the equality $x \land y = x \land z$ together with the dual relation $x \lor y = x \lor z$ imply the equality $y = z$;
- $x \land y = z \land y$ together with $x \lor y = z \lor y$ imply $x = z$.

Cancellation is equivalent to distributivity in the commutative case.

### 1.2 Green’s equivalence relations

On a skew lattice $(S, \land, \lor)$ the three canonical **Green’s equivalence relations** $R$, $L$ and $D$ on $S$ are defined by the equivalences

- $aRb \iff (a \land b = b \land a = a) \iff (a \lor b = a \land b = b)$
- $aLb \iff (a \land b = a \land b = a) \iff (a \lor b = b \lor a = a)$

and by the equivalences

- $aDb \iff (a \land b \land a = a \land b = b)$
- $\iff (a \lor b \lor a = a \land b = b)$,

for any points $a, b$ in $S$.

The **Leech’s First Decomposition Theorem** for skew lattices states that on any skew lattice $(S, \land, \lor)$ the Green’s relation $D$ is a congruence with respect to both the operations $\land, \lor$: each $D$-class is a rectangular band and the quotient space $S/D$ is a lattice, also referred to as the **maximal lattice image** of $S$. (See [13] for details).

### 1.3 Preorders induced by a skew lattice structure

On the underlying set $S$ the **preorder induced by the skew lattice structure** $(\land, \lor)$ is the relation $\preceq$ on $S$ defined by the equivalence

- $a \preceq b \iff a \land b = a \land b = a \iff b \lor a = b$.

The preorder $\preceq$ determines (in the standard way) an equivalence relation, its **indifference relation**, which is nothing but the Green’s equivalence $D$. Consequently, the preorder on $S$ induces a (partial) order on the lattice $S/D$. When the quotient $S/D$ is a chain with respect to that order, the skew lattice $S$ itself is called a **skew chain**.

The **natural (partial) order** $\preceq$ can be defined on $S$ by the lattice structure, defining the majoration $x \preceq y$ if and only if

- $x \land y = y \land x = x$. 

1.4 Right-handed and left-handed skew lattices

A skew lattice is *right-handed* if it satisfies the identities,
\[ x \land y \land x = y \land x, \quad x \lor y \lor x = x \lor y. \]

Hence the identities \( x \land y = y \) and \( x \lor y = x \) hold on each \( D \)-class. *Left-handed* skew lattices are defined by the dual identities.

The *Leech's Second Decomposition Theorem* for skew lattices [13] states that "On every skew lattice \((S, \land, \lor)\) the Green's relations \(R\) and \(L\) are congruencies with respect to both the operations \(\land, \lor\), and \(S\) is isomorphic to the fiber product of a left-handed and a right-handed skew lattice over a common maximal lattice image, specifically to the fiber product \(S/R \times S/D S/L\).

1.5 Cosets

A skew lattice consisting of only two \(D\)-classes is called *primitive*. The structure of primitive skew lattices was thoroughly studied in [14]. Let \(P\) be a primitive skew lattice with \(D\)-classes \(A\) and \(B\) and assume \(A > B\) on the quotient \(P/D\). For any point \(b \in B\), the set \(A \land b \land A = \{a \land b \land a' : a, a' \in A\}\) is said to be a *coset* of \(A\) in \(B\). Dually, a *coset* of \(B\) in \(A\) is any subset of the form \(B \lor a \lor B\), for some \(a \in A\).

All cosets of \(A\) in \(B\) and all cosets of \(B\) in \(A\) have equal power. It follows that, in the finite case, the power of each coset divides powers \(|A|\) and \(|B|\). The class \(B\) is partitioned by the cosets of \(A\). Given \(a \in A\), in each coset \(B_j\) of \(A\) in \(B\) there is exactly one element \(b \in B\) such that \(b < a\). Dually, given \(b \in B\), in each coset \(A_i\) of \(B\) in \(A\) there is exactly one element \(a \in A\) such that \(b < a\). Given cosets \(A_i\) in \(A\) and \(B_j\) in \(B\) there is a natural bijection of cosets \(\phi_{ij} : A_i \rightarrow B_j\), where \(\phi_{ij}(x)\) = \(y\) iff \(x \land y = y \land x = y\). Moreover, both operations \(\land\) and \(\lor\) on \(P\) are determined by the coset bijections. In the right handed case, the description of cosets can be simplified as it follows
\[ A \land b \land A = b \land A \quad \text{and} \quad B \lor a \lor B = B \lor a. \]

Indeed, for instance, \(a \land b \land a' = (a \land b) \land (b \land a') = b \land a'\).

2 The space of financial events

In [7] the *space of financial events* is defined as the usual Cartesian plane \(\mathbb{R}^2\). It is interpreted as the Cartesian product of a time-axis and a capital-axis. Every pair \(e = (t, c)\) belonging to this plane is called a *financial event with time* \(t\) and *capital* \(c\). If \(c > 0\) \([c \geq 0]\) then \(e\) is called a *strict credit* [weak credit], and if \(c < 0\) \([c \leq 0]\) then \(e\) is called a *strict debt* [weak debt]. If \(c = 0\) then \(e\) is said a *null event*.

Let \(i > -1\) and let
\[ f_i(t, c) = (1 + i)^{-t}c. \]
The function \( f_i \) induces a preorder \( \succeq_i \) on the space of financial events, defined by 
\[(t_0, c_0) \succeq_i (t, c) \text{ if and only if } f_i(t_0, c_0) \leq f_i(t, c), \]
which is further equivalent to 
\[c_0(1 + i)^{t - t_0} \leq c.\]
Following [7], the preorder \( \succeq_i \) is called the preorder induced by a separable capitalization factor of rate \( i \), since it corresponds to the separable capitalization factor of rate \( i \), that is the function 
\[f_i : h \mapsto (1 + i)^h.\]

The preorder \( \succeq_i \) induces an equivalence relation \( \sim_i \) on \( \mathbb{R}^2 \), defined by 
\[(t_0, c_0) \sim_i (t, c) \text{ if and only if } (t_0, c_0) \succeq_i (t, c) \text{ and } (t, c) \succeq_i (t_0, c_0), \]
or equivalently, 
\[\sim_i (t, c) \iff f_i(t_0, c_0) = f_i(t, c).\]

The equivalence class containing an event \((t_0, c_0)\) is given by 
\[[\cdot]_i = \{(t, (1 + i)^{t - t_0}c_0) \mid t \in \mathbb{R}\}\]
and represents a smooth curve in the plane \( \mathbb{R}^2 \).

### 3 The space of financial events as a skew lattice

**Definition 3.1.** Given a fixed real \( i > -1 \), we define non-commutative meet \( \wedge_i \) and non-commutative join \( \vee_i \) of the space of financial events as follows:

\[(t_0, c_0) \wedge_i (t, c) = \begin{cases} 
(t, (1 + i)^{t - t_0}c_0) & \text{if } (t_0, c_0) \succeq_i (t, c) \\
(t, c) & \text{if } (t, c) \succeq_i (t_0, c_0)
\end{cases}\]

and

\[(t_0, c_0) \vee_i (t, c) = \begin{cases} 
(t_0, (1 + i)^{t_0 - t}c) & \text{if } (t_0, c_0) \succeq_i (t, c) \\
(t_0, c_0) & \text{if } (t, c) \succeq_i (t_0, c_0).
\end{cases}\]

**Remark 3.1.** (Well posedness of the definitions). If \((t_0, c_0) \sim_i (t, c)\), then 
\[(t, (1 + i)^{t - t_0}c_0) = (t, c),\]
and the operations \( \wedge_i \) is well defined. A similar observation shows that operation \( \vee_i \) is well defined.

**Remark 3.2.** Note that the event \( e_0 \wedge_i e \) has the time of the second financial event \( e \) and the event \( e_0 \vee_i e \) has the time of first financial event \( e_0 \). It is evident that two events commute (with respect to the defined operations) if and only if they have the same time.

**Theorem 3.3.** Let operations \( \wedge_i \) and \( \vee_i \) on the space of financial events be defined as above. Then \( S_i = (\mathbb{R}^2, \wedge_i, \vee_i) \) is a skew lattice.

**Proof.** We prove idempotency and associativity for operation \( \wedge_i \). A dual proof can then be derived for operation \( \vee_i \). Idempotency is immediate:

\[(t_0, c_0) \wedge (t_0, c_0) = (t_0, (1 + i)^{t_0 - t_0}c_0).\]
To see that \( \land_i \) is associative, consider financial events \( e_0 = (t_0, c_0) \), \( e = (t, c) \) and \( e' = (t', c') \). Consider \( (e_0 \land_i e) \land_i e' \) and \( e_0 \land_i (e \land_i e') \). One must check several cases for the order of events \( e_0, e \) and \( e' \) in respect to \( \preceq_i \). We prove one of the non-trivial cases, the others are similar and shall be omitted. Assume that \( e \preceq_i e_0 \preceq_i e' \). Then

\[
(e_0 \land_i e) \land_i e' = e \land_i e'
\]

and

\[
e_0 \land_i (e \land_i e') = e \land_i e',
\]

because \( f_i(e \land_i e') = f_i(e) \leq f_i(e_0) \). The absorption follows from

\[
e_0 \land_i (e_0 \lor_i e) = (t_0, c_0) \land_i (t_0, (1 + i)h c) = (t_0, c_0)
\]

if \( e_0 \preceq_i e \), and

\[
e_0 \land_i (e_0 \lor_i e) = e_0 \land_i e_0 = e_0,
\]

if \( e \preceq_i e_0 \), and similar calculations. Therefore \( S_i \) is a skew lattice. \( \square \)

4 Dynamical interpretation of the skew lattice operations

The definitions of the two operations can be restated in the following dynamical way.

Proposition 4.1. (Dynamical meaning of the operations). Let

\[
\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2
\]

be the action of the additive group of the real numbers \((\mathbb{R}, +)\) upon the financial events plane defined by

\[
\mu(h, (t, c)) = (t + h, (1 + i)^h c),
\]

for every real \( h \) and for every financial event \( e = (t, c) \). Let us denote the financial event \( \mu(h, e) \) simply by \( h.e \). Then we have

\[
e_0 \land_i e = \begin{cases} (t - t_0).e_0 & \text{if } e_0 \preceq_i e \\ e & \text{if } e \preceq_i e_0 \end{cases}
\]

and

\[
e_0 \lor_i e = \begin{cases} (t_0 - t).e & \text{if } e_0 \preceq_i e \\ e_0 & \text{if } e \preceq_i e_0 \end{cases}
\]

for every couple of financial event \( e_0 = (t_0, c_0) \) and \( e = (t, c) \).

Proof. It is simply a rewriting of the definitions by means of the action \( \mu \). \( \square \)

Hence, the nature of the two definitions is dynamic.

Remark 4.2. For the use, in the context of financial events plane, of the dynamical systems, see [4], [6], [9] and [10]. Further research can be conducted by following [1] and [16].

Let us observe that the non commutativity of the lattice operations is a consequence of their dynamical nature. Let \( e_0 = (t_0, c_0) \) and \( e = (t, c) \) be two financial events, the difference \( h = t - t_0 \) is called the time vector sending \( e_0 \) into \( e \).
Theorem 4.3. (Dynamical meaning of the non-commutativity). Let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events and let $h = t - t_0$ be the time vector sending $e_0$ into $e$. Then, the two commutation relations hold true:

$$e_0 \wedge_i e = h \cdot (e \wedge_i e_0), \quad e_0 \lor_i e = (-h) \cdot (e \lor_i e_0).$$

Proof. We have, for what concerns the meet,

$$e_0 \wedge_i e = \begin{cases} (t - t_0) \cdot e_0 & \text{if } e_0 \preceq_i e \\ (t - t) \cdot e & \text{if } e \preceq_i e_0 \end{cases}$$

and

$$e \wedge_i e_0 = \begin{cases} (t_0 - t) \cdot e & \text{if } e \preceq_i e_0 \\ (t_0 - t_0) \cdot e_0 & \text{if } e_0 \preceq_i e \end{cases},$$

or, in equivalent form,

$$e_0 \wedge_i e = \begin{cases} (h) \cdot e_0 & \text{if } e_0 \preceq_i e \\ (0) \cdot e & \text{if } e \preceq_i e_0 \end{cases}$$

and

$$e \wedge_i e_0 = \begin{cases} (-h) \cdot e & \text{if } e \preceq_i e_0 \\ (-0) \cdot e_0 & \text{if } e_0 \preceq_i e \end{cases}.$$

It is clear, in each case, that $e_0 \wedge_i e = h \cdot (e \wedge_i e_0)$. In a symmetric fashion we obtain the second result. □

Remark 4.4. The relations of commutation of the preceding theorem mean that the nature of non-commutativity is dynamical at all.

5 Financial interpretation of the skew lattice operations

Remark 5.1. (Financial meaning of the operations). Let $e_0$ and $e$ be two financial events, we say that $e_0$ precedes $e$ if the time (first projection) of $e_0$ is less than the time of $e$. From the financial point of view, the two operations, when applied to a pair $(e_0, e)$ of financial events such that $e_0$ precedes $e$, describe the risk-aversion principle with respect to time. Indeed, let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events in the chronological order $(e_0, e)$, the meet of two events is always an event with time $t$ and the join is an event at time $t_0$, in other words the decision-maker prefers (as shadow maximum) the events closest in the time (indeed he prefers the state at $t_0$ of the $i$-best event), also in the case the two events are equivalent at the rate $i$, and symmetrically, the decision-maker finds worst the events which are far in the future, even in the case of equivalence. Further, the decision-maker valuates as shadow minimum the events furthermost in the time (indeed he valuates infimum the state at $t$ of the $i$-worst event), also in the case the two events are equivalent at the rate $i$, in other terms, the decision-maker finds worst the events which are far in the future, even in the case of equivalence.
Proposition 5.2. (Choice meaning of the operations). The meet operation
\[ \land_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto e_0 \land_i e, \]
is a choice function of the family of sets \( \{ \inf(e_0, e) \}_\mathbb{R}^2 \), that is, a function
\[ c : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto c(e_0, e), \]
such that \( c(e_0, e) \in \inf(e_0, e) \), for every pair \( (e_0, e) \) in \( \mathbb{R}^2 \times \mathbb{R}^2 \). Analogously, the join operation
\[ \lor_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto e_0 \lor_i e, \]
is a choice function of the family of sets \( \{ \sup(e_0, e) \}_\mathbb{R}^2 \), that is, a function
\[ c : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto c(e_0, e), \]
such that \( c(e_0, e) \in \sup(e_0, e) \), for every pair \( (e_0, e) \) in \( \mathbb{R}^2 \times \mathbb{R}^2 \).

Proof. Let \( e_0 = (t_0, c_0) \) and \( e = (t, c) \) be two financial events, let us determine the set of the infima of the couple \{\( e_0, e \)\}, with respect to the preorder \( \preceq \). We have
\[ \inf(e_0, e) = \begin{cases} [e_0]_i, & \text{if } e_0 \preceq_i e, \\ [e]_i, & \text{if } e \preceq_i e_0. \end{cases} \]

Observing that the meet \( e_0 \land_i e \) belongs to the set \( \inf(e_0, e) \), the proof is complete. \( \square \)

We can say more than the result of preceding proposition. Recall that, if \( e_0 = (t_0, c_0) \) is a financial event, the evolution curve of \( e_0 \) is, by definition, the curve
\[ \varepsilon(e_0) : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t - t_0),e_0. \]
Let
\[ \varepsilon : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R}^2) : e_0 \mapsto \varepsilon(e_0) \]
be the application sending each financial event into the corresponding evolution curve and let \( \varepsilon(\mathbb{R}^2) \) be the part of the function space \( \mathcal{F}(\mathbb{R}, \mathbb{R}^2) \) image of the financial events plane by means of the application \( \varepsilon \), i.e. the set of all the evolution curves in the financial events plane. The set \( \varepsilon(\mathbb{R}^2) \) can be endowed with the total (linear) order defined by
\[ \varepsilon(e_0) \preceq_i \varepsilon(e) \ if \ and \ only \ if \ e_0 \preceq_i e, \]
for any financial events \( e_0 \) and \( e \). Note that, for any two events \( e_0 \) and \( e \), the infimum \( \inf(\varepsilon(e_0), \varepsilon(e)) \) of the two corresponding evolution curves (which is also a minimum) is a curve of evolution (either \( \varepsilon(e_0) \) or \( \varepsilon(e) \)), and then a function of the time-axis \( \mathbb{R} \) into the plane of financial events \( \mathbb{R}^2 \).

Theorem 5.3. Let
\[ \varepsilon : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R}^2) : e_0 \mapsto \varepsilon(e_0) \]
be the application sending each financial event into the corresponding curve of evolution. Then we have
\[ e_0 \land_i e = \inf(\varepsilon(e_0), \varepsilon(e))(pr_1(e)), \quad e_0 \lor_i e = \sup(\varepsilon(e_0), \varepsilon(e))(pr_1(e)), \]
or in other terms
\[ e_0 \land_i e = \varepsilon(e_0) \land_i \varepsilon(e)(t), \quad e_0 \lor_i e = \varepsilon(e_0) \lor_i \varepsilon(e)(t), \]
for any event \( e_0, e \) with times \( t_0 \) and \( t \) respectively.
6 Basic properties of $S_i$

On a skew lattice $(S; \wedge, \vee)$ we introduce a right preorder $\geq_R$ defined by $a \geq_R b$ if and only if

$$(a \wedge b = b \text{ and } a \vee b = a)$$

and a left preorder $\leq_L$ defined by $a \leq_L b$ if and only if

$$(a \wedge b = a \text{ and } a \vee b = b),$$

for each $a, b$ in $S$. Clearly, the Green’s equivalence relations $R, L$ on $S$ are induced by those preorder respectively, as indifference relations.

**Theorem 6.1.** Let $\wedge_i$ and $\vee_i$ be the skew lattice operations on the space of financial events defined as above. Then:

1) given financial events $e_0$ and $e$, the relation $e_0 \leq_i e$ is equivalent to the equality $e_0 \wedge_i e \wedge_i e_0 = e_0$, which is further equivalent to $e \vee_i e_0 \wedge_i e = e$. In other terms the preorder induced by the skew lattice structure coincides with the preorder $\leq_i$;

2) the Green’s relation $D$ on $S_i$ coincides with the indifference relation $\sim_i$;

3) the right preorder induced by the skew lattice structure is $\geq_i$;

4) the Green’s equivalence $R$ coincides with the relation $\sim_i$;

5) the left preorder induced by the skew lattice structure is the natural order on each fiber $\{t\} \times \mathbb{R}$;

6) the maximal lattice image $S_i/D$ is isomorphic to the chain $(\mathbb{R}, \min, \max)$, and the space of financial events is a skew chain.

**Proof.** 1) To see that the inequality $e_0 \leq_i e$ is equivalent to the equality $e_0 \wedge_i e \wedge_i e_0 = e_0$, first assume that $e_0 \leq_i e$. Direct calculation yields

$$e_0 \wedge_i e \wedge_i e_0 = e_0.$$  

To prove the converse implication, let $e_0$ and $e$ be such that $e_0 \wedge_i e \wedge_i e_0 = e_0$. Assume that $e \leq_i e_0$, i.e. $f_i(t, e) \leq f_i(t_0, e_0)$. Then

$$e_0 = e_0 \wedge_i e \wedge_i e_0 = (t_0, (1 + i)^{-1}t_0),$$

which can only appear if $e \sim_i e_0$. So, if $e_0 \wedge_i e \wedge_i e_0 = e_0$, the only possibility is $e_0 \leq_i e$. That $e_0 \wedge_i e \wedge_i e_0 = e_0$ is equivalent to $e \vee_i e_0 \wedge_i e = e$ is a known fact in any skew lattice. 2) An immediate consequence is that relation $D$ coincides with $\sim_i$. 3) Indeed, the right preorder is defined by $e_0 \geq_R e$ if and only if

$$(e_0 \wedge_i e = e \wedge_i e_0 \wedge_i e = e_0),$$

which means

$$e = e_0 \wedge_i e = \begin{cases} (t - t_0) e_0 & \text{if } e_0 \leq_i e, \vspace{0.5cm} \\
(1 - t_0) e_0 & \text{if } e \leq_i e_0, \end{cases}$$

i.e. $e \leq_i e_0$ and

$$e_0 = e_0 \vee_i e = \begin{cases} (t_0 - t) e_0 & \text{if } e_0 \leq_i e, \vspace{0.5cm} \\
(t_0 - t) e_0 & \text{if } e \leq_i e_0, \end{cases}$$

i.e. $e \leq_i e_0$. 4) Similarly, the left preorder is defined by $e_0 \leq_L e$ if and only if

$$(e_0 \vee_i e = e \vee_i e_0 \vee_i e = e_0),$$

which means

$$e = e_0 \vee_i e = \begin{cases} (t - t_0) e_0 & \text{if } e \leq_i e_0, \vspace{0.5cm} \\
(1 - t_0) e_0 & \text{if } e_0 \leq_i e, \end{cases}$$

i.e. $e \leq_i e_0$ and

$$e_0 = e_0 \wedge_i e = \begin{cases} (t_0 - t) e_0 & \text{if } e \leq_i e_0, \\
(t_0 - t) e_0 & \text{if } e_0 \leq_i e, \end{cases}$$

i.e. $e \leq_i e_0$. 5) The maximal lattice image $S_i/D$ is isomorphic to the chain $(\mathbb{R}, \min, \max)$, and the space of financial events is a skew chain.
Skew lattice structures on the financial events plane

4) It immediately follows from the preceding property, taking into account that \( \sim_i \) is the indifference relation induced by the preorder \( \preceq_i \) and the equivalence \( \mathcal{R} \) is the indifference of the preorder \( \preceq_{\mathcal{R}} \).

5) Indeed, by definition of the left preorder, we have \( e_0 \preceq_L e \), if and only if
\[
(e_0 \land e = e_0 \quad \text{and} \quad e_0 \lor e = e),
\]
which means
\[
e_0 = e_0 \land_i e = \begin{cases} (t - t_0).c_0 & \text{if } e_0 \preceq_i e \\ e & \text{if } e \preceq_i e_0,
\end{cases}
\]
i.e., \( e_0 \preceq_i e \), that is \( t = t_0 \) and \( (e_0)_2 \preceq_i (e)_2 \); and
\[
e = e_0 \lor_i e = \begin{cases} (t_0 - t).e & \text{if } e_0 \preceq_i e \\ e_0 & \text{if } e \preceq_i e_0,
\end{cases}
\]
i.e., \( e = (t_0 - t).e \) and \( e_0 \preceq_i e \), that is \( t = t_0 \) and \( (e_0)_2 \preceq_i (e)_2 \).

6) The \( \mathcal{D} \)-classes are given by \( f_i \)-images. It is clear that any functional \( f_i \) is surjective, therefore we deduce the claimed isomorphism \( S_i/\mathcal{D} \cong (\mathbb{R}, \min, \max) \).

\[ \square \]

Corollary 6.2. Given \( i > -1 \), \( S_i \) is a cancellative skew lattice.

Proof. It was proved in [11] that all skew chains are cancellative.

\[ \square \]

Proposition 6.3. Given \( i > -1 \), the skew chain \( S_i \) is right handed.

Proof. Consider events \( e_0 = (t_0, c_0) \) and \( e = (t, c) \) and assume \( e_0 \preceq_i e \), then
\[
(e_0 \land_i e) \land_i e_0 = (t, (1 + i)^{t - t_0}c_0) \land_i (t_0, c_0)
\]
\[
= (t_0, (1 + i)^{t_0 - t}(1 + i)^{t - t_0}c_0) = e_0 = e \land_i e_0
\]
and
\[
(e \land_i e_0) \land_i e = e_0 \land_i e, \quad \text{as we claimed.}
\]

\[ \square \]

7 Binormality of \( S_i \)

Each equivalence class \([ (t_0, c_0) ] \) is determined by the value \( f(t_0, c_0) \). The set
\[
\{ (0, f(t, c)) \mid (t, c) \in \mathbb{R}^2 \}
\]
is a sub-lattice of the skew lattice \( \mathbb{R}^2 \), and is isomorphic to the maximal lattice image \((\mathbb{R}, \min, \max)\); such a lattice is called a lattice section.

A skew lattice \((S, \land, \lor)\) is called binormal if it satisfies the identities
\[
a \land b \land c \land a = a \land c \land b \land a \quad \text{and} \quad a \lor b \lor c \lor a = a \lor c \lor b \lor a.
\]

A right-handed skew lattice is binormal if and only if it satisfies
\[
b \land c \land a = c \land b \land a \quad \text{and} \quad a \lor b \lor c = a \lor c \lor b.
\]

It follows from [14] that any skew lattice in which any maximal primitive sub-algebra \( A \cup B \) has the property that \( A \) is a single coset of \( B \) in \( A \) and \( B \) is a single coset of \( A \) in \( B \), is binormal.
Theorem 7.1. Given any $i > -1$, the space of financial events $S_i$ is a binormal skew lattice.

Proof. Consider equivalence classes $A = [(t_A, c_A)]$ and $B = [(t_B, c_B)]$ with

$$f(t_B, c_B) < f(t_A, c_A).$$

Then $A \cup B$ is a primitive skew lattice, and $b \leq_i a$ for any $b \in B$ and any $a \in A$. When is $b \leq a$ in respect to the natural partial order? In this case we obtain

$$(t_B, c_B) = (t_B, c_B) \land (t_A, c_A) = (t_A, (1 + i)^{t_B - t_A} c_A),$$

which holds precisely when $t_A = t_B$. Therefore $A$ is the single coset of $B$ in $A$ and $B$ is the single coset of $A$ in $B$. $\square$

If $(S, \land_S, \lor_S)$ and $(T, \land_T, \lor_T)$ are skew lattice, then a homomorphism of skew lattices is any map $h : S \to T$ satisfying

$$h(x \land_S y) = h(x) \land_T h(y)$$

and the dual relation

$$h(x \lor_S y) = h(x) \lor_T h(y),$$

for all $x, y \in S$. A bijective homomorphism of skew lattices is called an isomorphism of skew lattices.

Corollary 7.2. Algebraically, each skew lattice $S_i$ is isomorphic to the direct product $\mathbb{R} \times C$ occurring when $i = 0$. Here $C = \{(0, c) \mid c \in \mathbb{R}\}$ is a right-rectangular skew lattice with the operations given by

$$(0, c) \land_0 (0, d) = (0, d) \quad \text{and} \quad (0, c) \lor_0 (0, d) = (0, c).$$

In particular these various $S_i$ are all isomorphic skew lattices.

8 A financial application

In this section we clarify the financial meaning of the skew lattice operations by means of the order of compound interest with total time-risk aversion, just introduced in the following subsection.

8.1 The order of compound capitalization with total time-risk aversion

Let $i > 0$ be a positive rate of interest and let $\leq'_i$ be the binary relation defined on the open half-plane of strict credits by $e_0 \leq'_i e$ if and only if

$$e_0 \leq_i e \quad \text{and} \quad t_0 \geq t,$$

for any two strict credits $e_0$ and $e$ of time $t_0$ and $t$ respectively. The relation $\leq'_i$ is an order, in fact it is a preorder since it is the conjunction of two preorders; moreover, it is an order since $e_0$ is indifferent, with respect to the preorder $\leq'_i$, to an event $e$ if and
Skew lattice structures on the financial events plane

only if \( e \) belongs to the set-curve of evolution generated by \( e_0 \) and \( t_0 = t \), considered that for each time there is only one event on a curve of evolution with that time.

From a financial point of view this new order represents the rationality of a decision-maker that takes into account not only the compound capitalization at rate \( i \) of the market but that is completely risk-averse in time, indeed if \( e_0 < i e \) but \( t_0 < t \), one does not consider \( e \) preferable to \( e_0 \) but incomparable with \( e_0 \), just for the inequality \( t_0 < t \).

\[ 8.2 \text{ The application} \]

The following theorem shows the relation between the preorder \( \leq' \) and the skew lattice operations.

**Theorem 8.1.** We have \( e_0 \leq'_i e \) if and only if

\[ e \wedge_i e_0 = e_0 \text{ and } t_0 \geq t, \]

or, equivalently,

\[ e \vee_i e_0 = e \text{ and } t_0 \geq t, \]

for any two strict credits \( e_0 \) and \( e \) of time \( t_0 \) and \( t \) respectively.

We present further a possible practical application. In decision problems one of the basic points of investigation is to find suprema and infima of the constraint with respect to a given preorder.

**Proposition 8.2.** a) Let \( K \) be a compact subset of the financial events plane contained in the open half-plane of the strict credits. Then the supremum of \( K \) with respect to the order \( \leq'_i \) is the non-commutative join of any event \( e \) with maximum \( f_i \)-value (at least one there exists by the Weierstrass theorem) with any event \( e_0 \) of \( K \) with maximum time (at least one exists by Weierstrass theorem) in the order \((e, e_0)\):

\[ \sup_{\leq'_i} K = e \vee_i e_0. \]

If \( e'_0 \) and \( e' \) are any two events such that \( f_i(e') = \min_K f_i \) and \( pr_1(e'_0) = \max_K pr_1 \), then

\[ \inf_{\leq'_i} K = e' \wedge_i e'_0. \]

b) Let \( K \) be a compact subset of the financial events plane contained in the open half-plane of the strict debts. Then the supremum of \( K \) with respect to the order \( \leq'_i \) is the non-commutative join of any event \( e \) with maximum \( f_i \)-value (at least one there exists by the Weierstrass theorem) with any event \( e_0 \) of \( K \) with maximum time (at least one exists by Weierstrass theorem) in the order \((e, e_0)\):

\[ \sup_{\leq'_i} K = e \vee_i e_0. \]

If \( e'_0 \) and \( e' \) are any two events such that \( f_i(e') = \min_K f_i \) and \( pr_1(e'_0) = \min_K pr_1 \), then

\[ \inf_{\leq'_i} K = e' \vee_i e'_0. \]

**Remark 8.3.** For the use and determination of extrema and Pareto boundaries, in the context of Decision Theory, see [3], [5] and [8].
References


Author’s address:

David Carfi  
Faculty of Economics, University of Messina,  
Via dei Verdi, 98122, Messina, Italy.  
E-mail: davidcarfi71@yahoo.it (corresponding author)  

Karin Cvetko-Vah  
Fakulteta za matematiko in fiziko, Oddelek za matematiko  
Jadranska 21, SI-1000 Ljubljana  
E-mail: karin.cvetko@fmf.uni-lj.si