Existence of quadrature surfaces for uniform density supported by a segment

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Abstract. Given are two strictly positive constants $a$ and $k$. We show that if $a \geq 3.92k$ then there exists an open and bounded set $\Omega$ in $\mathbb{R}^2$ which contains strictly the line segment $C$ ($C = [-1, 1] \times \{0\}$) such that the following overdetermined problem has a solution

$$-\Delta u = a\delta_C \text{ in } \Omega, \; u = 0 \; \text{and} \; -\frac{\partial u}{\partial \nu} = k \text{ on } \partial \Omega.$$ 

Here $\nu$ is the outward normal vector to $\partial \Omega$ and $\delta_C$ is the uniform density supported by $C$.

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1 Introduction and main theorem

Let $\mu$ be a positive measure with compact support $K_\mu$ and denote by $C_\mu$ the convex hull of $K_\mu$. Consider the following free boundary problem:

$$(FB_\mu) \begin{cases} 
\text{Find a domain } \Omega \text{ of } \mathbb{R}^N \text{ which strictly contains } K_\mu \\
\text{and a function } u_\Omega \in H^1_0(\Omega) \text{ such that:} \\
-\Delta u_\Omega = \mu \text{ in } \Omega \\
u_{\Omega} = 0 \text{ on } \partial \Omega \\
-\frac{\partial u_\Omega}{\partial \nu} = k \text{ on } \partial \Omega \text{ (overdetermined condition).} 
\end{cases}$$

where $\nu$ is the outward normal vector to $\partial \Omega$ and $k \in \mathbb{R}^+_\ast$.

This problem is known as the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-shaw flows, electromagnetic shaping, gravitational problems etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the introduction in [6].

In [2], the authors gave sufficient condition of existence for the problem $(FB_\mu)$ with $\mu \in L^2(\mathbb{R}^N)$ ($N \geq 2$) and $K_\mu$ has a nonempty interior.
Existence of quadrature surfaces

This paper concerns the case where \( N = 2 \) and \( \mu = a\delta_{[-1,1] \times \{0\}} \) (\( a > 0 \)).

By using the moving plane method [5], H. Shahgholian showed in [9] that if the problem \( (FB_\mu) \) admits a solution \( (\Omega, u_\Omega) \) such that \( \Omega \) is of class \( C^2 \) and \( u_\Omega \in C^2(\overline{\Omega}) \), then all the inward normals at the boundary \( \partial \Omega \) of \( \Omega \) meet \( C_\mu \). Since we relate the existence of a solution for Problem \( (FB_\mu) \) to the existence of a minimum of some shape optimization problem, it is natural to resolve this one in a class of domains with this geometric normal property (see below).

In [1], the author studied bounded domains with the property he denoted by \( C\text{-}\text{gnp} \) (Geometric Normal Property w.r.t \( C \)). Namely, for a given compact convex set \( C \), the bounded domain \( \omega \) satisfies \( C\text{-}\text{gnp} \) if

1. \( \omega \supset \text{int}(C) \),
2. \( \partial \omega \setminus C \) is locally Lipschitz,
3. for any \( c \in \partial C \) there is an outward normal ray \( \Delta_c \) such that \( \Delta_c \cap \omega \) is connected, and
4. for every \( x \in \partial \omega \setminus C \) the inward normal ray to \( \omega \) (if exists) meets \( C \).

Let \( D \) be a ball of \( \mathbb{R}^2 \) with large radius in order to contain all the sets we will use. Let \( C = [-1,1] \times \{0\} \) and set

\[ O_C = \{ \omega \subset D : \omega \text{ satisfies } C\text{-}\text{gnp} \}, \]

and

\[ J(\omega) = -\frac{1}{2} \int_\omega |\nabla u_\omega(x)|^2 dx + \frac{k^2}{2} \int_\omega dx, \]

where \( u_\omega \) is the solution of the following Dirichlet problem \( P(\omega) : \)

\[ -\Delta u_\omega = a\delta_C \text{ in } \omega, \quad u_\omega = 0 \text{ on } \partial \omega. \]

Remark 1.1. \( \delta_C \) is a distribution belonging to \( H^{-1}(\omega) \) and thus the solution \( u_\omega \) of \( P(\omega) \) is a priori only in \( H_0^1(\omega) \). Nevertheless \( u_\omega \) is harmonic (and thus it is \( C^\infty \)) outside the line segment \( C \) and one can prove that it is continuous in \( \omega \).

Our aim here is to prove the following

**Theorem 1.2.** 1. If \( a \geq 3.92k \) then there exists \( \Omega \in O_C \) which contains strictly \( C \) and such that \( J(\Omega) = \min_{\omega \in O_C} J(\omega) \) and

\[ \begin{cases} -\Delta u_\Omega = a\delta_C & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial \Omega \end{cases} \]
2. If $\Omega$ is of class $C^2$ then
\[ -\frac{\partial u}{\partial \nu} = k \quad \text{on} \quad \partial \Omega. \]

For the first point, we will use the shape optimization tool in order to get the minimum $\Omega$ of $J$ then the Standard Maximum Principle and the Fourier expansion will enable us to have a sufficient condition that $C$ is strictly contained in $\Omega$. For the second point, the shape derivative together with the characterization of the $C$-GNP (see Proposition 2.8 below) will give the overdetermined condition.

**Remark 1.3.** Theorem 1.2 says that if the line segment in the complex plane is provided with a uniform density above a certain level, then there will exist a domain containing compactly the line segment such that the given measure on the line segment is equigravitational to the arc-length measure of the domain.

## 2 Preliminary results

**Definition 2.1.** Let $K_1$ and $K_2$ be two compact subsets of $D$. We call a Hausdorff distance of $K_1$ and $K_2$ (or briefly $d_H(K_1, K_2)$), the following positive number:
\[ d_H(K_1, K_2) = \max \{ \rho(K_1, K_2), \rho(K_2, K_1) \}, \]
where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j)$, $i, j = 1, 2$ and $d(x, K_j) = \min_{y \in K_j} |x - y|$.

**Definition 2.2.** Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. Let $K_n$ and $K$ be their complements in $D$. We say that the sequence $\omega_n$ converges in the Hausdorff sense, to $\omega$ (or briefly $\omega_n \overset{H}{\to} \omega$) if
\[ \lim_{n \to +\infty} d_H(K_n, K) = 0. \]

**Definition 2.3.** Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. We say that the sequence $\omega_n$ converges in the compact sense, to $\omega$ (or briefly $\omega_n \overset{K}{\to} \omega$) if
- every compact subset of $\omega$ is included in $\omega_n$, for $n$ large enough, and
- every compact subset of $\overline{\omega}$ is included in $\overline{\omega_n}$, for $n$ large enough.

**Definition 2.4.** Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. We say that the sequence $\omega_n$ converges in the sense of characteristic functions, to $\omega$ (or briefly $\omega_n \overset{L}{\to} \omega$) if $\chi_{\omega_n}$ converges to $\chi_{\omega}$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$, ($\chi_{\omega}$ is the characteristic function of $\omega$).

**Theorem 2.5.** If $\omega_n \in \mathcal{O}_C$, then there exists an open subset $\omega \subset D$ and a subsequence (still denoted by $\omega_n$) such that
1. $\omega_n \overset{H}{\to} \omega$
2. $\omega_n \xrightarrow{K} \omega$

3. $\chi_{\omega_n}$ converges to $\chi_\omega$ in $L^1(D)$

4. $\omega \in O_C$

5. $u_n$ converges strongly in $H^1_0(D)$ to $u_\omega$ (supposed solutions of $P(\omega_n)$ and $P(\omega)$).

Furthermore, the assertions (1), (2) and (3) are equivalent.

For the proof of this theorem, see Theorem 3.1 and Theorem 4.3 in [1]. For the equivalence between (1), (2) and (3), see Propositions 3.4, 3.5, 3.6, 3.7 and 3.8 in [1]. Notice that, in general, we do not have the equivalence between (1), (2) and (3) (see for instance [7]).

**Definition 2.6.** Let $C$ be a convex set. We say that an open subset $\omega$ has the C-sp, if and only if

1. $\omega \supset \text{int}(C)$,

2. $\partial \omega \setminus C$ is locally Lipschitz,

3. for any $c \in \partial C$ there is an outward normal ray $\Delta_c$ such that $\Delta_c \cap \omega$ is connected, and

4. $\forall x \in \partial \omega \setminus C \ K_x \cap \omega = \emptyset$, where $K_x$ is the closed cone defined by

$$\{ y \in \mathbb{R}^N : (y - x)(z - x) \leq 0, \forall z \in C \}.$$

**Remark 2.7.** $K_x$ is the normal cone to the convex hull of $C$ and $\{x\}$.

**Proposition 2.8.** $\omega$ has the C-gnp if and only if $\omega$ satisfies the C-sp.

For the proof of this proposition see Proposition 2.3 in [1].

**Theorem 2.9.** Let $L$ be a compact subset of $\mathbb{R}^N$. Let $f_n$ be a sequence of functions defined on $L$. We assume that the $f_n$ are of class $C^3$ and

$$\left| \frac{\partial f_n}{\partial x_i} \right| \leq M, \quad \left| \frac{\partial^2 f_n}{\partial x_i \partial x_j} \right| \leq M, \quad \left| \frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k} \right| \leq M,$$

where $M$ is a strictly positive constant and is independent of $n$.

Define a sequence $\Omega_n$, by $\Omega_n = \{ x \in L : f_n(x) > 0 \}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all $x$ in $L$. If the $\Omega_n$ have the C-gnp, then there exists $\Omega$ of class $C^2$ and a subsequence (still denoted by $\Omega_n$) such that $\Omega_n$ converges in the compact sense, to $\Omega$.

For the proof of this theorem, see [2].

**Remark 2.10.** The aim of Theorem 2.9 is to give the $C^2$ regularity of the minimum $\Omega$ of $J$ defined below. This in order to use the shape derivative and so to resolve Problem $(FB)_\mu$. The proof of this theorem uses the following lemma
Lemma 2.11. Let $L$ be a compact subset of $\mathbb{R}^N$. Let $f_n$ be a sequence of functions defined as Theorem 2.9. Suppose that $\Omega$ is an open subset of $L$ such that

$$
\Omega = \{ x \in L : h(x) > 0 \} \quad \text{and} \quad \partial \Omega = \{ x \in L : h(x) = 0 \},
$$

where $h$ is a continuous function defined in $L$. If the $f_n$ converge uniformly to $h$ in $L$, then the $\Omega_n$ converge in the compact sense, to $\Omega$.

3 Proof of Theorem 1.2

3.1 $\Omega$ contains strictly $C$

Using the variational formulation of the Dirichlet problem $P(\omega)$, we get

$$
\int_\omega |\nabla u_\omega(x)|^2 \, dx = a \int_C u_\omega.
$$

If $u_D$ denotes the solution of the Dirichlet problem $P(D)$, by the maximum principle, $0 \leq u_\omega \leq u_D$ and so

$$
J(\omega) = -\frac{a}{2} \int_C u_\omega + \frac{k^2}{2} \int_\omega dx \geq -\frac{a}{2} \int_D u_D,
$$

and $\inf J$ exists. Let $\Omega_n$ be a minimizing sequence in $\mathcal{O}_C$. One can choose $\Omega_n$ as in Theorem 2.9 above and get the existence of a subsequence $\Omega_{n_k}$ and of $\Omega$ which is of class $C^2$ such that $\Omega_{n_k} \xrightarrow{H} \Omega$. Then, from Theorem 2.5, item 1 implies $\Omega_{n_k} \xrightarrow{H} \Omega$, item 4 gives $\Omega \in \mathcal{O}_C$ and by item 3 $\int_{\Omega_{n_k}} dx$ converges to $\int_\Omega dx$. Now if $u_{n_k}$ and $u_\Omega$ are respectively the solutions of $P(\Omega_{n_k})$ and $P(\Omega)$ then item 3 together with item 5 of Theorem 2.5 implies that $\int_{\Omega_{n_k}} u_{n_k}$ converges to $\int_\Omega u_\Omega$ when $k$ tends to infinity. Hence $J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega)$.

Now suppose, by contradiction, that $\partial \Omega$ intersects $C$ at a point $c$. As we will reason locally, we can suppose that $c$ is in the origin. Let $\varepsilon > 0$, put $\Omega_\varepsilon = \Omega \cup B(0, \varepsilon)$ and $I_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. Let $u_{\Omega_\varepsilon}$ be the solution of the Dirichlet problem $P(\Omega_\varepsilon)$. By the maximum principle we have $u_{\Omega_\varepsilon} > u_\Omega$ in $\Omega$. Then, as $(C \cap \Omega) \setminus I_\varepsilon \subset \Omega$ and $u_\Omega = 0 \leq u_{\Omega_\varepsilon}$ on $C \setminus \Omega$, we get

$$
-\frac{a}{2} \int_{C \setminus I_\varepsilon} u_{\Omega_\varepsilon} < -\frac{a}{2} \int_{C \setminus I_\varepsilon} u_\Omega.
$$

Therefore

$$
(3.1) \quad J(\Omega_\varepsilon) - J(\Omega) \leq \frac{k^2}{2} \left( \int_{\Omega_\varepsilon} \, dx - \int_{\Omega} \, dx \right) + \frac{a}{2} \int_{I_\varepsilon} u_\Omega - \frac{a}{2} \int_{I_\varepsilon} u_{\Omega_\varepsilon},
$$

$$
(3.2) \quad \leq \frac{\pi k^2 \varepsilon^2}{2} + \frac{a}{2} \int_{I_\varepsilon} u_\Omega - \frac{a}{2} \int_{I_\varepsilon} u_{\Omega_\varepsilon}.
$$

To get a contradiction of our assumption, we need to prove the two following lemmas.
Lemma 3.1.

\[
\int_{I_\varepsilon} u_\Omega \leq (k + \varepsilon) \varepsilon^2.
\]

Proof. Since \(0 \in \partial \Omega \cap C\), the optimality condition gives: \(-\frac{\partial u_\Omega}{\partial \nu}(0) \leq k\) (see Remark 3.3 below). Now, as \(u_\Omega = v - \frac{\varepsilon^2}{2}|y|\) where \(v\) is harmonic in \(\Omega\) then \(u_\Omega\) is of class \(C^4\) on the closed higher half-plane. Consequently, for all \(\varepsilon > 0\) there is a neighborhood \(V_\varepsilon\) of 0 in the higher half-plane, such that

\[
\forall x \in V_\varepsilon : \|\nabla u_\Omega(x)\| \leq k + \varepsilon.
\]

The Mean-Value Theorem applied to the line segments \((-h,0),(0,0))\) and \((h,0),(0,0))\) \((0 < h < \varepsilon)\), (3.4) implies

\[
u_\Omega(-h,0) \leq (k + \varepsilon)h \text{ and } u_\Omega(h,0) \leq (k + \varepsilon)h.
\]

Therefore

\[
\int_{-\varepsilon}^{\varepsilon} u_\Omega(h,0) \, dh \leq (k + \varepsilon)\varepsilon^2
\]

which gives the inequality (3.3).

Lemma 3.2.

\[
\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \frac{2\varepsilon^2}{\pi} \left[ \frac{1}{2} + \frac{\pi^2}{8} \right].
\]

Proof. Let \(v_\varepsilon\) be the solution of the following Dirichlet problem

\[
(P_\varepsilon) \left\{ \begin{array}{l}
-\Delta v_\varepsilon = a\delta_{I_\varepsilon} \text{ in } B(0,\varepsilon) \\
v_\varepsilon = 0 \text{ on } \partial B(0,\varepsilon).
\end{array} \right.
\]

By the maximum principle, we have \(u_{\Omega_\varepsilon} > v_\varepsilon\) in \(B(0,\varepsilon)\) and thus

\[
\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \int_{I_\varepsilon} v_\varepsilon.
\]

Let \(w = -\frac{\varepsilon}{2}|y|\) be the fundamental solution of \(-\Delta w = a\delta_{I_\varepsilon}\). If \(w_\varepsilon = v_\varepsilon - w\) then \(w_\varepsilon\) satisfies:

\[
\left\{ \begin{array}{l}
-\Delta w_\varepsilon = 0 \text{ in } B(0,\varepsilon) \\
w_\varepsilon = \frac{a\varepsilon}{2} |\sin(\theta)| \text{ on } \partial B(0,\varepsilon).
\end{array} \right.
\]

Since \(w_\varepsilon(r,0) = v_\varepsilon(r,0)\) for all \(r \in [0,\varepsilon]\), then

\[
\int_{I_\varepsilon} v_\varepsilon = \int_{I_\varepsilon} w_\varepsilon.
\]

But

\[
w_\varepsilon(r,\theta) = \sum_{n \geq 0} a_n r^n \cos(n\theta) + \sum_{n \geq 1} b_n r^n \sin(n\theta),
\]
\[ w_\varepsilon(\varepsilon, \theta) = \frac{a\varepsilon}{2} |\sin(\theta)| \]

and

\[ |\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{p \geq 1} \frac{1}{(2p)^2 - 1} \cos(2p\theta) \]

then

\[ w_r(r, \theta) = \frac{a}{\pi} \left( \varepsilon - 2 \sum_{p \geq 1} \frac{\varepsilon^{1-2p}}{(2p)^2 - 1} r^{2p} \cos(2p\theta) \right). \]

Therefore

\[
\int_{I_\varepsilon} w_\varepsilon(r, 0) \, dr = \int_{-\varepsilon}^{\varepsilon} (a_0 + \sum_{p \geq 1} a_{2p} r^{2p}) \, dr \\
= 2a_0 \varepsilon + \sum_{p \geq 1} \frac{2a_{2p} \varepsilon^{2p+1}}{2p + 1} \\
= \frac{2a\varepsilon^2}{\pi} \left[ 1 - 2 \sum_{p \geq 1} \frac{1}{(2p + 1)((2p)^2 - 1)} \right].
\]

Now, since

\[
\sum_{p \geq 1} \frac{1}{(2p + 1)((2p)^2 - 1)} = \frac{3}{4} - \frac{\pi^2}{16}
\]

then

\[
\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \int_{I_\varepsilon} w_\varepsilon(r, 0) \, dr = \frac{2a\varepsilon^2}{\pi} \left[ \frac{1}{2} + \frac{\pi^2}{8} \right].
\]

\[ \square \]

**End of the proof of Theorem 1.2**

Now thanks to (3.3) and (3.5), the inequality 3.2 becomes

\[ J(\Omega_\varepsilon) - J(\Omega) < \frac{1}{2} \left( ak + \pi k^2 - \frac{\pi^2}{4} - \frac{a^2}{4\pi} \right) \varepsilon^2 + \frac{a}{2} \varepsilon^3. \]

or again

\[ J(\Omega_\varepsilon) - J(\Omega) < \frac{k^2}{2} \left( \frac{a}{k} + \pi - 0.46 \frac{a^2}{k^2} \right) \varepsilon^2 + \frac{a}{2} \varepsilon^3. \]

If we put \( t = \frac{a}{k} \), the sign of

\[ P(t) = -0.46t^2 + t + \pi \]

is negative if \( t \geq 3.92 \).

It follows that if \( a \geq 3.92k \) then \( J(\Omega_\varepsilon) < J(\Omega) \) which contradicts the minimality of \( \Omega \).
3.2 The overdetermined condition: \( -\frac{\partial u_\Omega}{\partial \nu} = k \) on \( \partial \Omega \)

Put \( J(\Omega) = \frac{1}{2} [J_1(\Omega) + k^2 V(\Omega)] \) where

\[
J_1(\Omega) = -\int_\Omega |\nabla u_\Omega(x)|^2 \, dx,
\]

and

\[
V(\Omega) = \int_\Omega dx.
\]

Let us consider a deformation field \( \theta \in C^2(\mathbb{R}^N; \mathbb{R}^N) \). Since \( \Omega \) is of class \( C^2 \), then the classical Hadamard formula gives the derivative of \( J \) with respect to the displacement \( \theta \) (or in the direction \( \theta \)) (see \([8, 10]\)).

\[
dJ_1(\Omega; \theta) = -\int_{\partial \Omega} \left( -\frac{\partial u_\Omega}{\partial \nu} \right)^2 \theta \cdot \nu \, d\sigma - 2 \int_{\Omega} \nabla u_\Omega \cdot \nabla u'(x) \, dx
\]

where \( \nu \) is the outward normal vector to \( \partial \Omega \) and \( u' \) the derivative of \( u_\Omega \) which is defined as the solution of the following problem:

\[
\begin{aligned}
-\Delta u' &= 0 \quad \text{in } \Omega \\
\frac{\partial u_\Omega}{\partial \nu} - \theta \cdot \nu &= \left( -\frac{\partial u_\Omega}{\partial \nu} \right) \theta \cdot \nu \quad \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
dV(\Omega; \theta) = \int_{\partial \Omega} \theta \cdot \nu \, d\sigma.
\]

Then

\[
dJ(\Omega; \theta) = \frac{1}{2} \left[ \int_{\partial \Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial \Omega} \left( -\frac{\partial u_\Omega}{\partial \nu} \right)^2 \theta \cdot \nu \, d\sigma \right] - \int_{\Omega} \nabla u_\Omega \cdot \nabla u'(x) \, dx.
\]

Using the Green formula,

\[
\begin{aligned}
dJ(\Omega; \theta) &= \frac{1}{2} \left[ \int_{\partial \Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial \Omega} \left( -\frac{\partial u_\Omega}{\partial \nu} \right)^2 \theta \cdot \nu \, d\sigma \right] \\
&\quad + \int_{\Omega} u_\Omega \Delta u'(x) \, dx - \int_{\partial \Omega} u_\Omega \frac{\partial u'}{\partial \nu} \theta \cdot \nu \, d\sigma.
\end{aligned}
\]

According to (3.6) and (3.7),

\[
dJ(\Omega; \theta) = \frac{1}{2} \left[ \int_{\partial \Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial \Omega} \left( -\frac{\partial u_\Omega}{\partial \nu} \right)^2 \theta \cdot \nu \, d\sigma \right].
\]

Now since \( \Omega \) is the minimum of \( J \), then \( dJ(\Omega; \theta) \geq 0 \) for every admissible displacement \( \theta \). Therefore

\[
\int_{\partial \Omega} \left( k^2 - \left( -\frac{\partial u_\Omega}{\partial \nu} \right)^2 \right) \theta \cdot \nu \, d\sigma \geq 0 \quad \text{for every admissible displacement } \theta.
\]
We mean by admissible displacement the one which allows us to keep the C-gnp or the C-sp (according to Proposition 2.8 above). Since $\Omega$ has the C-gnp, it satisfies the C-sp. This together with the fact that $C$ is strictly contained in $\Omega$ implies

$$\forall x \in \partial \Omega \ K_x \cap \Omega = \emptyset.$$ 

For $t$ sufficiently small, let $\Omega_t = \Omega + t\theta(\Omega)$ be the deformation of $\Omega$ in the direction $\theta$. Let $x_t \in \partial \Omega_t$. There exists $x \in \partial \Omega$ s.t $x_t = x + t\theta(x)$. Using the definition of $K_x$, and the equality above, it is obvious to get (for $t$ small enough and for every displacement $\theta$):

$$\forall x_t \in \partial \Omega_t \ K_{x_t} \cap \Omega_t = \emptyset,$$

which means that $\Omega_t$ satisfies the C-sp (and so the C-gnp) for every displacement $\theta$ when $t$ is sufficiently small. Then, using $\theta$ and $-\theta$, and the fact that the set of the functions $\theta \cdot \nu$ is dense in $L^2(\partial \Omega)$, we deduce

$$-\frac{\partial u_{\Omega}}{\partial \nu}(x) = k \text{ on } \partial \Omega.$$

**Remark 3.3.** In the case where $\partial \Omega \cap C \neq \emptyset$, if there exists $x \in \partial \Omega \cap C$ such that $\theta(x) \cdot \nu(x) \leq 0$ then the inward normal at $x_t$ doesn’t intersect the line segment $C$. So to keep the C-gnp the displacements $\theta$ must satisfy $\theta(x) \cdot \nu(x) \geq 0$ for all $x \in \partial \Omega \cap C$. So (3.8) implies

$$-\frac{\partial u_{\Omega}}{\partial \nu}(x) \leq k \quad \forall x \in \partial \Omega \cap C.$$

### 4 Final remarks

**Remark 4.1.** Using the notion of quadrature domains, B. Gustafsson and H. Shalgholian showed in [6] that in the case where $\mu = a\delta_{[-1,1] \times \{0\}}$ the problem $(FB_{\mu})$ admits a solution if $a \geq 24\pi k$.

**Remark 4.2.** It is not hard to see that in the case where $\mu = a\delta_{[-1,1] \times \{0\}}$, if $(\Omega, u_{\Omega})$ is a regular solution of the problem $(FB_{\mu})$ then $a > 2k$. **Question:** Is the converse true? The answer seems to be no.

**Remark 4.3.** Set

$$\mathcal{O}_P = \{ \omega \in \mathcal{O}_C \text{ and } |\partial \omega| \leq \text{cst} \},$$

where $|\partial \omega|$ denotes the perimeter of $\omega$. Using the same arguments as above, one can prove that:

1. If $a \geq 3.92k$ then there exists $\Omega \in \mathcal{O}_P$ which contains strictly $C$ and such that $J(\Omega) = \min_{\omega \in \mathcal{O}_P} J(\omega)$ and

\[
\begin{cases}
-\Delta u_{\Omega} = a\delta_C & \text{in } \Omega \\
u_{\Omega} = 0 & \text{on } \partial \Omega
\end{cases}
\]
2. If $\Omega$ is of class $C^2$ then there exists a Lagrangian multiplier $\lambda(\Omega)$ s.t.

$$-\frac{\partial u_\Omega}{\partial \nu} = \sqrt{\lambda(\Omega) H_{\partial \Omega}} + k^2$$

on $\partial \Omega$.

where $H_{\partial \Omega}$ is the mean curvature of $\partial \Omega$.

**Remark 4.4.** In [3], the authors gave sufficient condition of existence for the following free boundary problem

$$\begin{cases}
-\Delta u_\Omega = \mu & \text{in } \Omega \\
u_\Omega = 0 & \text{on } \partial \Omega \\
-\frac{\partial u_\Omega}{\partial \nu} = \sqrt{\lambda H_{\partial \Omega}} + k^2 & \text{on } \partial \Omega
\end{cases}$$

where $\text{Supp}\mu$ has a nonempty interior and $\lambda$ and $k$ are two positive constants.

**References**


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