# Existence and uniqueness for optimal control of Oxygen absorption in aquatic system

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**Abstract.** In this paper, we consider the mathematical formulation and analysis of an optimal control problem for a nonlinear Dissolve Oxygen(DO)/ Biological Oxygen Demand(BOD) system with logistic growth term. In the BOD/DO system studied by Bermúdez [2] this logistic growth term is not included. We study the existence and uniqueness of solution for the nonlinear system as well as the existence and uniqueness of optimal solutions.

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**Key words**: Optimal control, dissolve oxygen, biological Oxygen demand, water pollution.

## 1 Introduction

Most often, our waterways are being polluted by municipal, agricultural and industrial wastes, including many toxic synthetic chemicals which cannot be broken down at all by natural processes. Even in tiny amounts, some of these substances can cause serious harm.

Microorganisms such as bacteria are responsible for decomposing organic waste. When organic matter such as dead plants, leaves, grass clippings, manure, sewage, or even food waste is present in a water supply, the bacteria will begin the process of breaking down this waste. When this happens, much of the available dissolved oxygen is consumed by aerobic bacteria, robbing other aquatic organisms of the oxygen they need to live [17].

Biological oxygen demand (BOD) is an indicator for the concentration of biodegradable organic matter present in a sample of water. It can be used to infer the general quality of the water and its degree of pollution. BOD measures the rate of uptake of oxygen by micro-organisms in the sample of water at a fixed temperature and over a given period of time.

If the pollution level is not too high this need can be satisfied by the dissolved oxygen. Notice that the oxygen is very sensitive to wastewater discharges namely the thermal ones. Indeed, at high temperature solubility of oxygen decreases while activity of microorganisms which are oxygen consuming increases. If the quantity of

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organic matter increases beyond a maximum value the dissolved oxygen is not enough to decompose it leading to modification in the ecosystem.

In recent times a number of researchers have worked on the BOD/DO parabolic models. Streeter and Phelps [16] gave the classical model for BOD/DO. Bermúdez [2] considered Streeter model, and in [3] gave an optimal location for wastewater outfall for steady case parabolic equation. Martínez [12] gave the theoretical analysis for the optimal control problem related to wastewater treatment resulting in pointwise control for both the objective functional and state constraint stating the existence of unique solution. Alvarez-Vazquez [1] threated the case of evolution parabolic equation of [3] for an optimal location of wastewater outfall. Piasecki [14] and [15] in his work modified the Streeter's model to include Sediment Oxygen Demand(SOD).

This paper concerns the application of a distributed parameter control for a diffusive-convective population, whose growth is governed by logistic terms. The growth of microorganisms population have been shown to follow logistic growth pattern [4]. In recent years, several authors have studied population models with logistic growth terms, Clark [5] and Murray [13]. Lenhart and Bhat [10] treated wildlife management problem with logistic growth terms and Lenhart [11] considered degenerate parabolic equation having logistic growth terms. Similarly, Garvie [6], [7] and [8] considered population models with logistic growth terms. The present work modifies the classical model of Streeter and Phelps by including the logistic growth terms.

Mathematical models play a major role in predicting the pollution level in the regions under consideration. Obviously, the knowledge of mathematical models for the evolution of pollutant concentration is an unavoidable first step if one wants to use optimal control techniques. So, the first part of this work is devoted to the study of Biological Oxygen Demand (BOD) and Dissolved Oxygen (DO), which is frequently used in the case of domestic discharges. Next, in section 3 we discuss the existence and uniqueness of solution for this nonlinear system of equations and in the subsequent section, we consider the existence and uniqueness of solution for the optimal control of the coupled system.

## 2 Pollutant Dispersion: The BOD-DO Model

We consider a domain occupied by shallow water of polluted wastewater. Firstly, in order to simulate the water quality in the domain ,we have to choose some indicators of pollution levels. Two of the most important (especially in the case of domestic discharges) are the Dissolved Oxygen (DO) and the organic matter, which can be measured in terms of the need of oxygen to decompose it, the so called Biological Oxygen Demand (BOD). If the pollution level is not too high the BOD can be satisfied by the DO. However, if the organic matter increases beyond a maximum value the DO is not enough for its decomposition, leading to important modifications (anaerobic processes) in the ecosystem. To avoid this situation a threshold value of BOD may not be exceeded and a minimum level of DO must be guaranteed.

The evolution of the BOD and the DO in the domain $\Omega \subset I\!R^2$ is governed by a system of partial differential equations (see Streeter and Phelps [16], A. Bermúdez, [2]). We give a modification of the model given by Bermúdez, [2] to include a logistic growth for BOD and we also suppose that there are no source term . Let us denote by $\rho_1(x,t)$ and $\rho_2(x,t)$ the concentrations of BOD and DO at point $x \in \Omega$ and at time $t \in [0,T]$, respectively. Then, these concentrations are obtained as the solution of the following two initial-boundary value problems:

$$(2.1)\quad \left.\begin{array}{ll}
\dfrac{\partial \rho_1}{\partial t} + v.\nabla\rho_1 - \beta_1\Delta\rho_1 = \rho_1(a - b\rho_1) - k_1\rho_1 & \text{in } Q \\
\rho_1(x,0) = \rho_{10}(x) & \text{in } \Omega \\
\dfrac{\partial \rho_1}{\partial n} = 0 & \text{on } \Sigma \\
\dfrac{\partial \rho_2}{\partial t} + v.\nabla\rho_2 - \beta_2\Delta\rho_2 = -k_1\rho_1 + \dfrac{1}{h}k_2(d_s - \rho_2) & \text{in } Q \\
\rho_2(x,0) = \rho_{20}(x) & \text{in } \Omega \\
\dfrac{\partial \rho_2}{\partial n} = 0 & \text{on } \Sigma
\end{array}\right\}$$

where $u$ and $h$ can be obtained from the shallow water equation, $\beta_1$ and $\beta_2$ (horizontal viscosity coefficients) are positive parameters, $k_1, k_2$ (kinetic coefficients related to BOD elimination and oxygen transfer through the surface, respectively) and $d_s$ (oxygen saturation density) can be obtained from experimental measurements.

In the domain $\Omega$, it is necessary to assure water quality, i.e. pollution concentration must be lower than a given maximum level. If we take BOD and DO as indicators of the water quality, then the environmental constraints on it can be written as:

$$(2.2)\qquad \rho_1|_\Omega \leq \sigma \qquad \text{and} \qquad \rho_2|_\Omega \geq \zeta$$

where $\sigma$ and $\zeta$ are, respectively, critical levels for BOD and DO. For this system, we prove, in the subsequent section, the existence and uniqueness of solution.

# 3 Analysis of the State System

Let $\Omega \subset I\!R^2$ be a bounded domain with boundary smooth enough and $(0,T)$ an open interval. $Q$ denotes the cylindrical domain $\Omega \times (0,T)$, while $\Sigma = \partial\Omega \times (0,T)$ which is the lateral boundary of $Q$. We make the following assumptions of the problem data adapted from Martínez, Rodríguez, Vázquez-Méndez [12])

$$v \in [L^\infty(\bar{\Omega} \times [0,T])]^2, \quad h \in C(\bar{\Omega} \times [0,T]) \quad h(x,t) \geq \alpha > 0 \quad \forall (x,t) \in \bar{\Omega} \times [0,T])$$

$$\rho_{10}, \rho_{20} \in C^2(\bar{\Omega}), \quad a \in L^\infty[\Omega \times [0,T]], \quad b \in L^\infty[\Omega \times [0,T]]$$

**Definition 3.1.** Adapting the definition in Ladyzenskaja, Solonnikov and Uraltseva [9], the weak solution of the system (2.1) defined as a function $\rho = (\rho_1, \rho_2) \in [L^2(0,T;H^1(\Omega))]^2$, is called the solution of (2.1) if the following holds:

$$\int_Q \{-\frac{\partial\eta_1}{\partial t}\rho_1 - \frac{\partial\eta_2}{\partial t}\rho_2 + \beta_1\nabla\eta_1\nabla\rho_1 + \beta_2\nabla\eta_2\nabla\rho_2 + v\eta_1\nabla\rho_1 + v\eta_2\nabla\rho_2 + (a-b\rho_1)\rho_1\eta_1$$
$$+k_1\eta_1\rho_1 + k_1\eta_2\rho_1 + \frac{1}{h}k_2\eta_2\rho_2\}dxdt = \int_Q \frac{1}{h(x,t)}k_2 d_s\eta_2 dxdt + \int_\Omega \eta(x,0)\rho_0(x)dx$$
$$\forall \ \ \eta = (\eta_1,\eta_2) \in L^2(0,T;H^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \ \text{ with } \eta|_{\mathrm{t=T}} = 0, \ \eta|_\Gamma = 0.$$

Our main result for this section is the following:

**Theorem 3.1.**
*There exists a unique pair* $\rho = (\rho_1,\rho_2) \in [L^2(0,T;H^1(\Omega))]^2 \cap [L^2(0,T;L^2(\Omega))]^2$ *with* $\rho_t \in [[L^2(0,T;H^{-1}(\Omega))]^2]^2$ *of the state equation (2.1) satisfying the following*

$$||\rho||^2_{[[L^2(0,T;H^1(\Omega))]^2]^2} + ||\rho||^2_{[[L^2(0,T;L^2(\Omega))]^2]^2} + ||\rho_t||^2_{[[L(0,T;H^{-1}(\Omega))]^2]^2} \le C_4(||\rho_{10}||_{C(\bar\Omega)} + ||\rho_{20}||_{C(\bar\Omega)})$$

*Proof.* Let us consider $\rho^k \equiv \rho_i, \ i=1,2$ the solution of the approximate system

$$\left.\begin{array}{ll}
\frac{\partial\rho_1^k}{\partial t} + v.\nabla\rho_1^k - \beta_1\Delta\rho_1^k = \rho_1^k(a-b\rho_1^k) - k_1\rho_1^k & \text{in } Q \\
\rho_1^k(x,0) = \rho_{10}(x) & \text{in } \Omega \\
\frac{\partial\rho_1^k}{\partial n} = 0 & \text{on } \Sigma \\
\frac{\partial\rho_2^k}{\partial t} + v.\nabla\rho_2^k - \beta_2\Delta\rho_2^k = -k_1\rho_1^k + \frac{1}{h}k_2(d_s-\rho_2^k) & \text{in } Q \\
\rho_2^k(x,0) = \rho_{20}(x) & \text{in } \Omega \\
\frac{\partial\rho_2^k}{\partial n} = 0 & \text{on } \Sigma
\end{array}\right\} \tag{3.1}$$

Multiplying (3.1) by $\rho^k \equiv \rho_i, \ i=1,2$ and integrating over $Q$ we have

$$\begin{aligned}\int_Q \left\{\frac{\partial\rho_1^k}{\partial t}\rho_1^k + \frac{\partial\rho_2^k}{\partial t}\rho_2^k\right\}dxdt = \int_Q \Big\{&\beta_1\Delta\rho_1^k\rho_1^k + \beta_2\Delta\rho_2^k\rho_2^k - v\rho_1^k\nabla\rho_1^k \\ &-v\rho_2^k\nabla\rho_2^k + (a-b\rho_1^k)\rho_1^k\rho_1^k - k_1\rho_1^k\rho_1^k - k_1\rho_2^k\rho_1^k + \Big\}dxdt \\ &+\int_Q \frac{1}{h(x,t)}k_2(d_s-\rho_2^k)\rho_2^k dxdt\end{aligned} \tag{3.2}$$

resulting in,

(3.3)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\rho^k||^2_{[[L^2(\Omega)]^2]^2} \le \beta_i||\nabla\rho^k||^2_{[[L^2(\Omega)]^2]^2} &+ ||v||_{(L^\infty(\bar\Omega)\times[0,T])}||\nabla\rho^k||_{[[L^2(\Omega)]^2]^2}||\rho^k||_{[[L^2(\Omega)]^2]^2} \\ &+k_1||\rho^k||^2_{[[L^2(\Omega)]^2]^2} + ||a||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||^2_{[L^2(\Omega)]^2} \\ &+||b||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||_{[L^2(\Omega)]^2}||\rho_1^k||^2_{[L^2(\Omega)]^2} \\ &+\frac{1}{\alpha}k_2 d_s||\rho_2^k||_{[L^2(\Omega)]^2} + \frac{1}{\alpha}k_2||\rho_2^k||^2_{[L^2(\Omega)]^2}\end{aligned}$$

where $\beta_i \equiv \beta_i,\ i = 1, 2$. By Cauchy - Schwartz inequality and Poincar$\acute{e}$ inequality then (3.3) simplifies to

$$\begin{aligned}(3.4)\qquad &\frac{d}{dt}||\rho^k||^2_{[[L^2(\Omega)]^2]^2} \le C_1||\rho^k||^2_{[[H^1(\Omega)]^2]^2} + C_2||\rho^k||^2_{[[L^2(\Omega)]^2]^2} \\ &+||b||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||_{[L^2(\Omega)]^2}||\rho_1^k||^2_{[L^2(\Omega)]^2} + C_3||\rho_2^k||^2_{[L^2(\Omega)]^2}\end{aligned}$$

where $C_1 = 2(\beta_i + ||v||_{L^\infty(\bar{\Omega})}),\ C_2 = 2(||v||_{L^\infty(\bar{\Omega})} + 2k_1 + ||a||_{L^\infty(\Omega)} + \frac{1}{\alpha}k_2)$ and $C_3 = \frac{1}{\alpha}k_2 d_s$

then by Gronwall's inequality

$$||\rho^k||^2_{[[L^2(\Omega)]^2]^2} \le C_4||\rho_0||^2_{C(\bar{\Omega})}$$

then integrating (3.3) from 0 to $T$ and using Cauchy - Schwartz inequality, Poincar$\acute{e}$ inequality and Gronwall's inequality, we have

$$||\rho^k||^2_{[[L^2(0,T;L^2(\Omega))]^2]^2} \le C_4||\rho_0||^2_{C(\bar{\Omega})}$$

integrating (3.4) from 0 to $T$ and employing Gronwall's inequality, we have

$$||\rho^k||^2_{[[L^2(0,T;H^1(\Omega))]^2]^2} \le C_4||\rho_0||^2_{C(\bar{\Omega})}$$

for any $\eta = (\eta_1, \eta_2) \in C_0^\infty(Q)$, we have

$$\begin{aligned}\int_Q\{\frac{\partial\rho_1^k}{\partial t}\eta_1 + \frac{\partial\rho_2^k}{\partial t}\eta_2\}dxdt = \int_Q\{&-\beta_1\Delta\rho_1^k\eta_1 - \beta_2\Delta\rho_2^k\eta_2 - v\eta_1\nabla\rho_1^k \\ &-v\eta_2\nabla\rho_2^k + (a - b\rho_1^k)\rho_1^k\eta_1 - k_1\eta_1\rho_1^k - k_1\eta_2\rho_1^k + \}dxdt \\ &+\int_Q\frac{1}{h(x,t)}k_2(d_s - \rho_2^k)\eta_2 dxdt\end{aligned}$$

Consequently

$$\begin{aligned}||\frac{\partial\rho^k}{\partial t}||_{[[H^{-1}(\Omega)]^2]^2} + \le \beta_i||\nabla\rho^k||^2_{[[L^2(\Omega)]^2]^2} + ||v||_{(L^\infty(\bar{\Omega})\times[0,T])}||\nabla\rho^k||_{[[L^2(\Omega)]^2]^2}||\rho||^2_{[[L^2(\Omega)]^2]^2} \\ +k_1||\rho^k||^2_{[[L^2(\Omega)]^2]^2} + ||a||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||^2_{[L^2(\Omega)]^2} \\ +||b||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||_{[L^2(\Omega)]^2}||\rho_1^k||^2_{[L^2(\Omega)]^2} + \frac{1}{\alpha}k_2 d_s||\rho_2^k||^2_{[L^2(\Omega)]^2} + \frac{1}{\alpha}k_2||\rho_2^k||^2_{[L^2(\Omega)]^2}\end{aligned}$$

Thus

$$\begin{aligned}\int_0^T||\frac{\partial\rho^k}{\partial t}||_{[[H^{-1}(\Omega)]^2]^2}dt + \le \beta_i||\nabla\rho^k||^2_{[[L^2(\Omega)]^2]^2} + ||v||_{(L^\infty(\bar{\Omega})\times[0,T])}||\nabla\rho^k||_{[[L^2(\Omega)]^2]^2}||\rho||^2_{[[L^2(\Omega)]^2]^2} \\ +k_1||\rho^k||^2_{[[L^2(\Omega)]^2]^2} + ||a||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||^2_{[L^2(\Omega)]^2} + ||b||_{(L^\infty(\Omega)\times[0,T])}||\rho_1^k||_{[L^2(\Omega)]^2}||\rho_1^k||^2_{[L^2(\Omega)]^2} \\ +\frac{1}{\alpha}k_2 d_s||\rho_2^k||^2_{[L^2(\Omega)]^2} + \frac{1}{\alpha}k_2||\rho_2^k||^2_{[L^2(\Omega)]^2}\end{aligned}$$

and therefore

$$||\rho_t^k||^2_{[[L^2(0,T;H^{-1}(\Omega))]^2]^2} \leq C_4||\rho_0||^2_{C(\bar{\Omega})}$$

Consequently, there exists a subsequent $\rho^k$ of $\rho$ such that

$$\begin{array}{ll} \rho^k \rightharpoonup \rho & \text{in } [L^2(\Omega)]^2 \text{ and in } [L^2(0,T:H^1(\Omega))]^2 \\ \rho_t^k \rightharpoonup \rho & \text{in } [L^2(0,T:H^{-1}(\Omega))]^2 \end{array}$$

On passing to limit in (3.1), we see that $\rho$ is also a solution of (2.1) by the definition of 3.1.

To prove uniqueness of solution, we let $w = \rho - \hat{\rho}$ the difference of two possible solutions. Then define for any $t_1 \in [0,T]$, set

$$\eta = \begin{cases} -\int_{t_1}^{t} w \; d\tau & t \in [0,t_1) \\ 0 & t \in [t_1,T] \end{cases} \quad \text{where } \mathrm{t_1 \in (0,T)} \text{ and } \mathrm{w = (w_1, w_2)}$$

By the definition of solution we have

$$\int_{Q_{t_1}} \left\{ -\frac{\partial \eta}{\partial t} w_1 - \frac{\partial \eta}{\partial t} w_2 + v\eta\nabla w_1 + v\eta\nabla w_2 + \beta_1\nabla\eta\nabla w_1 + \beta_1\nabla\eta\nabla w \right\} dxdt$$

$$= \int_{Q_{t_1}} \left\{ (a - b(\rho_1 + \hat{\rho_1}))w_1\eta - 2k_1\eta w_1 \right\} dxdt + \int_{Q_{t_1}} \left\{ \frac{1}{\bar{h}(x,t_1)}(k_2 d_s - w_2)\eta \right\} dxdt$$
$$+ \int_{\Omega} \eta(x,0) w_0(x) dx$$

Then, it follows that

$$\int_{Q_{t_1}} \{\eta_t^2 + \eta_t^2 - v\eta\eta_{xt} - v\eta\eta_{xt} - \beta_1\eta_x\eta_{xt} - \beta_2\eta_x\eta_{xt}\} dxdt = \int_{Q_{t_1}} \Big\{ -(a - b(\rho_1 + \hat{\rho_1}))\eta\eta_t$$
$$+2k_1\eta\eta_t \Big\} dxdt + \int_{Q_{t_1}} \left\{ \frac{1}{\bar{h}(x,t_1)}(k_2 d_s\eta + \eta\eta_t) \right\} dxdt + \int_{\Omega} \eta(x,0)\rho_0(x)dx$$

which gives,

$$||\eta_t||^2_{[[L^2(Q_{t_1})]^2]^2} + c_1||\eta_x(x,0)||^2_{[[L^2(\Omega)]^2]^2} \leq c_2||\eta||_{[[L^2(Q_{t_1})]^2]^2}||\eta_t||_{[[L^2(Q_{t_1})]^2]^2}$$
$$+\frac{1}{2}\frac{1}{\alpha}d_s||\eta||^2_{[[L^2(Q_{t_1})]^2]^2} + ||\eta(x,0)||_{[[L^2(\Omega)]^2]^2}||\rho_0(x)||^2_{[C(\bar{\Omega})]^2}$$

where $c_1 = \frac{1}{2}\beta_i$ and $c_2 = \frac{1}{2}((a - b(\rho_1 + \hat{\rho_1}))) + 2k_1 + \frac{1}{\alpha}k_2$, then by Cauchy - Schwartz inequality we have

$$||\eta_t||^2_{[[L^2(Q_{t_1})]^2]^2} + c_1||\eta_x(x,0)||^2_{[[L^2(\Omega)]^2]^2} \leq c_2||\eta||_{[[L^2(Q_{t_1})]^2]^2} + c_2||\eta_t||_{[[L^2(Q_{t_1})]^2]^2}$$
$$+\frac{1}{2}\frac{1}{\alpha}d_s||\eta||^2_{[[L^2(Q_{t_1})]^2]^2} + ||\eta(x,0)||_{[[L^2(\Omega)]^2]^2}||\rho_0(x)||^2_{[C(\bar{\Omega})]^2}$$

and hence

$$(1-c_2)||\eta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_3||\eta||^2_{[[L^2(Q_{t_1})]^2]^2}$$

where $c_3 = \frac{1}{2}\frac{1}{\alpha}d_s + c_2$

thus,

$$||\eta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_4||\eta||^2_{[[L^2(Q_{t_1})]^2]^2} \tag{3.5}$$

where $c_4 = \frac{(1-c_2)}{c_3}$

Now, we let $\zeta(x,t) = \int_0^t w(x,\xi)d\xi$. Then, for $t \in [0,t_1]$

$$\zeta_t = w = -\eta_t, \qquad \eta(x,t) = \zeta(x,t_1) - \zeta(x,t) \tag{3.6}$$

and on the other hand

$$||\zeta(.\ ,t_1)||^2_{[[L^2(\Omega)]^2]^2} = 2\int_Q \int_0^{t_1} \zeta\zeta_t dtdx \le ||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} + ||\zeta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \tag{3.7}$$

and

$$||\eta||^2_{[[L^2(Q_{t_1})]^2]^2} \le 2||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} + 2t_1||\zeta(x,t_1)||^2_{[[L^2(\Omega)]^2]^2} \tag{3.8}$$

Hence, combining (3.5) to (3.7), we obtain

$$||\zeta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_4||\eta||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_4||\zeta(.\ ,t_1)||^2_{[[L^2(Q_{t_1})]^2]^2} + c_4||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2}$$

In view of (3.8), we have

$$||\zeta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_4t_1||\zeta||^2_{[[L^2(\Omega)]^2]^2} + c_4||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_5t_1||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} + c_4||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2}$$

$$-c_5t_1||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} + ||\zeta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le c_4||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2}$$

For a $t$ satisfying the condition

$$0 \le t_1 \le \frac{1}{2c_5}$$

hence

$$||\zeta_t||^2_{[[L^2(Q_{t_1})]^2]^2} \le ||\zeta||^2_{[[L^2(Q_{t_1})]^2]^2} \qquad \forall\ \ t \in [0,t_1]$$

$$\int_\Omega |\zeta(x,t)|^2 dx = 2\int_0^t \int_Q \zeta(x,t)\zeta_t(x,t)dxdt \le C\int_0^t \int_Q |\zeta|dxdt, \qquad t \in\ \ [0,t_1] \tag{3.9}$$

Then, by Gronwall's inequality, we end up with $\zeta = 0$ for $t \in [0,t_1]$. This implies $\rho = \hat{\rho}$ for $t \in [0,t_1]$. Continuing the argument for $t \in [t_1, 2t_1]$ and so on, we conclude that $\rho = \hat{\rho}$ on $Q$

□

# 4 Optimal Control

In this section, we prove the existence and uniqueness of optimal solution to the state equations (2.1), given an appropriate objective functional stated below.

## 4.1 The Optimization Problem

We state the optimal control problem. We look for a $(\rho, m) \in H^1(\Omega) \times \mathcal{U}_{ad}$ such that the cost functional:

$$J(\rho, m) = \frac{1}{2}\int_Q |\rho - \rho_d|^2 \, dxdt + \frac{\xi}{2}\int_0^T m^2 dt \tag{4.1}$$

is minimized subject to the constraints

$$\left.\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + v.\nabla\rho_1 - \beta_1\Delta\rho_1 = \rho_1(a - b\rho_1) - k_1\rho_1 + m && \text{in } Q\\
&\rho_1(x, 0) = \rho_{10}(x) && \text{in } \Omega\\
&\frac{\partial \rho_1}{\partial n} = 0 && \text{on } \Sigma\\
&\frac{\partial \rho_2}{\partial t} + v.\nabla\rho_2 - \beta_2\Delta\rho_2 = -k_1\rho_1 + \frac{1}{h}k_2(d_s - \rho_2) && \text{in } Q\\
&\rho_2(x, 0) = \rho_{20}(x) && \text{in } \Omega\\
&\frac{\partial \rho_2}{\partial n} = 0 && \text{on } \Sigma
\end{aligned}\right\} \tag{4.2}$$

Now, let $\mathcal{U}_{ad}$, the admissible space of control be defined as

$$\mathcal{U}_{ad} = \{m : 0 < J(m) \leq \text{ and (4.2) are satisfied}\}.$$

The control problem is to find the values of $m > 0$, in such a way that they satisfy (4.2) and they minimize the objective function, i.e.,

$$J(\bar{\rho}, \bar{m}) \leq J(\rho, m) \qquad \forall\ (\rho, m) \in H^1(\Omega) \times \mathcal{U}_{ad} \tag{4.3}$$

Any element $\bar{m} \in \mathcal{U}_{ad}$, satisfying (4.1) is called an optimal control and the corresponding state, denoted by $\bar{\rho}$ is called an optimal state.

## 4.2 The Existence of an Optimal Solution

We now show the existence of an optimal solution and give the following theorem

**Theorem 4.1.** *If there exists a feasible control $\bar{m} \in \mathcal{U}_{ad}$ such that $\rho_1|_\Omega \leq \sigma$ and $\rho_2|_\Omega \geq \zeta$ then the optimal problem has, at least, a solution.*

**Proof:** The set $\mathcal{U}_{ad}$ is nonempty, thus may choose $(\rho(m^k), m^k)$ in $H^1(\Omega) \times \mathcal{U}_{ad}$ such that

$$\lim_{k\to\infty} J(\rho^k, m^k) \ = \inf_{(\rho,m)\in H^1(\Omega)\times\mathcal{U}_{ad}} J(\rho, m).$$

Set $\rho(m^k) = \rho^k$. By the definition of $\mathcal{U}_{ad}$ we have

$$(4.4)\quad \left.\begin{array}{ll}
\dfrac{\partial \rho_1^k}{\partial t} + v.\nabla\rho_1^k - \beta_1\Delta\rho_1^k = \rho_1^k(a - b\rho_1^k) - k_1\rho_1^k & \text{in } Q \\
\rho_1^k(x,0) = \rho_{10}(x) & \text{in } \Omega \\
\dfrac{\partial \rho_1^k}{\partial n} = 0 & \text{on } \Sigma \\
\dfrac{\partial \rho_2^k}{\partial t} + v.\nabla\rho_2^k - \beta_2\Delta\rho_2^k = -k_1\rho_1^k + \dfrac{1}{h}k_2(d_s - \rho_2^k) & \text{in } Q \\
\rho_2^k(x,0) = \rho_{20}(x) & \text{in } \Omega \\
\dfrac{\partial \rho_2^k}{\partial n} = 0 & \text{on } \Sigma
\end{array}\right\}$$

Let $\{m^k\}_{n\in N} \in \mathcal{U}_{ad}$ be a minimizing sequence. From boundedness of the sequence we can deduce the existence of a subsequence (still denoted the same way) that converges weakly in $(L^2(0,T))$ to an element $m \in \mathcal{U}_{ad}$. From theorem (3.1) we have that the sequences $\rho^k = (\rho_1^k, \rho_2^k) = (\rho_1^k(m^k), \rho_2^k(m^k))$ are uniformly bounded. Since the embedding from $[H^1(\Omega)]^2 \hookrightarrow\hookrightarrow [L^2(\Omega)]^2$ is compact and using the compactness lemma, it follows that we may extract a subsequence, denoted again by $\rho^k$ such that

$$\begin{array}{ll}
\rho_k \rightharpoonup \rho & \text{in } \ [L^2(\Omega)]^2 \\
\rho_k \rightharpoonup \rho & \text{in } [L^2(0, T : H^1(\Omega))]^2
\end{array}$$

So, passing to the limit, we obtain that $\rho$ satisfies (4.4) and consequently, $\rho = \rho(m)$. since $J(\rho, m)$ is lower semi continuous we conclude that $(\bar{\rho}, \bar{m})$ is an optimal solution, i.e.

$$J(\bar{\rho}, \bar{m}) \ = \inf_{(\rho,m)\ \in\ \mathcal{U}_{ad}} J(\rho, m).$$

Thus, we have shown that an optimal solution belonging to $\mathcal{U}_{ad}$ exists.

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