Some statistically convergent difference sequence spaces defined over real 2-normed linear spaces

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Abstract. In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [5] and Schoenberg [26]. The main aim of this article is to study the concept of statistical convergence from difference sequence spaces point of view which are defined over real linear 2-normed spaces.

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1 Introduction

Throughout the article $w(X)$, $c(X)$, $c_0(X)$, $\bar{c}(X)$, $\bar{c}_0(X)$, $\ell_\infty(X)$, $m(X)$ and $m_0(X)$ will represent the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null X valued sequences spaces, where $(X, \|.,.\|)$ is a real linear 2-normed space. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, \dots)$, where θ is the zero element of X.

The notion of difference sequence space was introduced by Kizmaz [18], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [2] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [31], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy, Esi and Tripathy [32] generalized the above notions and unified these as follows:

Let m, n be non- negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

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Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [2]. Taking $n = 1$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [31]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [18].

Recently H. Dutta introduced another type of difference operator $\Delta^n_{(v,m)}$, where m, n are non-negative integers and $v = (v_k)$ is a sequence of non-zero scalars. For details, one may refer to Dutta [4].

The concept of 2-normed spaces was introduced and studied by Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language published in Mathematische Nachrichten, see for example references [3, 9, 10, 11, 12]. This notion which seems to be a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. [34] of USA in 1969 entitled 2-Banach spaces. In the same year Gähler [12] published another paper on this theme in the same journal. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S.C. Gupta [17] of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by A.H. Siddiqi [28]. For other works in this direction one may refer to [7, 8, 13, 16, 24, 29]. In the recent years, a number of articles devoted to statistical convergence (see Gürdal and Pehlivan [14]) and its generalization, ideal convergence(see Gürdal and Şahiner [15]) using 2-norm, have been published.

Let X be a real vector space of dimension d, where $2 \le d$. A real-valued function $\|.,.\|$ on X^2 satisfying the following four conditions:

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,
(2) $\|x_1, x_2\|$ is invariant under permutation,
(3) $\|\alpha x_1, x_2\| = |\alpha|\|x_1, x_2\|$, for any $\alpha \in R$,
(4) $\|x + x', x_2\| \le \|x, x_2\| + \|x', x_2\|$

is called a 2-*norm* on X, and the pair $(X, \|.,.\|)$ is called a 2-normed space.

The notion of statistical convergence was studied at the initial stage by Fast [5] and Schoenberg [26] independently. Later on it was further investigated by Šalàt [25], Fridy [6], Buck [1], Sen and Tripathy [33] and many others. Gürdal and Pehlivan [14] studied statistical convergence in 2-Banach space.

A subset E of N is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E.

The following inequality will be used throughout the article:

Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le \sup p_k = G$, $D = \max\left\{1, 2^{G-1}\right\}$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \le D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}$$

and for $\lambda \in C$,

$$|\lambda|^{p_k} \le \max\left\{1, |\lambda|^G\right\}.$$

The notion of paranormed sequence space was studied at the initial stage by Simons [27] and Nakano [22]. Later on it was further investigated by Maddox [21], Lascarides [19], Lascarides and Maddox [20], Nanda [23], B.c. Tripathy [30], Tripathy and Sen [33] and a number of workers in the field of sequence spaces.

2 Definitions and background

A sequence space E is said to be *solid* (or *normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be *symmetric* if $\left(x_{\pi(k)}\right) \in E$ whenever $(x_k) \in E$, where π is a permutation on N.

A sequence (x_k) is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta\left(\{k \in N : |x_k - L| \geq \varepsilon\}\right) = 0$.

For $L = 0$, we say this is *statistically null*.

Throughout $\bar{c}$, $\bar{c_0}$ denote the classes of all statistically convergent and statistically null sequences respectively.

A sequence (x_k) in a 2-normed space $(X, \|.,.\|)$ is said to *converge* to some $L \in X$ in the 2-norm if

$$\lim_{k\to\infty} \|x_k - L, u_1\| = 0, \text{ for every } u_1 \in X.$$

A sequence (x_k) in a 2-normed space $(X, \|.,.\|)$ is said to be *Cauchy* with respect to the 2-norm if

$$\lim_{k,\ l\to\infty} \|x_k - x_l, u_1\| = 0, \text{ for every } u_1 \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

We introduce the following definitions in this article. Let m and n be two non-negative integers and $p = (p_k)$ be a sequence of strictly positive real numbers. Then

$$\bar{c}\left(\|.,.\|, \Delta^n_{(m)}, p\right) =$$

$$\left\{(x_k) \in w(X) : (\|\Delta^n_{(m)} x_k - L, z\|)^{p_k} \xrightarrow{stat} 0, \text{ for every } z \in X \text{ and some } L \in X\right\},$$

$$\bar{c}_0\left(\|.,.\|, \Delta^n_{(m)}, p\right) = \left\{(x_k) \in w(X) : (\|\Delta^n_{(m)} x_k, z\|)^{p_k} \xrightarrow{stat} 0, \text{ for every } z \in X\right\},$$

We procure the following definition for the sake of completeness:

$$\ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right) = \left\{(x_k) \in w(X) : \sup_{k\geq 1}(\|\Delta^n_{(m)} x_k, z\|)^{p_k} < \infty, \text{ for every } z \in X\right\},$$

The following definition is introduced:

$$W\left(\|.,.\|, \Delta^n_{(m)}, p\right) =$$

$$\left\{(x_k) \in w(X) : \lim_{j\to\infty} \sum_{k=1}^{j} (\|\Delta^n_{(m)} x_k - L, z\|)^{p_k} = 0, \text{ for every } z \in X \text{ and some } L \in X\right\}.$$

We write

$$m\left(\|.,.\|,\Delta^n_{(m)},p\right)=\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)\cap\ell_\infty\left(\|.,.\|,\Delta^n_{(m)},p\right)$$

and

$$m_0\left(\|.,.\|,\Delta^n_{(m)},p\right)=\bar{c}_0\left(\|.,.\|,\Delta^n_{(m)},p\right)\cap\ell_\infty\left(\|.,.\|,\Delta^n_{(m)},p\right),$$

where $(\Delta^n_{(m)}x_k)=(\Delta^{n-1}_{(m)}x_k-\Delta^{n-1}_{(m)}x_{k-m})$ and $\Delta^0_{(m)}x_k=x_k$ for all $k\in N$, which is equivalent to the following binomial representation:

$$\Delta^n_{(m)}x_k=\sum_{v=0}^{n}(-1)^v\binom{n}{v}x_{k-mv}.$$

In the above expansion we take $x_k=0$ for non-positive values of k.

The main aim behind considering the generalized difference operator $\Delta^n_{(m)}$ is that we can derive several other spaces from the above constructed spaces for particular values of m and n. In particular for $n=0$, the above spaces reduce to the spaces $\bar{c}\,(\|.,.\|,p)$, $\bar{c}_0\,(\|.,.\|,p)$, $\ell_\infty\,(\|.,.\|,p)$, $W\,(\|.,.\|,p)$, $m\,(\|.,.\|,p)$ and $m_0\,(\|.,.\|,p)$ respectively.

Again if we replace the base space X, which is a real linear 2-normed space by C, complete normed linear space, we get the spaces $\bar{c}\left(\Delta^n_{(m)},p\right)$, $\bar{c}_0\left(\Delta^n_{(m)},p\right)$, $\ell_\infty\left(\Delta^n_{(m)},p\right)$, $W\left(\Delta^n_{(m)},p\right)$, $m\left(\Delta^n_{(m)},p\right)$ and $m_0\left(\Delta^n_{(m)},p\right)$ respectively.

Further if we take $X=C$, $p_k=l$, a constant for all $k\in N$ and $n=0$, we get the spaces $\bar{c}$, $\bar{c}_0$, ℓ_∞, W, m and m_0 respectively.

First we procure some known results; those will help in establishing the results of this article.

Lemma 2.1. ([33]) *For two sequences p_k and (t_k) we have $m_0(p)\supseteq m_0(t)$ if and only if* $\lim\inf_{k\in K}\frac{p_k}{t_k}>0$, *where* $K\subseteq N$ *such that* $\delta(K)=1$.

Lemma 2.2. ([33]) *Let $h=\inf p_k$ and $G=\sup p_k$, then the following are equivalent:*
(i) $G<\infty$ *and* $h>0$,
(ii) $m(p)=m$.

We now cite the following two known 2-normed spaces.

Example 2.1. *Consider the spaces Z where $Z=\ell_\infty$, c and c_0 of real sequences. Let us define:*

$$\|x,y\|=\sup_{i\in N}\sup_{j\in N}|x_iy_j-x_jy_i|,\ \textit{where } x=(x_1,x_2,\dots)\textit{ and } y=(y_1,y_2,\dots)\in Z.$$

Then $\|.,.\|_E$ is a 2-norm on Z.

Example 2.2. *Let us take $X=R^2$ and consider the function on X defined as:*

$$\|x_1,x_2\|_E=abs\left(\begin{vmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{vmatrix}\right),\ \textit{where } x_i=(x_{i1},x_{i2})\in R^2\ \textit{for each } i=1,2.$$

Then $\|.,.\|_E$ is a 2-norm on X known as Euclidean 2-norm.

Remark 2.1. *Every closed linear subspace of an arbitrary linear normed space E, different from E, is a nowhere dense set in E.*

3 Main results

In this section we mainly investigate several linear topological and algebraic properties relevant to the spaces $\bar{c}_0\left(\|.,.\|,\Delta^n_{(m)},p\right)$, $\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)$, $m\left(\|.,.\|,\Delta^n_{(m)},p\right)$ and $m_0\left(\|.,.\|,\Delta^n_{(m)},p\right)$ respectively.

Theorem 3.1. *Let $p=(p_k)$ be a bounded sequence of strictly positive real numbers. Then the classes of sequences $\bar{c}_0\left(\|.,.\|,\Delta^n_{(m)},p\right)$, $\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)$, $m_0\left(\|.,.\|,\Delta^n_{(m)},p\right)$ and $m\left(\|.,.\|,\Delta^n_{(m)},p\right)$ are linear spaces.*

Proof. We prove the theorem only for the space $\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)$ and for the other spaces it will follow on applying similar arguments.

Let $(x_k),(y_k)\in\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)$. Then there exist $L,J\in X$ such that for every $z\in X$

$$\left(\|\Delta^n_{(m)}x_k-L,z\|\right)^{p_k}\xrightarrow{stat}0$$

and

$$\left(\|\Delta^n_{(m)}y_k-J,z\|\right)^{p_k}\xrightarrow{stat}0.$$

Let α,β be scalars. Then we have for every $z\in X$

$$\left(\|\Delta^n_{(m)}(\alpha x_k+\beta y_k)-(\alpha L+\beta J),z\|\right)^{p_k}=\left(\|\Delta^n_{(m)}\alpha(x_k-L)+\Delta^n_{(m)}\beta(y_k-J),z\|\right)^{p_k}$$

$$\le\left(|\alpha|\|\Delta^n_{(m)}x_k-L,z\|+|\beta|\|\Delta^n_{(m)}y_k-J,z\|\right)^{p_k}$$

$$\le D|\alpha|^G\left(\|\Delta^n_{(m)}x_k-L,z\|\right)^{p_k}+D|\beta|^G\left(\|\Delta^n_{(m)}y_k-J,z\|\right)^{p_k},\ \text{where } G=\sup p_k.$$

$$\xrightarrow{stat}0,\ \text{as } k\to\infty.$$

Hence $\bar{c}\left(\|.,.\|,\Delta^n_{(m)},p\right)$ is a linear space. □

Theorem 3.2. *The spaces $m_0\left(\|.,.\|,\Delta^n_{(m)},p\right)$ and $m\left(\|.,.\|,\Delta^n_{(m)},p\right)$ are paranormed spaces, paranormed by*

$$g(x)=\sup_{k\in N,\ z\in X}\left(\|\Delta^n_{(m)}x_k,z\|\right)^{\frac{p_k}{H}},\ \textit{where } H=\max\{1,\sup_k p_k\}.$$

Proof. Clearly $g(x)=g(-x)$; $x=\theta$ implies $g(\theta)=0$. Now

$$g(x+y)=\sup_{k\in N,\ z\in X}\left(\|\Delta^n_{(m)}(x_k+y_k),z\|\right)^{\frac{p_k}{H}}.$$

$$\le\sup_{k\in N,\ z\in X}\left(\|\Delta^n_{(m)}x_k,z\|\right)^{\frac{p_k}{H}}+\sup_{k\in N,\ z\in X}\left(\|\Delta^n_{(m)}y_k,z\|\right)^{\frac{p_k}{H}}.$$

This implies that

$g(x+y)\le g(x)+g(y)$.

The continuity of the scalar multiplication follows from the following equality:

$$g(\lambda x) = \sup_{k\in N,\ z\in X} \left(\|\Delta^n_{(m)}(\lambda x_k), z\|\right)^{\frac{p_k}{H}}.$$
$$= \sup_{k\in N,\ z\in X} \left(|\lambda|\|\Delta^n_{(m)} x_k, z\|\right)^{\frac{p_k}{H}}.$$
$$\leq \max(1, |\lambda|) \sup_{k\in N,\ z\in X} \left(|\lambda|\|\Delta^n_{(m)} x_k, z\|\right)^{\frac{p_k}{H}}.$$
$$= \max(1, |\lambda|)\, g(x).$$

Hence the spaces $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and $m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ are paranormed by g. □

Remark 3.1. *For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two 2-norms $\|.,.\|_1$ and $\|.,.\|_2$ on X we have $Z\left(\|.,.\|_1, \Delta^n_{(m)}, p\right) \cap Z\left(\|.,.\|_2, \Delta^n_{(m)}, t\right) \neq \phi$, where $Z = \bar{c}, m, \bar{c}_0, m_0$.*

Proof. The proof follows from the fact that the zero element belongs to each of the classes of sequences involved in the intersection. □

Theorem 3.3. *The spaces $Z\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ are not solid in general, where $Z = \bar{c}, m, \bar{c}_0, m_0$.*

Proof. To show that the spaces are not solid in general, consider the following examples.

Example 3.1. *Let $m = 3, n = 1$ and consider the 2-normed space as defined in Example 2.1. Let $p_k = 5$ for all $k \in N$. Consider the sequence (x_k), where $x_k = (x^i_k)$ is defined by $(x^i_k) = (k, k, k, \dots)$ for each fixed $k \in N$. Then $(x_k) \in Z\left(\|.,.\|, \Delta^1_{(3)}, p\right)$ for $Z = \bar{c}, m$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z\left(\|.,.\|, \Delta^1_{(3)}, p\right)$ for $Z = \bar{c}, m$. Thus $Z\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ for $Z = \bar{c}, m$ are not solid in general.* □

Example 3.2. *Let $m = 3, n = 1$ and consider the 2-normed space as defined in Example 2.1. Let $p_k = 1$ for all k odd and $p_k = 2$ for all k even. Consider the sequence (x_k), where $x_k = (x^i_k)$ is defined by $(x^i_k) = (3, 3, 3, \dots)$ for each fixed $k \in N$. Then $(x_k) \in Z\left(\|.,.\|, \Delta^1_{(3)}, p\right)$ for $Z = \bar{c}_0, m_0$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z\left(\|.,.\|, \Delta^1_{(3)}, p\right)$ for $Z = \bar{c}_0, m_0$. Thus $Z\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ for $Z = \bar{c}_0, m_0$ are not solid in general.* □

Theorem 3.4. *The spaces $Z\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ are not symmetric in general, where $Z = \bar{c}, m, \bar{c}_0, m_0$.*

Proof. To show that the spaces are not symmetric in general, consider the following example.

Example 3.3. *Let $m = 2, n = 2$ and consider the 2-normed space as defined in Example 2.2. Let $p_k = 2$ for all k odd and $p_k = 1$ for all k even. Consider the sequence (x_k) defined by $x_k = (k, k)$ for each fixed $k \in N$. Then $\Delta^2_{(2)} x_k = x_k -$*

$2x_{k-2} + x_{k-4}$, $k \in N$. *Hence* $(x_k) \in Z\left(\|.,.\|, \Delta^2_{(2)}, p\right)$ *for* $Z = \bar{c}, m, \bar{c}_0, m_0$. *Let* (y_k) *be a rearrangement of* (x_k)*, which is defined as follows:*

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \dots\}.$$

Then $(y_k) \notin Z\left(\|.,.\|, \Delta^2_{(2)}, p\right)$ *for* $Z = \bar{c}, m, \bar{c}_0, m_0$. *Hence for* $Z = \bar{c}, m, \bar{c}_0, m_0$*, the spaces* $Z\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ *are not symmetric in general.* □

Remark 3.2. *For two sequences* (p_k) *and* (t_k) *we have*

$$m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right) \supseteq m_0\left(\|.,.\|, \Delta^n_{(m)}, t\right)$$

if and only if $\lim\inf_{k\in K} \frac{p_k}{t_k} > 0$*, where* $K \subseteq N$ *such that* $\delta(K) = 1$.

Proof. If we take $(y_k) = \left(\|\frac{\Delta^n_{(m)} x_k}{\rho}, z\|\right)$ for all $k \in N$, then the result follows from the Lemma 2.1. □

Remark 3.3. *For two sequences* (p_k) *and* (t_k) *we have*

$$m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right) = m_0\left(\|.,.\|, \Delta^n_{(m)}, t\right)$$

if and only if $\lim\inf_{k\in K} \frac{p_k}{t_k} > 0$ *and* $\lim\inf_{k\in K} \frac{t_k}{p_k} > 0$*, where* $K \subseteq N$ *such that* $\delta(K) = 1$.

Proof. This result is a consequence of the above result. □

Remark 3.4. *Let* $h = \inf p_k$ *and* $G = \sup p_k$*, then the following are equivalent:*
(i) $G = \sup p_k$ *and* $h > 0$,
(ii) $m\left(\|.,.\|, \Delta^n_{(m)}, p\right) = m\left(\|.,.\|, \Delta^n_{(m)}\right)$

Proof. Taking $(y_k) = \left(\|\frac{\Delta^n_{(m)} x_k}{\rho}, z\|\right)$ for all $k \in N$ and using the Lemma 2.2, we get the result. □

Theorem 3.5. *Let* $p = (p_k)$ *be a sequence of non-negative bounded real numbers such that* $\inf p_k > 0$. *Then* $m\left(\|.,.\|, \Delta^n_{(m)}, p\right) = W\left(\|.,.\|, \Delta^n_{(m)}, p\right) \cap \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$.

Proof. Let $(x_k) \in W\left(\|.,.\|, \Delta^n_{(m)}, p\right) \cap \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. Then for a given $\varepsilon > 0$, we have

$$\sum_{k=1}^{j}\left(\|\Delta^n_{(m)} x_k - L, z\|\right)^{p_k} \geq \text{card}\left\{k \leq j : (\|\Delta^n_{(m)} x_k - L, z\|)^{p_k} \geq \varepsilon\right\}\varepsilon.$$

From the above inequality, it follows that $(x_k) \in m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$.

Conversely let $(x_k) \in m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and $\rho > 0$ be such that

$$(\|\Delta^n_{(m)} x_k - L, z\|)^{p_k} \xrightarrow{stat} 0, \text{ for every } z \in X \text{ and some } L \in X.$$

For a given $\varepsilon > 0$, let $B = \sup_k \left(\|\Delta^n_{(m)} x_k - L, z\|\right)^{\frac{p_k}{H}} < \infty$, where $H = \max\{1, \sup p_k\}$. Let $L_j = \left\{k \le j : (\|\Delta^n_{(m)} x_k - L, z\|)^{p_k} \ge \frac{\varepsilon}{2}\right\}$. Since $(x_k) \in m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$, so $\frac{Card\{L_j\}}{j} \to 0$ as $j \to \infty$. Let $n_0 > 0$ be such that $\frac{Card\{L_j\}}{j} < \frac{\varepsilon}{2B^H}$ for all $j > n_0$. Then for all $j > n_0$, we have

$$\begin{aligned}\frac{1}{j}\sum_{k=1}^{j}\left(\|\Delta^n_{(m)} x_k - L, z\|\right)^{p_k} &= \frac{1}{j}\sum_{k\notin L_j}\left(\|\Delta^n_{(m)} x_k - L, z\|\right)^{p_k} + \frac{1}{j}\sum_{k\in L_j}\left(\|\Delta^n_{(m)} x_k - L, z\|\right)^{p_k} \\ &\le \frac{j - Card\{L_j\}}{j}.\frac{\varepsilon}{2} + \frac{Card\{L_j\}}{j}.B^H \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $(x_k) \in W\left(\|.,.\|, \Delta^n_{(m)}, p\right) \cap \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. □

The following result is a consequence of the above theorem.

Corollary 3.1. *Let (p_k) and (t_k) be two bounded sequences of real numbers such that $\inf p_k > 0$ and $\inf t_k > 0$. Then*

$$W\left(\|.,.\|, \Delta^n_{(m)}, p\right) \cap \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right) = W\left(\|.,.\|, \Delta^n_{(m)}, t\right) \cap \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, t\right).$$

Theorem 3.6. *Let $(X, \|.,.\|)$ be a 2-Banach space, then the spaces $m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ are complete.*

Proof. We prove the result for the space $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and for the other space it will follow on applying similar arguments. Let (x^i) be a Cauchy sequence in $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. Then for a given $\varepsilon (0 < \varepsilon < 1)$, there exists a positive integer n_0 such that $g(x^i - x^j) < \varepsilon$, for all $i, j \ge n_0$. This implies that

$$\sup_{k\in N,\ z\in X}\left(\|\Delta^n_{(m)} x^i_k - \Delta^n_{(m)} x^j_k, z\|\right)^{\frac{p_k}{H}} < \varepsilon,$$

for all $i, j \ge n_0$. It follows that for every $z \in X$,

$$\left(\|\Delta^n_{(m)}(x^i_k - x^j_k), z\|\right) < \varepsilon, \text{ for each } k \ge 1 \text{ and } i, j \ge n_0.$$

Hence $(\Delta^n_{(m)} x^i_k)$ is a Cauchy sequence in the 2-Banach space X for all $k \in N$. Thus $(\Delta^n_{(m)} x^i_k)$ is convergent in X for all $k \in N$. For simplicity, let $\lim_{i\to\infty} \Delta^n_{(m)} x^i_k = y_k$ for each $k \in N$. Let $k = 1$, then we have

$$\lim_{i\to\infty} \Delta^n_{(m)} x^i_1 = \lim_{i\to\infty}\sum_{v=0}^{n}(-1)^v\binom{n}{v}x^i_{1-mv} = \lim_{i\to\infty} x^i_1 = y_1. \tag{3.1}$$

Similarly, we have,

$$\lim_{i\to\infty} \Delta^n_{(m)} x^i_k = \lim_{i\to\infty} x^i_k = y_k, \text{ for } k = 1, \ldots, nm. \tag{3.2}$$

Thus from (3.1) and (3.2), we have $\lim_{i\to\infty} x^i_{1+nm}$ exists. Let $\lim_{i\to\infty} x^i_{1+nm} = x_{1+nm}$. Proceeding in this way inductively, we have $\lim_{i\to\infty} x^i_k = x_k$, say exists for each $k \in N$. Now we have for all $i, j \geq n_0$,

$$\sup_{k\in N,\ z\in X} \left(\|\Delta^n_{(m)}(x^i_k - x^j_k), z\|\right)^{\frac{p_k}{H}} < \varepsilon$$

$$\Rightarrow \lim_{j\to\infty} \left\{ \sup_{k\in N,\ z\in X} \left(\|\Delta^n_{(m)}(x^i_k - x^j_k), z\|\right)^{\frac{p_k}{H}} \right\} < \varepsilon, \text{ for all } i \geq n_0$$

$$\Rightarrow \sup_{k\in N,\ z\in X} \left(\|\Delta^n_{(m)}(x^i_k - x_k), z\|\right)^{\frac{p_k}{H}} < \varepsilon, \text{ for all } i \geq n_0.$$

It follows that $(x^i - x) \in m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. Since $(x^i) \in m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ is a linear space, so we have $x = x^i - (x^i - x) \in m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. This completes the proof. □

As consequence, it follows that $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ and $m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ are closed subspaces of $\ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$. Since the inclusion relations

$$m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right) \subset \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right), \quad m\left(\|.,.\|, \Delta^n_{(m)}, p\right) \subset \ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$$

are strict, we have the following result.

Corollary 3.2. *The spaces* $m_0\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ *and* $m\left(\|.,.\|, \Delta^n_{(m)}, p\right)$ *are nowhere dense subsets of* $\ell_\infty\left(\|.,.\|, \Delta^n_{(m)}, p\right)$.

4 Conclusions

In this paper we introduce the difference sequence spaces $\bar{c}_0(\|.,.\|, \Delta^n_{(m)}, p)$, $\bar{c}(\|.,.\|, \Delta^n_{(m)}, p)$, $m(\|.,.\|, \Delta^n_{(m)}, p)$, $m_0(\|.,.\|, \Delta^n_{(m)}, p)$, $\ell_\infty(\|.,.\|, \Delta^n_{(m)}, p)$ and $W(\|.,.\|, \Delta^n_{(m)}, p)$ with base space a real linear 2-normed space. We study the spaces $\bar{c}_0(\|.,.\|, \Delta^n_{(m)}, p)$, $\bar{c}(\|.,.\|, \Delta^n_{(m)}, p)$, $m(\|.,.\|, \Delta^n_{(m)}, p)$ and $m_0(\|.,.\|, \Delta^n_{(m)}, p)$ with the help of the spaces $\ell_\infty(\|.,.\|, \Delta^n_{(m)}, p)$ and $W(\|.,.\|, \Delta^n_{(m)}, p)$ for different properties including linearity, existence of paranorm and investigate the spaces for solidity and symmetricity. Further we prove that the spaces $m(\|.,.\|, \Delta^n_{(m)}, p)$ and $m_0(\|.,.\|, \Delta^n_{(m)}, p)$ are complete paranormed spaces when the base space is a 2-Banach space.

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