The Boubaker polynomials expansion scheme BPES for solving a standard boundary value problem

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Abstract. In this note, we propose an analytical solution to a nonlinear second-order boundary value problem using the 4q-Boubaker Polynomials Expansion Scheme (BPES). The results are plotted, and compared with exact solutions proposed elsewhere, in order to evaluate accuracy.

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1 Introduction

Nonlinear second-order boundary value equation systems are present in different applied physics domains like non-linear mechanics [21], fluid dynamics [13], heat diffusion/transfer [10, 16] and applied mathematics analyses [7, 12, 15, 20].

Among the different formulations, the well-known standard second-order boundary value problem (BVP) is given by F. Geng et al. [9] by the system:

(1.1)
$$\begin{cases} a_0(\xi)f'' + a_1(\xi)f + a_2(\xi)f + a_3(\xi)g'' + a_4(\xi)g' + a_5(\xi)g + F_1(\xi, f, g) = u(\xi) \\ b_0(\xi)f'' + b_1(\xi)f + b_2(\xi)f + b_3(\xi)g'' + b_4(\xi)g' + b_5(\xi)g + F_2(\xi, f, g) = v(\xi) \\ f(\xi)|_{\xi=0} = f(\xi)|_{\xi=1} = 0 \\ g(\xi)|_{\xi=0} = g(\xi)|_{\xi=1} = 0, \end{cases}$$

where $\xi \in [0, 1]$, f and f are ξ-dependent unknown functions, $a_k|_{k=1..5}$, $b_k|_{k=1..5}$, $u(\xi)$ and $v(\xi)$ are given absolutely continuous real-valued functions, F_1 and F_2 are two given nonlinear functions.

In this study, an attempt to give analytical solution to a nonlinear second-order boundary value equation systems is performed. The used protocol involves, as a main step, an original polynomial scheme.

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2 Proposed and exact solutions derivation

The resolution protocol is based on the Boubaker polynomials expansion scheme (BPES) [3, 19, 11, 5, 6, 1, 14, 18, 2, 4, 8]. The first step of this scheme consists of application of the expressions:

$$f(\xi) = \frac{1}{2N_0}\sum_{k=1}^{N_0} \lambda_k \times B_{4k}(\xi \times r_k). \tag{2.1}$$

and

$$g(\xi) = \frac{1}{2N_0}\sum_{k=1}^{N_0} \lambda'_k \times B_{4k}(\xi \times r_k). \tag{2.2}$$

where B_{4k} are the 4k-order Boubaker polynomials, r_k are B_{4k} minimal positive roots, N_0 is a prefixed integer, λ_k and λ'_k are unknown pondering real coefficients.

The main advantage of these formulations is the fact of verifying the four boundary conditions, in advance to problem resolution. Due to the properties of the Boubaker polynomials [19, 1, 2], and since r_k are B_{4k}, the following conditions stand, for i.e. $\lambda_{k+1} = -\lambda_k|_{k=1..N_0-1}$

$$\begin{cases} f(\xi)|_{\xi=0} = \frac{1}{N_0}\sum_{k=1}^{N_0}\lambda_k, \quad f(\xi)|_{\xi=1} = \frac{1}{2N_0}\sum_{k=1}^{N_0}\lambda_k \times B_{4k}(r_k) = 0 \\ g(\xi)|_{\xi=0} = \frac{1}{N_0}\sum_{k=1}^{N_0}\lambda'_k, \quad g(\xi)|_{\xi=1} = \frac{1}{2N_0}\sum_{k=1}^{N_0}\lambda'_k \times B_{4k}(r_k) = 0 \end{cases} \tag{2.3}$$

The studied example [9] is:

$$\begin{cases} f''(\xi) + \xi f(\xi) + 2\xi g(\xi) + \xi f^2(\xi) = G(\xi) \\ G(\xi) = 2\xi sin(\xi \times \pi) - 2\xi^5 - 2\xi^4 + \xi^2 - 2 \\ g'(\xi) + g(\xi) + \xi^2 f(\xi) + sin(\xi) g^2(\xi) = \xi^3(1-\xi) + sin(\xi \times \pi) H(\xi) \\ H(\xi) = (1 + sin(\xi) sin(\xi \times \pi) + \pi cos(\xi \times \pi)) \\ f(\xi)|_{\xi=0} = f(\xi)|_{\xi=1} = 0, \quad g(\xi)|_{\xi=0} = g(\xi)|_{\xi=1} = 0 \end{cases} \tag{2.4}$$

By introducing the first expressions in the main system, and by majoring and integrating along the interval $[0,1]$, f and g and are confined, through the coefficients $\lambda_k|_{k=1..N_0}$ and $\lambda'_k|_{k=1..N_0}$, to be weak solutions of the system:

$$\begin{cases} \sum_{k=1}^{N_0}\lambda_k \times M_k + \sum_{k=1}^{N_0}\lambda'_k \times P_k = \int_0^1 G(\xi)d\xi = \frac{20-19\pi}{10\pi} \\ M_k = \int_0^1 \left(\frac{d^2 B_{4k}(\xi \times r_k)}{d\xi^2} + \xi B_{4k}(\xi \times r_k)\right) d\xi \\ P_k = \int_0^1 (2\xi B_{4k}(\xi \times r_k))d\xi, \quad M'_k = \int_0^1 (\xi^2 B_{4k}(\xi \times r_k))d\xi \\ P'_k = \int_0^1 \left(\frac{dB_{4k}(\xi \times r_k)}{d\xi} + B_{4k}(\xi \times r_k)\right) d\xi. \end{cases} \tag{2.5}$$

The set of solutions $\left.\widehat{\lambda}_k\right|_{k=1..N_0}$ and $\left.\widehat{\lambda}'_k\right|_{k=1..N_0}$ is the one which minimizes the global minimum square function G_{MS}:

$$G_{MS} = \left(\sum_{k=1}^{N_0} \widehat{\lambda}_k \times M_k + \sum_{k=1}^{N_0} \widehat{\lambda}'_k \times P_k - \frac{20-19\pi}{10\pi}\right)^2 + \left(\sum_{k=1}^{N_0} \widehat{\lambda}_k \times M'_k + \sum_{k=1}^{N_0} \widehat{\lambda}'_k \times P'_k - \frac{40+\pi}{20\pi}\right)^2.$$

The correspondent solutions are represented in the Fig. 1, along with the exact solution given by F. Geng et al. [9] and A. Saadatmandi et al. [17]:

(2.6) $$f(\xi) = \xi - \xi^2, \quad g(\xi) = sin(\xi \times \pi)$$

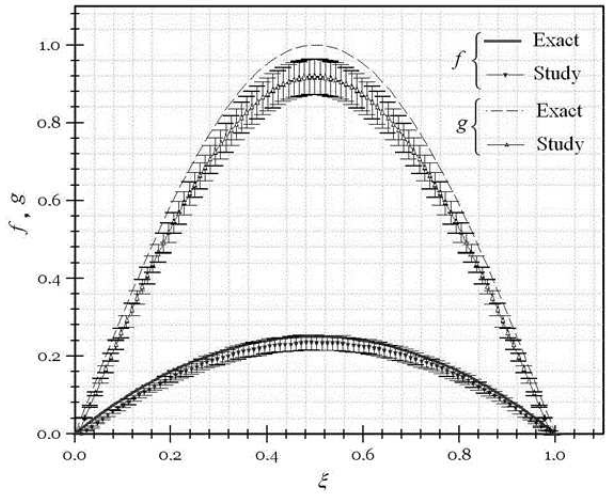

Figure 1: Exact and proposed solutions f and g.

Table 1.

ξ	$g(\xi)_{Ref.[9]}$	$f(\xi)_{Ref.[21]}$	$g(\xi)_{BPES}$	$f(\xi)_{BPES}$	Err. (g)	Err.(f)
0.00	0.00	0.00	0.00	0.00589	—	—
0.05556	0.17365	0.00868	0.05247	0.17169	0.05343	1.12531
0.11111	0.34202	0.0171	0.09877	0.33502	0.09877	2.04668
0.17778	0.52992	0.0265	0.14617	0.52319	0.1459	1.26982
0.24444	0.69466	0.03473	0.18469	0.68874	0.18416	0.85249
0.31111	0.82904	0.04145	0.21432	0.82439	0.21357	0.56005
0.37778	0.92718	0.04636	0.23506	0.92418	0.23416	0.32368
0.44444	0.98481	0.04924	0.24691	0.9837	0.24593	0.11252
0.51111	0.99939	0.04997	0.24988	1.00026	0.24888	0.08647
0.57778	0.9703	0.04851	0.24395	0.97307	0.24303	0.28607
0.64444	0.89879	0.04494	0.22914	0.90325	0.22836	0.49545
0.71111	0.78801	0.0394	0.20543	0.7938	0.20487	0.73438
0.77778	0.64279	0.03214	0.17284	0.64945	0.17253	1.03607
0.84444	0.46947	0.02347	0.13136	0.47647	0.13133	1.48998
0.91111	0.27564	0.01378	0.08099	0.28242	0.08124	2.46017

3 Results and discussions

The results show a good agreement between the exact and the proposed solutions (Fig. 1) since the mean absolute error is less than 2.5 perc. (see Table 1). On the other hand, the absolute errors values are not far from those recently published for similar equations, by A. Saadatmandi et al. [17]. This means that it is possible to obtain analytical solutions even when an exact one seems not to be obtainable.

4 Conclusion

This work proposes an analytical solution to well known applied-physics-related Klein-Gordon equation. A given example gives good fundaments to the performed Boubaker Polynomials Expansion Scheme (BPES), particularly when exact solutions expressions are difficult to establish.

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