A new type of difference sequence spaces

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Abstract. In this article we introduce a new sequence space denoted by $m(\triangle_v^u, \phi, p)$. We give some topological properties and inclusion relations on this space. The results herein proved are analogous to those from [1].

M.S.C. 2000: 40C05, 46A45.
Key words: Difference sequence; solid spaces; symmetric space.

1. Introduction

Let ℓ^0 be the set of all complex sequences and l_∞, c and c_0 be the sets of all bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$||x||_\infty = \sup_k |x_k|, \quad \text{where} \quad k \in \mathbb{N} = \{1, 2, \cdots\}.$$

A sequence space X with linear topology is called a K-space if each of the maps $P_k : X \to \mathbb{C}$ defined by $P_k(x) = x_k$ is continuous for $k = 1, 2, \cdots$. A K- space X is called a BK-space provided X is a Banach space.

The idea of difference sequence space was introduced by Kizmaz [12]. In 1981, Kizmaz [12] defined the sequence spaces:

$$l_\infty(\triangle) = \{x = \{x_k\} \in \ell^0 : (\triangle x_k) \in l_\infty\},$$

$$c(\triangle) = \{x = \{x_k\} \in \ell^0 : (\triangle x_k) \in c\},$$

and

$$c_0(\triangle) = \{x = \{x_k\} \in \ell^0 : (\triangle x_k) \in c_0\},$$

where $\triangle x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

Et and Colak [5] generalized the above sequence spaces to the sequence spaces

$$X(\triangle^r) = \{x = \{x_k\} \in \ell^0 : \triangle^r x_k \in X\},$$

for $X = l_\infty$, c and c_0, where $r \in \mathbb{N}$,

$$\triangle^0 x = (x_k), \quad \triangle x = (x_k - x_{k+1}), \quad \triangle^r x = (\triangle^r x_k - \triangle^r x_{k+1})$$

Applied Sciences, Vol.12, 2010, pp. 102-108.

and so that

$$\triangle^r x_k = \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix} \quad x_{k+i}.$$

Difference sequence spaces have been studied by Colak and Et [2], Et [4], Et and Esi [6], Vakeel A. Khan [8,9,10,11] and many others.

Let $X, Y \subset \ell^0$. Then we shall write

$$M(X,Y) = \bigcap_{x \in X} \quad x^{-1} * Y = \{a \in \ell^0 : ax \in Y \text{ for all } x \in X\}.$$

The set

$$X^\alpha = M(X, l_1)$$

is called Köthe - Toeplitz dual space or α - dual of X(see [16]).

Let X be a sequence space. Then X is called

(i) solid (or normal), if $(\alpha_k x_k) \in X$, whenever $(x_k) \in X$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
(ii) symmetric, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.
(iii)perfect, if $X = X^{\alpha\alpha}$.
(iv) sequence algebra, if $x.y \in X$, whenever $x, y \in X$.

It is well known that if X is perfect then X is normal (see [7]).

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further,

$$\mathcal{C}_s = \left\{ \sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s \right\} (cf[13]),$$

the set of those σ whose support has cardinality at most s, and

$$\Phi = \left\{ \phi = \{\phi_k\} \in \ell^0 : \phi_1 > 0, \Delta\phi_k \geq 0 \text{ and } \Delta\left(\frac{\phi_k}{k}\right) \leq 0 \quad (k = 1, 2, \ldots) \right\},$$

where $\Delta\phi_k = \phi_k - \phi_{k-1}$; and ℓ^0 is the set of all real sequences.

For $\phi \in \Phi$, we define the following sequence space, introduce in [14],

$$m(\phi) = \left\{ x = \{x_k\} \in \ell^0 : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} | x_k | \right) < \infty \right\}.$$

Recently the space $m(\phi)$ was extended to $m(\phi, p)$ by Tripathy and Sen [15] as follows:

$$m(\phi, p) := \left\{ x = \{x_k\} \in \ell^0 : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} | x_k |^p < \infty \right\}.$$

It is easy to see that:

$$\| x \|_{m(\phi,p)} = \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} | x_k |^p \right)^{\frac{1}{p}}.$$

Remark 1. The space $m(\phi, p)$ is a Banach space with the norm

$$\| x \|_{m(\phi,p)} = \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k\in\sigma} | x_k |^p \right)^{\frac{1}{p}}.$$

Remark 2. As in [14], we have

(i) If $\phi_n = 1$ for all $n \in \mathbb{N}$ then $m(\phi, p) = l_p$. Moreover

$$l_p \subseteq m(\phi, p) \subseteq l_\infty.$$

(ii) If $p = 1$, then $m(\phi, p) = m(\phi)$. Also

$$m(\phi) \subseteq m(\phi, p).$$

(iii) $m(\phi, p) \subseteq m(\psi, p)$ if and only if $\sup_{s\geq 1} \left(\frac{\phi_s}{\psi_s}\right) < \infty$.

2. Main Results

Let u be a fixed positive integer and $v = (v_k)$ be any fixed sequence of non zero complex numbers (see [3]). Now we define the sequence space $m(\triangle_v^u, \phi, p)$ as follows:

$$m(\triangle_v^u, \phi, p) := \left\{ x = \{x_k\} \in \ell^0 : \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} (|\triangle_v^u x_k|^p) < \infty, \qquad 0 \leq p < \infty \right\}.$$

where

$$\triangle_v^0 x_k = (v_k x_k),$$
$$\triangle_v x_k = (v_k x_k - v_{k+1} x_{k+1}),$$
$$\triangle_v^u x_k = (\triangle_v^{u-1} x_k - \triangle_v^{u-1} x_{k+1})$$

such that

$$\triangle_v^u x_k = \sum_{i=0}^{u} (-1)^i \begin{bmatrix} u \\ i \end{bmatrix} \quad v_{k+i} x_{k+i}.$$

It is clear that if $u = 0$, $v = (1, 1, 1, \cdots)$ and $p = 1$, we have $m(\phi)$, which was defined by Sargent [13].

Theorem 2.1. *The sequence space $m(\triangle_v^u, \phi, p)$ is a Banach space for $\phi \in \Phi$ normed by*

$$(2.1.1) \qquad ||x||_{\triangle_1} = \sum_{i=1}^{u} |x_i| + \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} ((|\triangle_v^u x_k|^p))^{1/p}, \quad 1 \leq p < \infty,$$

and a complete p-normed space by p-norm

$$(2.1.2) \qquad ||x||_{\triangle_2} = \sum_{i=1}^{u} |x_i|^p + \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} (|\triangle_v^u x_k|^p), \quad 0 < p < 1.$$

Proof. It is clear that $m(\triangle_v^u, \phi, p)$ is a normed linear space normed by (2.1.1) for $1 \leq p < \infty$ and a p - normed space by p - norm (2.1.2) for $0 < p < 1$. We need to show that $m(\triangle_v^u,\phi,p)$ is complete. Let $\{x^{(l)}\}$ be a Cauchy sequence in $m(\triangle_v^u,\phi,p)$ where $x^l=(x_k^l)_k=(x_1^l,x_2^l,\cdots)\in m(\triangle_v^u,\phi,p)$ for each $l\in\mathbb{N}$. Then for given $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$||x^l-x^t||_{\triangle_1}=\sum_{i=1}^{u}|x_i^l-x_i^t|+\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left((|\triangle_v^u(x_k^l-x_k^t)|^p)\right)^{1/p}<\epsilon,\quad\text{for}\quad\text{all }l,t>n_0.$$

Now we obtain

$$|x_k^l-x_k^t|\to 0,\quad as\quad l,t\to\infty,\ \text{for each } k\in\mathbb{N}.$$

Therefore $(x_k^l)_l=(x_k^1,x_k^2,\cdots)$ is a Cauchy sequence in $\mathbb{C}$ for each k. Since $\mathbb{C}$ is complete, it is convergent

$$\lim_l x_k^l=x_k\quad (say)\ \text{ for each }\ k\in\mathbb{N}.$$

Taking limit as $t\to\infty$ in (2.1.3), we get

$$\sum_{i=1}^{u}|x_i^l-x_i|+\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(|\triangle_v^u(x_k^l-x_k)|^p\right)^{1/p}<\epsilon,\quad\text{for}\quad\text{all}\quad n>n_0.$$

Hence $(x_k^l-x_i)\in m(\triangle_v^u,\phi,p)$. We know that $m(\triangle_v^u,\phi,p)$ is a linear space and (x_k^l), $(x_k^l-x_k)$ are in $m(\triangle_v^u,\phi,p)$, it follows that

$$(x_k)=(x_k^l)-(x_k^l-x_k)\in m(\triangle_v^u,\phi,p).$$

Hence $m(\triangle_v^u,\phi,p)$ is complete. Similarly, we can show that $m(\triangle_v^u,\phi,p)$ is complete space p-normed by (2.1.1) for $0<p<1$. □

Theorem 2.2. *For any $\phi\in\Phi$ the space $m(\triangle_v^u,\phi,p)$ is a K - space.*

The proof is straightforward.

Theorem 2.3. *$m(\triangle_v^u,\phi)\subset m(\triangle_v^u,\phi,p)$, for any $\phi\in\Phi$.*

Proof. Let $x\in m(\triangle_v^u,\phi)$. Then there exist a positive number K such that

$$\sum_{k\in\sigma}|\triangle_v^u x_k|^p\leq K\phi_s,\qquad \sigma\in\phi_s\quad\text{for each fixed s.}$$

Hence

$$\sum_{k\in\sigma}|\triangle_v^u x_k|^p<K\phi_s,\qquad \sigma\in\phi_s\quad\text{for each p ¿ 0}\ \ and\ \ \sigma\in\phi_s.$$

Thus $x\in m(\triangle_v^u,\phi,p)$. □

Theorem 2.4. *For any two sequences (ϕ_s) and (ψ_s) of real numbers, we have $m(\triangle_v^u,\phi,p)\subset m(\triangle_v^u,\psi,p)$ if and only if*

$$\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)<\infty.$$

Proof. Let $x \in m(\triangle_v^u, \phi, p)$. Then $\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |\triangle_v^u x_k|^p < \infty$. Suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$. Then $\phi_s \leq K\psi_s$ and so that $\frac{1}{\psi_s} \leq \frac{K}{\phi_s}$ for some positive number K and for all s. Therefore we have

$$\frac{1}{\psi_s}\sum_{k\in\sigma} |\triangle_v^u x_k|^p \leq \frac{K}{\phi_s}\sum_{k\in\sigma} |\triangle_v^u x_k|^p \quad \text{for each s.}$$

Now

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\psi_s}\sum_{k\in\sigma} |\triangle_v^u x_k|^p \leq K \sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma} |\triangle_v^u x_k|^p.$$

Hence $\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\psi_s} \sum_{k\in\sigma} |\triangle_v^u x_k|^p < \infty$. Therefore $x \in m(\triangle_v^u, \psi, p)$.

Conversely, let $m(\triangle_v^u, \phi, p) \subseteq m(\triangle_v^u, \psi, p)$ and suppose that $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \infty$. Then there exists aN increasing sequence (s_i) of natural numbers such that $\lim\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) = \infty$. Now for every $B \in \mathbb{R}^+$ (the set of positive real numbers), there exists $i_0 \in \mathbb{N}$ such that $\frac{\phi_{s_i}}{\psi_{s_i}} > B$ for all $s_i \geq i_0$. Hence $\frac{1}{\psi_{s_i}} > \frac{B}{\psi_{s_i}}$ and

$$\frac{1}{\psi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p > \frac{B}{\psi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p$$

for all $s_i \geq i_0$. Now taking supremum over $s_i \geq i_0$ and $\sigma \in \mathcal{C}_s$ we get

$$\sup_{s_i\geq i_0}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\psi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p > B \sup_{s_i\geq i_0}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p. \tag{2.2.1}$$

Since (2.2.1) holds for all $B \in \mathbb{R}^+$ (we may take B sufficiently large) we have

$$\sup_{s_i\geq i_0}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\psi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p = \infty$$

when $x \in m(\triangle_v^u, \phi, p)$ with

$$0 < \sup_{s_i\geq i_0}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_{s_i}}\sum_{k\in\sigma} |\triangle_v^u x_k|^p < \infty.$$

Therefore $x \notin m(\triangle_v^u, \psi, p)$. This contradicts to $m(\triangle_v^u, \phi, p) \subseteq m(\triangle_v^u, \psi, p)$, whence $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$. □

From Theorem 2.4, we get the following result.

Corollary 2.1. $m(\triangle_v^u, \phi, p) = m(\triangle_v^u, \psi, p)$ *if and only if*

$$0 < \inf_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) \leq \sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty.$$

Theorem 2.5. $m(\triangle_v^{u-1}, \phi, p) \subset m(\triangle_v^u, \phi, p)$ *and the inclusion is strict.*

Proof. The proof follows from the following inequality and Minkowski's inequality

$$|\triangle_v^u x| = |\triangle_v^{u-1} x_k - \triangle_v^{u-1} x_{k+1}| \leq |\triangle_v^{u-1} x_k| + |\triangle_v^{u-1} x_{k+1}|.$$

To show that the inclusion is strict consider the following example.

Example 2.1. Let $\phi_n = 1$ for all $n \in \mathbb{N}$, $x = (k^{u-1})$ and $v = (1, 1, 1, \cdots)$, then

$$x \in l_p(\triangle_v^u) \setminus l_p(\triangle_v^{u-1}).$$

Theorem 2.6. *The sequence space $m(\triangle_v^u, \phi, p)$ is not sequence algebra, is not solid and is not symmetric, for $u \geq 1$.*

Proof. For the proof of this theorem, consider the following examples:

Example 2.2. Let $x = (k^{u-1})$, $y = (k^{u-1})$ and $v = (1, 1, 1, \cdots)$. Then $x, y \in m(\triangle_v^u, \phi, p)$, but $x.y \notin m(\triangle_v^u, \phi, p)$. Hence $m(\triangle_v^u, \phi, p)$ is not sequence algebra.

Example 2.3. Let $x = (k^{u-1})$, $v = (1, 1, 1, \cdots)$ and $\alpha_k = (-1)^k$. Then $x = (k^{u-1}) \in m(\triangle_v^u, \phi, p)$, but

$$(\alpha_k x_k) \notin m(\triangle_v^u, \phi, p) \text{ for } \alpha = (\alpha_k) = (-1)^k.$$

Hence $m(\triangle_v^u, \phi, p)$ is not solid.

Example 2.4. Let $x = (k^{u-1})$ and $v = (1, 1, 1, \cdots)$. Let (y_k) be a arrangement of (x_k) which is defined as follows :

$$y_k = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \cdots\}.$$

Then $y \notin m(\triangle_v^u, \phi, p)$. Hence $m(\triangle_v^u, \phi, p)$ is not symmetric.

The following result is a consequence of Theorem 2.6.

Corollary 2.2. *The sequence space $m(\triangle_v^u, \phi, p)$ is not perfect.*

Theorem 2.7. $l_p(\triangle_v^u) \subseteq m(\triangle_v^u, \phi, p) \subseteq l_\infty(\triangle_v^u)$.

Proof. Since $m(\triangle_v^u, \phi, p) = l_p(\triangle_v^u)$ for $\phi_n = 1$, for all $n\mathbb{N}$, then

$$l_p(\triangle_v^u) \subseteq m(\triangle_v^u, \phi, p).$$

Now suppose that $x \in m(\triangle_v^u, \phi, p)$. Then we have

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |\triangle_v^u x_k|^p < \infty,$$

and hence

$$|\triangle_v^u x_k| < K\phi_1, \text{ for all } n \in \mathbb{N} \text{ and for some positive integer K.}$$

Thus $x \in l_\infty(\triangle_v^u)$. This completes the proof of Theorem. □

Corollary 2.3. *If $0 < p < q$, then $m(\triangle_v^u, \phi, p) \subseteq m(\triangle_v^u, \phi, q)$.*

Proof. For the proof of this theorem follows from the following inequality

$$\left(\sum_{k=1}^n |x_k|^q \right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}}, \quad (0 < p < q).$$

References

[1] R. Colak and M. Et, *On some difference sequence sets and their topological properties*, Bull. Malaysian Math.Sci.Soc. 2 (2005), 125-130.

[2] R. Colak and M. Et, *On some generalized difference sequence spaces and related matrix transformations*, Hokkoido Mathematical Journal 26, 3 (1997), 483-492.

[3] A. Esi and M.Isik, *Some generalized difference sequence spaces*, Thai Journal of Mathematics, 3 (2005), 241-247.

[4] M. Et, *On some topological properties of generalized difference sequence spaces*, Internat. J. Math. and Math. Sci. 24, 11 (2000), 785-791.

[5] M. Et and R. Colak, *On some generalized difference sequence spaces*, Soochow J. of Math. 21 (1995), 377-386.

[6] M. Et and A. Esi, *On Köthe Toeplitz duals of generalized difference sequence spaces*, Bull. Malaysian Math. Sci. Soc. 24, 11 (2000), 25-32.

[7] P. Kampthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker Inc. 1980.

[8] V. A. Khan, *Some inclusion relations between the difference sequence spaces defined by sequence of moduli*, Indian Mathematical Society 73 (2006), 77-81.

[9] V. A. Khan and Q.M.D. Lohani, *Difference sequence spaces defined by a sequence of moduli*, Southeast Asian Bull. Math. 30 (2006), 1061-1067.

[10] V.A. Khan and Q.M.D. Lohani, *New lacunary strong convergence difference sequence spaces defined by sequence of moduli*, Kyungpook Mathematical Journal 46 (2006), 591-595.

[11] V.A. Khan, *New lacunary strongly summable difference sequences and $\triangle_v^m$ - lacunary almost statistical convergence*, Vietnam Journal of Mathematics 36 (2008), 405-413.

[12] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. 24, 2 (1981), 169-176.

[13] M. Mursaleen, *Some geometric properties of a sequence space related to l_p*, Bull. Aust. Math. Soc. 67, 2 (2003), 343-347.

[14] W.L.C. Sargent, *Some sequence spaces related to the ℓ^p spaces*, J. London Math. Soc. 35 (1960), 161-171.

[15] B.C. Tripathy and M. Sen, *On a new class of sequences related to the space l_p*, Tamkang J. Math. 33, 2 (2002), 167-171.

[16] A. Wilansky, *Summability Through Functional Analysis*, North Holland Mathematics Studies 85, North Holland, 1984.

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