Folding on the wedge sum of graphs and their fundamental group

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Abstract. In this article, we introduce the definition of a new type of the wedge sum of two graphs. Also, we will deduce the fundamental group of the folding on some types of the Dual graph. The relation between the wedge sum of two dual graphs is obtained. The effect of folding on the wedge sum of finite numbers of graphs and their fundamental group will be achieved. Theorems and corollaries governing these studies will be achieved.

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Key words: Graph; wedge sum; folding; fundamental group.

1 Introduction

The graph theory is being applied in many different fields such as engineering system science, social science and human relations, business administrations and scientific management, political science, physical and organization systems, the electrical circuits and networks, route maps, architectural floor plans, chemistry, ecology, transportation theory, system diagnosis, music, etc.

The folding of a manifold introduced in [15], also some studies of the folding are obtained in [6]-[9],[11]. The fundamental groups of some types of a manifold are discussed in [1]-[5], [12]-[14]. The study of graphs are discussed and defined in [10, 16, 17].

2 Definitions and background

In this section, we give the definitions which are needed especially in this paper.

Definition 2.1. The set of homotopy classes of loops based at the point x_0 with the product operation $[f][g] = [f \cdot g]$ is called the fundamental group and denoted by $\pi_1(X, x_0)$ [14].

Definition 2.2. Let M and N be two Riemannian manifolds of dimension m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for

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every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ [15]. If f does not preserve length it is called topological folding [11].

Definition 2.3. An abstract graph G is a diagram consisting of a finite non-empty set of elements called vertices together with a set of unordered pairs of these elements called edges [17].

Definition 2.4. A connected graph is a graph in one piece, whereas one which splits into several pieces is disconnected [17].

Definition 2.5. A planar graph is a graph which can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their end points [16].

Definition 2.6. A dual graph of a given planar graph G is a graph which has a vertex for each plane region of G, and an edge for each edge in G joining two neighboring regions, for a certain embedding of G and denoted by dual(G) [17].

Definition 2.7. An infinite graph is graph such that the edge and vertex sets each have infinite cardinality [17].

3 The main results

Aiming to our study, we will introduce the following:

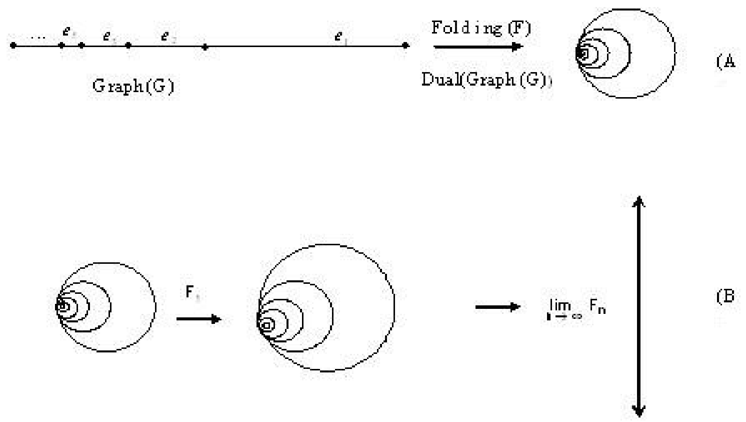
Definition 3.1. Given graphs G_1 and G_2 with chosen vertices $v_1 \in G_1$ and $v_2 \in G_2$, then the wedge sum $G_1 \vee G_2$ is the quotient of the disjoint union $G_1 \cup G_2$ obtained by identifying v_1 and v_2 to a single point.

Theorem 3.2. *If G_n is connected planar graph with n- vertices, without loops then $\pi_1\left(dual(G_n)\right)$ is a free group of rank $n-1$.*

Proof. Let be G_n connected planar graph with n vertices, then $dual(G_n)$ has n faces, but n faces form $n-1$ loops and these loops are homotopy equivalent to $(n-1)$-leaved rose, so $\pi_1\left(dual(G_n)\right) \approx \pi_1(\bigvee_{i=1}^{n-1} S_i^1) = \mathop{*}_{i=1}^{n-1} \pi_1(S_i^1)$, thus $\pi_1\left(dual(G_n)\right) \approx \pi_1(\text{ Z} * \text{Z} * \ldots \infty)$. Hence $\pi_1\left(dual(G_n)\right)$ is a free group of rank $n-1$. □

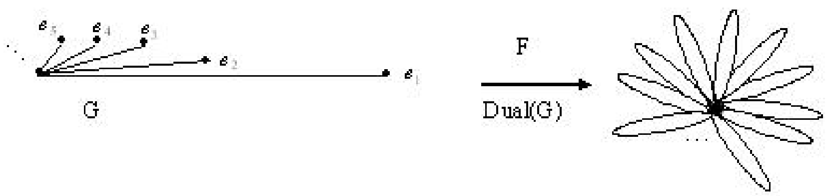
Theorem 3.3. *Let G be a connected path graph which can be represented it as the closed interval $[0,1]$ where $length\,(e_{\ n}) = \frac{1}{n(n+1)}$ for, $n = 1, 2, 3, \ldots$. Then*
i) $\pi_1(dual(G))$ is uncountable.
ii) There is a folding $F : G \longrightarrow G'$ which induces folding $\overline{F} : \pi_1(G) \longrightarrow \pi_1(G')$ such that $\overline{F}\ (\pi_1(G))$ is uncountable.
iii) $\lim\limits_{n\to\infty}(\overline{F}_n(\pi_1(dual(G)))) = \lim\limits_{n\to\infty}(\overline{F}_n(\pi_1(G))) = 0$.

Proof. i) Let G be a connected path graph which can be represented it as the closed interval $[0,1]$, where $length(e_n) = \frac{1}{n(n+1)}$ for, $n = 1, 2, 3, \ldots$. Then $dual(G) \subseteq R^2$ is the union of the circles C_n of radii $\frac{1}{n}$ and centered at $(\frac{1}{n}, 0)$, $n = 1, 2, \ldots$, as in Fig. (1.a). Now we want to show that $\pi_1(dual(G))$ is uncountable. Consider the retraction $r_n : dual(G) \longrightarrow C_n$ which collapsing all C_i except C_n to origin. Each r_n induces a surjection $\overline{r_n} : \pi_1(dual(G))) \longrightarrow \pi_1(C_n) \approx Z$, where the origin is a base point. Then the product of the $\overline{r_n}$ is a homomorphism $r : \pi_1(dual(G)) \longrightarrow \prod_{\infty} Z$ to the direct product of infinite number of copies of Z. Since r is onto and $\prod_{\infty} Z$ uncountable it follows that $\pi_1(dual(G))$ is uncountable. ii) Let $F : G \longrightarrow G'$ be a folding such that each edges can be folded to loop as in Fig. (1.a) thus we get the induced folding $\overline{F} : \pi_1(G) \longrightarrow \pi_1(G')$ such that $\overline{F}(\pi_1(G)) = \pi_1(F(G))$, it follows from $\pi_1(F(G)) = \pi_1(dual(G))$ that $\overline{F}(\pi_1(G))$ is uncountable. iii) $\lim_{n\to\infty}(F_n(dual(G)))$ is a line as in Fig. (1.b) so $\pi_1(\lim_{n\to\infty}(F_n(dual(G)))) = 0$, thus $\lim_{n\to\infty}(\overline{F}_n(\pi_1(dual(G)))) = \pi_1(\lim_{n\to\infty}(F_n(dual(G)))) = 0$. Also, it follows from $\lim_{n\to\infty}(F_n(G))$ is a line that $\pi_1(\lim_{n\to\infty}(F_n(G))) = 0$. Hence $\lim_{n\to\infty}(\overline{F}_n(\pi_1(G))) = \pi_1(\lim_{n\to\infty}(F_n(G))) = 0$. Therefore, $\lim_{n\to\infty}(\overline{F}_n(\pi_1(dual(G)))) = \lim_{n\to\infty}(\overline{F}_n(\pi_1(G))) = 0$. □



Theorem 3.4. *Let G be an infinite connected graph that is the union of edges e_n where $length(e_n) = \frac{1}{n} length(e_1))$ for, $n = 2, 3, \ldots$ with common one vertex. Then*
i) $\pi_1(dual(G))$ is a free group on a countable set of generators.
ii) There is a folding $F : G \longrightarrow G'$ such that $F(G) = dual(G)$ which induces folding $\overline{F} : \pi_1(G) \longrightarrow \pi_1(G')$ such that $\overline{F}(\pi_1(G)) = \pi_1(dual(G))$.

Proof. i) It follows from $dual(G) = \bigvee_{\infty} S^1$ that $\pi_1(dual(G) \approx \pi_1(\bigvee_{\infty} S^1)$ and so $\pi_1(dual(G) \approx \pi_1($ Z $*$Z$* \ldots \infty)$ thus, $\pi_1(dual(G))$ is a free group on a countable set of generators.
ii) Consider the folding $F : G \longrightarrow G'$ which can be folded all edges to loops as in Fig. 2 thus $F(G) = dual(G)$, so we get the induced folding $\overline{F} : \pi_1(G) \longrightarrow \pi_1(G')$ such that $\overline{F}(\pi_1(G)) = \pi_1(dual(G))$. □



Theorem 3.5. *Let G be a connected planar graph with n- vertices***,** *then there is a folding $F : dual(G) \to dual(G)$ which induce folding $\overline{F} : \pi_1(dual(G)) \to \pi_1(dual(G))$ such that $\overline{F}(\pi_1(dual(G))) \approx Z$.*

Proof. Let G be a connected planar graph with n vertices; then $dual(G)$ has n faces, but n faces form $n-1$ loops. Now, consider the folding from $dual(G)$ into itself which folded all loops to only one loop, so $F(dual(G)) \approx S^1$, thus $\overline{F}(\pi_1(dual(G))) = \pi_1(F(dual(G))) \approx \pi_1(S^1)$. Hence there is a folding $F : dual(G) \to dual(G)$ which induces foldings $\overline{F} : \pi_1(dual(G)) \to \pi_1(dual(G))$ such that $\overline{F}(\pi_1(dual(G))) \approx Z$. □

Lemma 3.6. *Let G_1 and G_2 be two graphs, then $dual\,(G_1 \vee G_2) = dual(G_1) \vee dual(G_2)$.*

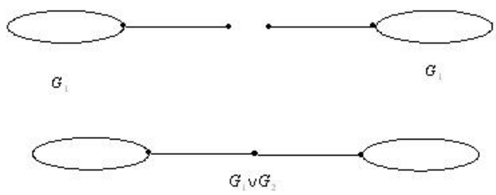
Proof. The proof of this lemma follows immediately from the definition of the dual graph. □

Theorem 3.7. *Let P_n be a path graph with n vertices. Then $\pi_1(dual\ (\overset{\infty}{\underset{n=1}{\vee}} P_n))$ is a free group on a countable set of generators.*

Proof. It follows from $dual\ (\overset{\infty}{\underset{n=1}{\vee}} P_n) = \overset{\infty}{\underset{n=1}{\vee}} dual\ (P_n) = \overset{\infty}{\underset{n=1}{\vee}} S_n$ that $\pi_1(dual\ (\overset{\infty}{\underset{n=1}{\vee}} P_n)) = \pi_1(\overset{\infty}{\underset{n=1}{\vee}} S_n)$, thus $\pi_1(\overset{\infty}{\underset{n=1}{\vee}} S_n) = \overset{\infty}{\underset{n=1}{*}} S_n \approx \overset{\infty}{\underset{n=1}{*}} Z$. Hence, $\pi_1(dual\ (\overset{\infty}{\underset{n=1}{\vee}} P_n))$ is a free group on a countable set of generators. □

Theorem 3.8. *Let G_1, G_2 be two disjoint connected graphs, then there is a folding $F : G_1 \cup G_2 \longrightarrow G_1' \cup G_2'$ which induces folding $\overline{F} : \pi_1(G_1 \cup G_2) \longrightarrow \pi_1(G_1' \cup G_2')$ such that $\overline{F}(\pi_1(G_1 \cup G_2)) = \pi_1(F(G_1)) * \pi_1(F(G_2))$.*

Proof. let $F : G_1 \cup G_2 \longrightarrow G_1' \cup G_2'$ be a folding such that $F(G_1 \cup G_2) = F(G_1) \vee F(G_2)$ as in Fig. 3, then we get the induced folding $\overline{F} : \pi_1(G_1 \cup G_2) \longrightarrow \pi_1(G_1' \cup G_2')$ such that $\overline{F}(\pi_1(G_1 \cup G_2)) = \overline{F}(\pi_1(G_1 \vee G_2))$, so $\overline{F}(\pi_1(G_1 \cup G_2)) = \overline{F}(\pi_1(G_1)) * \overline{F}(\pi_1(G_2))$. Since, $\overline{F}(\pi_1(G_i)) = \pi_1(F(G_i))\, for,\ i = 1, 2$, it follows that $\overline{F}(\pi_1(G_1 \cup G_2)) = \pi_1(F(G_1)) * \pi_1(F(G_2))$. □



Theorem 3.9. *If $G_1, G_2, \ldots, G_n$ are connected graphs and $F : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ is a folding mapping from $\overset{n}{\underset{i=1}{\vee}} G_i$ into itself, then there is an induced folding*

$$\overline{F} : \overset{n}{\underset{i=1}{*}} \pi_1(F(G_i)) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(F(G_i)),$$

which reduces the rank of $\overset{n}{\underset{i=1}{*}} \pi_1(F(G_i))$.

Proof. Let $F : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ be a folding such that $F(\overset{n}{\underset{i=1}{\vee}} G_i) = G_1 \vee G_2 \vee \ldots \vee F(G_s) \vee \cdots \vee G_n$, for $s = 1, 2, \ldots, n$. Then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} G_i)) \approx \pi_1(G_1) * \pi_1(G_2) * \cdots * \pi_1(F(G_s)) * \cdots * \pi_1(G_n)$. Since rank ($\pi_1(F(G_s))) \leq$ rank ($\pi_1(G_s)$), it follows that $\overline{F}$ reduce the rank of $\overset{n}{\underset{i=1}{*}} \pi_1((G_i)$.Also, if $F(\overset{n}{\underset{i=1}{\vee}} G_i) = G_1 \vee G_2 \vee \cdots \vee F(G_s) \vee \cdots \vee F(G_k) \vee \cdots \vee G_n$ for $k = 1, 2, \ldots, n, s < k$ then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} G_i)) \approx \pi_1(G_1) * \pi_1(G_2) * \cdots * \pi_1(F(G_s)) * \cdots * \pi_1(F(G_k)) * \cdots * \pi_1(G_n)$ and so $\overline{F}$ reduce the rank of $\overset{n}{\underset{i=1}{*}} \pi_1((G_i)$ .Moreover, by continuing this process if $F(\overset{n}{\underset{i=1}{\vee}} G_i) = \overset{n}{\underset{i=1}{\vee}} F(G_i)$.Then, $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} G_i)) = \pi_1(\overset{n}{\underset{i=1}{\vee}} F(G_i)) \approx \overset{n}{\underset{i=1}{*}} \pi_1(F(G_i))$. Hence $\overline{F}$ reduce the rank of $\overset{n}{\underset{i=1}{*}} \pi_1(F(G_i))$. □

Theorem 3.10. *If $G_1, G_2, \ldots, G_n$ are connected graphs and G_i is homeomorphic to S_i^1 for $i = 1, 2, \ldots, n$, then for every k, there is a folding $F_k : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ into itself which induces a folding $\overline{F}_k$ of $\overset{n}{\underset{i=1}{*}} \pi_1(G_i)$ into itself such that $F_k(\overset{n}{\underset{i=1}{*}} \pi_1(G_i))$ is a free group of rank $n-k$ for $k = 1, 2, \ldots, n-1$ and $\overline{F}_n(\overset{n}{\underset{i=1}{*}} \pi_1(G_i))$ is a free group of rank 1.*

Proof. Let $F_1 : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ be a folding such that $F_1(\overset{n}{\underset{i=1}{\vee}} G_i) = G_1 \vee G_2 \vee \ldots \vee F_1(G_t) \vee \ldots \vee G_n$ for $t = 1, 2, \ldots, n$ and $F_1(G_t) = G_i \, i \neq t$ folded loop to another loop, then consider the induced folding $\overline{F}_1 : \overset{n}{\underset{i=1}{*}} \pi_1(G_i) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(G_i)$ such that $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) = \pi_1(F_1(\overset{n}{\underset{i=1}{\vee}} G_i))$ and so $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) \approx \pi_1(G_1) * \pi_1(G_2) * \cdots * \pi_1(F_1(G_t)) * \cdots * \pi_1(G_n)$, thus $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(G_i)) \approx \overset{n-1}{\underset{i=1}{*}} \pi_1(G_i)$. Since $\pi_1(S_i^1) \approx Z$, it follows that $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1))$ is a free group of rank $n-1$. Also, let $F_2 : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ be folding such that $F_2(\overset{n}{\underset{i=1}{\vee}} G_i) = G_1 \vee G_2 \vee \ldots \vee F_2(G_s) \vee \ldots \vee F_2(G_t) \vee \ldots \vee G_n$ for $s, t = 1, 2, \ldots, n, \; s \prec t$, and $F_2(G_s) = G_i \;\; F_2(G_t) = G_i, \; i \neq s, i \neq t$ then we can get the induced folding $\overline{F}_2 : \overset{n}{\underset{i=1}{*}} \pi_1(G_i) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(G_i)$ such that $\overline{F}_2(\overset{n}{\underset{i=1}{*}} \pi_1(G_i))$ is a free group of rank $n-2$. By continuing this process we obtain the folding $F_n : \overset{n}{\underset{i=1}{\vee}} G_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} G_i$ such that $F_n(\overset{n}{\underset{i=1}{\vee}} G_i) = \overset{n}{\underset{i=1}{\vee}} F_n(G_i)$ and $F_n(G_i) = G_1, i = 1, 2, \ldots, n$ which induces a folding $\overline{F}_n : \overset{n}{\underset{i=1}{*}} \pi_1(G_i) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(G_i)$ such that $\overline{F}_n(\overset{n}{\underset{i=1}{*}} \pi_1(G_i))$ is a free group of rank 1. □

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